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LACUNARY STATISTICAL CONVERGENCE FOR SEQUENCE OF SETS IN INTUITIONISTIC FUZZY METRIC SPACE

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Abstract. We investigate the concept of lacunary statistical convergence and lacunary strongly convergence for sequence of sets in intuitionistic fuzzy metric space (IFMS) and examine their characterization. We obtain some inclusion relation relating to these concepts. Further some necessary and sufficient conditions for equality of the sets of statistical convergence and lacunary statistical convergence for sequence of sets in IFMS have been established. The concept of strong Cesàro summability in IFMS has been defined and some results are established.

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1. Introduction and Background

Theory of fuzzy sets (FSs) was firstly given by Zadeh [1]. The publication of the paper affected deeply all the scientific fields. This notion is significant for real-life conditions, but has not adequate solution to some problems. Such problems lead to original quests.

The notion of a fuzzy norm on a linear space was first originated by Katsaras [2]. Felbin [3] gave an alternative idea of a fuzzy norm whose concerned metric is of Kaleva and Seikkala [4] type.

Intuitionistic fuzzy sets was examined by Atanassov [5] is appropriate for such a situation. The notion of intuitionistic fuzzy metric space has been introduced by Park [6]. Furthermore, the concept of intuitionistic fuzzy normed space is given by Saadati and Park [7]. A lot of improvement has been made in the area

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of intuitionistic fuzzy normed space after the studies of [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Statistical convergence of a real number sequence was firstly originated by Fast [21]. It became a noteworthy topic in summability theory after the work of Fridy [22] and Šalát [23].

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. Using lacunary sequence, Fridy and Orhan [24] examined the concept of lacunary statistical convergence. The publication of the paper affected deeply all the scientific fields. Some works in lacunary statistical convergence can be found in [25, 26, 27, 28, 29].

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see [30, 31, 32, 33, 34, 35]). Nuray and Rhoades [36] extended the notion of convergence of set sequences to statistical convergence and gave some fundamental theorems.

Ulusu and Nuray [37] introduced the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was studied by Nuray and Rhoades [36].

Firstly, we recall some definitions used throughout the paper.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, A).$$

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this cace we write $W - \lim A_k = A$ ([34]).

As an example, consider the following sequence of circles in the (x, y)-plane: $A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}$. As $k \to \infty$ the sequence is Wijsman convergent to the y-axis $A = \{(x, y) : x = 0\}$.

Let (X, d) a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

$$WS := \{\{A_k\} : st - \lim {}_W A_k = A\}$$

where WS denotes the set of Wijsman statistical convergence sequences ([36]).

Let (X,ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary statistical convergent to A if $\{d(x, A_k)\}$ is lacunary statistically convergent to $d(x, A)$; i.e., for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_r \frac{1}{h_r} |k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon| = 0.$$

In this case we write $S_\theta - \lim_W A_k = A$ or $A_k \to A(WS_\theta)$ ([37]).

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$, for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$ ([7]).

A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\Diamond$ satisfies the following conditions:

(i) $\Diamond$ is commutative and associative,
(ii) $\Diamond$ is continuous,
(iii) $a \Diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$ ([7]).

Park [6] initiated the concepts of intuitionistic fuzzy metric spaces as a natural generalization of fuzzy metric spaces due to the George and Veeramani [38] which is defined as follows:

Let $\mathcal{X}$ be a non-empty set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and ϕ, ω are fuzzy sets on $\mathcal{X}^2 \times (0,\infty)$. Then, the five-tuple $(\mathcal{X},\phi,\omega,*,\Diamond)$ is known as an intuitionistic fuzzy metric space (for short, IFMS) satisfying the following conditions for every $x, y, z \in \mathcal{X}$ and for all $s, t > 0$:

(i) $\phi(x,y,t) + \omega(x,y,t) \leq 1$,
(ii) $\phi(x,y,t) > 0$,
(iii) $\phi(x,y,t) = 1$ if and only if $x = y$,
(iv) $\phi(x,y,t) = \phi(y,x,t)$,
(v) $\phi(x,y,t) * \phi(y,z,s) \leq \phi(x,z,t+s)$,
(vi) $\phi(x,y,.) : (0,\infty) \to [0,1]$ is continuous;
(vii) $\omega(x,y,t) > 0$,
$(viii)$ $\omega(x,y,t) = 0$ if and only if $x = y$,
(ix) $\omega(x,y,t) = \omega(y,x,t)$
(x) $\omega(x,y,t) \Diamond \omega(y,z,s) \geq \omega(x,z,t+s)$,
(xi) $\omega(x,y,.) : (0,\infty) \to [0,1]$ is continuous in t;

In case (ϕ,ω) is called intuitionistic fuzzy metric on $\mathcal{X}$.

Let $(\mathcal{X},\phi,\omega,*,\Diamond)$ be an IFMS. A sequence $x = (x_k)$ in $\mathcal{X}$ is convergent to $x \in \mathcal{X}$ iff $\phi(x_n,x,t) \to 1$ and $\omega(x_n,x,t) \to 0$ as $n \to \infty$, for each $t > 0$ ([6]).

Let $(\mathcal{X},\phi,\omega,*,\Diamond)$ be an IFMS. A sequence $x = (x_k)$ in $\mathcal{X}$ is called a Cauchy sequence if for every $\varepsilon > 0$ and $t > 0$, there exists $m_0 \in \mathbb{N}$ such that $\phi(x_k - x_m, t) > 1 - \varepsilon$ and $\omega(x_k - x_m, t) < \varepsilon$ for all $k, m \geq m_0$ ([6]).

The aim of the present paper is to introduce the concepts of lacunary statistical convergence and lacunary strongly convergence for sequence of sets in intuitionistic fuzzy metric space (IFMS) and obtain some important results on these concepts. Also, we have introduced the concept of lacunary statistically Cauchy sequences of sets in IFMS and given some new characterizations of it. Further some inclusion relations between the concepts of statistical convergence and lacunary statistical convergence for sequence of sets in IFMS have been established.

2. MAIN RESULTS

In this section, we introduce and study the notion of Wijsman lacunary statistically convergence for a sequence of sets in IFMS and obtain some basic results.

Definition 2.1. Let $(\mathcal{X},\phi,\omega,*,\diamondsuit)$ be an IFMS and θ be a lacunary sequence. Let T, $\{T_k\}$ be any nonempty closed subsets of $\mathcal{X}$, one say that the sequence $\{T_k\}$ is Wijsman lacunary statistically convergent to T with regards to the IFM (ϕ,ω), if for every $\varepsilon\in(0,1)$, for each $x\in\mathcal{X}$ and for all $p>0$,

$$\delta_\theta\left(\left\{\begin{array}{c} k\in\mathbb{N}:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\leq 1-\varepsilon \\ \text{or } \left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\geq\varepsilon \end{array}\right\}\right)=0, \tag{1}$$

or equivalently

$$\delta_\theta\left(\left\{\begin{array}{c} k\in\mathbb{N}:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|> 1-\varepsilon \\ \text{and } \left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|<\varepsilon \end{array}\right\}\right)=1. \tag{2}$$

In this case, we write $WS_\theta^{(\phi,\omega)}-\lim T_k=T$ or $T_k\overset{(\phi,\omega)}{\to}T(WS_\theta)$. We indicate the set of all Wijsman S_θ-convergent set sequences with regards to IFM (ϕ,ω) by $WS_\theta^{(\phi,\omega)}$.

By using (1) and (2), we easily get the following lemma.

Lemma 2.2. *Let $(\mathcal{X},\phi,\omega,*,\diamondsuit)$ be an IFMS and θ be a lacunary sequence. For every $\varepsilon>0$ and $p>0$, the following statements are equivalent:*

(a) $WS_\theta^{(\phi,\omega)}-\lim T_k=T$;

(b)

$$\begin{aligned}&\delta_\theta\left(\left\{k\in\mathbb{N}:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\leq 1-\varepsilon\right\}\right)\\ &\quad=\delta_\theta\left(\left\{k\in\mathbb{N}:\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\geq\varepsilon\right\}\right)=0;\end{aligned}$$

(c)

$$\delta_\theta\left(\left\{\begin{array}{c} k\in\mathbb{N}:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|> 1-\varepsilon \\ \text{and } \left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|<\varepsilon \end{array}\right\}\right)=1;$$

(d)

$$\begin{aligned}&\delta_\theta\left(\left\{k\in\mathbb{N}:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|> 1-\varepsilon\right\}\right)\\ &\quad=\delta_\theta\left(\left\{k\in\mathbb{N}:\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|<\varepsilon\right\}\right)=1;\end{aligned}$$

(e)

$$WS_{\theta}^{(\phi,\omega)} - \lim |\phi(x,T_k,p) - \phi(x,T,p)| = 1$$
$$\text{and } WS_{\theta}^{(\phi,\omega)} - \lim |\omega(x,T_k,p) - \omega(x,T,p)| = 0.$$

Theorem 2.3. *If $WS_{\theta}^{(\phi,\omega)} - \lim T_k = T$, then $WS_{\theta}^{(\phi,\omega)}$-limit is unique.*

Proof. Assume that there are two distinct sets $T_1, T_2 \in X$ such that $WS_{\theta}^{(\phi,\omega)} - \lim T_k = T_1$ and $WS_{\theta}^{(\phi,\omega)} - \lim T_k = T_2$. Given $\varepsilon > 0$, select $\gamma > 0$ such that $(1-\gamma) * (1-\gamma) > 1-\varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. Hence, for any $p > 0$, take the following sets as:

$$\mathcal{K}_{\phi,1}(\gamma,p) = \left\{k \in \mathbb{N} : \left|\phi\left(x,T_k,\tfrac{p}{2}\right) - \phi\left(x,T_1,\tfrac{p}{2}\right)\right| \leq 1-\gamma\right\},$$

$$\mathcal{K}_{\phi,2}(\gamma,p) = \left\{k \in \mathbb{N} : \left|\phi\left(x,T_k,\tfrac{p}{2}\right) - \phi\left(x,T_2,\tfrac{p}{2}\right)\right| \leq 1-\gamma\right\},$$

$$\mathcal{K}_{\omega,1}(\gamma,p) = \left\{k \in \mathbb{N} : \left|\omega\left(x,T_k,\tfrac{p}{2}\right) - \omega\left(x,T_1,\tfrac{p}{2}\right)\right| \geq \gamma\right\},$$

$$\mathcal{K}_{\omega,2}(\gamma,p) = \left\{k \in \mathbb{N} : \left|\omega\left(x,T_k,\tfrac{p}{2}\right) - \omega\left(x,T_2,\tfrac{p}{2}\right)\right| \geq \gamma\right\}.$$

Since $WS_{\theta}^{(\phi,\omega)} - \lim T_k = T_1$, we have by Lemma 2.2

$$\delta_\theta(\mathcal{K}_{\phi,1}(\gamma,p)) = \delta_\theta(\mathcal{K}_{\omega,1}(\gamma,p)) = 0 \text{ for all } p > 0.$$

Futhermore, utilizing $WS_{\theta}^{(\phi,\omega)} - \lim T_k = T_2$, we get

$$\delta_\theta(\mathcal{K}_{\phi,2}(\gamma,p)) = \delta_\theta(\mathcal{K}_{\omega,2}(\gamma,p)) = 0 \text{ for all } p > 0.$$

Now let

$$\mathcal{K}_{\phi,\omega}(\gamma,p) = (\mathcal{K}_{\phi,1}(\gamma,p) \cup \mathcal{K}_{\phi,2}(\gamma,p)) \cap (\mathcal{K}_{\omega,1}(\gamma,p) \cup \mathcal{K}_{\omega,2}(\gamma,p)).$$

Then, observe that $\delta_\theta(\mathcal{K}_{\phi,\omega}(\gamma,p)) = 0$ which gives $\delta_\theta(\mathbb{N} \setminus \mathcal{K}_{\phi,\omega}(\gamma,p)) = 1$. If $k \in \mathbb{N} \setminus \mathcal{K}_{\phi,\omega}(\gamma,p)$, then we have two possible cases.

Case (i) $k \in \mathbb{N} \setminus (\mathcal{K}_{\phi,1}(\gamma,p) \cup \mathcal{K}_{\phi,2}(\gamma,p))$,

Case (ii) $k \in \mathbb{N} \setminus (\mathcal{K}_{\omega,1}(\gamma,p) \cup \mathcal{K}_{\omega,2}(\gamma,p))$.

We first think that $k \in \mathbb{N} \setminus (\mathcal{K}_{\phi,1}(\gamma,p) \cup \mathcal{K}_{\phi,2}(\gamma,p))$. Then, we have

$$\phi(T_1 - T_2, p) \geq \left|\phi\left(x,T_k,\tfrac{p}{2}\right) - \phi\left(x,T_1,\tfrac{p}{2}\right)\right|$$
$$* \left|\phi\left(x,T_k,\tfrac{p}{2}\right) - \phi\left(x,T_2,\tfrac{p}{2}\right)\right| > (1-\gamma) * (1-\gamma).$$

Since $(1-\gamma) * (1-\gamma) > 1-\varepsilon$, it follows that $\phi(T_1 - T_2, p) > 1-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\phi(T_1 - T_2, p) = 1$ for all $p > 0$, which yields $T_1 = T_2$. On the other side, if $k \in \mathbb{N} \setminus (\mathcal{K}_{\omega,1}(\gamma,p) \cup \mathcal{K}_{\omega,2}(\gamma,p))$, then we can write

$$\omega(T_1 - T_2, p) < \left|\omega\left(x,T_k,\tfrac{p}{2}\right) - \omega\left(x,T_1,\tfrac{p}{2}\right)\right|$$
$$\Diamond \left|\omega\left(x,T_k,\tfrac{p}{2}\right) - \omega\left(x,T_2,\tfrac{p}{2}\right)\right| < \gamma \Diamond \gamma.$$

Now utilizing the fact that $\gamma\Diamond\gamma < \varepsilon$, we see that $\omega(T_1 - T_2, p) < \varepsilon$. Since arbitrary $\varepsilon > 0$, we obtain $\omega(T_1 - T_2, p) = 0$ for all $p > 0$. This gives that $T_1 = T_2$. So, we conclude that $WS_\theta^{(\phi,\omega)}$-limit is unique. □

Definition 2.4. Let $(\mathcal{X},\phi,\omega,*,\Diamond)$ be an IFMS. Then a sequence of sets $\{T_k\}$ is said to be Wijsman convergent to T with regards to the IFM (ϕ,ω), if for every $\varepsilon \in (0,1)$, for each $x \in \mathcal{X}$ and for all $p > 0$, there exists $k_0 \in \mathbb{N}$ such that $|\phi(x,T_k,p) - \phi(x,T,p)| > 1-\varepsilon$ and $|\omega(x,T_k,p) - \omega(x,T,p)| < \varepsilon$ for all $k \geq k_0$. In this case, we write $(\phi,\omega)_W - \lim T_k = T$.

Theorem 2.5. *Let $(\mathcal{X},\phi,\omega,*,\Diamond)$ be an IFMS and θ be a lacunary sequence. If $(\phi,\omega)_W - \lim T_k = T$, then $WS_\theta^{(\phi,\omega)} - \lim T_k = T$.*

Proof. Let $(\phi,\omega)_W - \lim T_k = T$. Then, for every $\varepsilon > 0$ and $p > 0$, there is a number $k_0 \in \mathbb{N}$ such that

$$|\phi(x,T_k,p) - \phi(x,T,p)| > 1-\varepsilon$$

and

$$|\omega(x,T_k,p) - \omega(x,T,p)| < \varepsilon$$

for all $k \geq k_0$. Hence, the set

$$\left\{ \begin{array}{c} k \in \mathbb{N} : |\phi(x,T_k,p) - \phi(x,T,p)| \leq 1-\varepsilon \\ \text{or } \ |\omega(x,T_k,p) - \omega(x,T,p)| \geq \varepsilon \end{array} \right\}$$

has finite number of terms. Since every finite subsets of $\mathbb{N}$ has density zero and hence

$$\delta_\theta\left(\left\{ \begin{array}{c} k \in \mathbb{N} : |\phi(x,T_k,p) - \phi(x,T,p)| \leq 1-\varepsilon \\ \text{or } \ |\omega(x,T_k,p) - \omega(x,T,p)| \geq \varepsilon \end{array} \right\}\right) = 0,$$

that is, $WS_\theta^{(\phi,\omega)} - \lim T_k = T$. □

Definition 2.6. Let $(\mathcal{X},\phi,\omega,*,\Diamond)$ be an IFMS. Let T, $\{T_k\}$ be any nonempty closed subsets of $\mathcal{X}$, one say that the sequence $\{T_k\}$ is Wijsman statistically convergent to T with regards to the IFM (ϕ,ω), if for every $\varepsilon \in (0,1)$, for each $x \in \mathcal{X}$ and for all $p > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{ \begin{array}{c} k \leq n : |\phi(x,T_k,p) - \phi(x,T,p)| \leq 1-\varepsilon \\ \text{or } \ |\omega(x,T_k,p) - \omega(x,T,p)| \geq \varepsilon \end{array} \right\}\right| = 0.$$

In this case, we write $WS^{(\phi,\omega)} - \lim T_k = T$ or $T_k \overset{(\phi,\omega)}{\rightarrow} T(WS)$. We indicate the set of all Wijsman statistically convergent set sequences with regards to IFM (ϕ,ω) by $WS^{(\phi,\omega)}$.

Let $WS^{(\phi,\omega)}$ and $WS_\theta^{(\phi,\omega)}$ indicate the sets of Wijsman statistically and Wijsman lacunary statistically convergent sequences respectively in IFMS $(\mathcal{X},\phi,\omega,*,\Diamond)$.

For $x \in \mathcal{X}$, $p > 0$ and $\beta \in (0,1)$, the ball centered at x with radius β is introduced by

$$B^y(x,\beta,p) = \{y \in \mathcal{X}\colon\ \phi(x-y,p) > 1-\beta \text{ and } \omega(x-y,p) < \beta\}.$$

Theorem 2.7. *For any lacunary sequence* θ, $WS_{\theta}^{(\phi,\omega)} \subseteq WS^{(\phi,\omega)}$ *iff* $\limsup_r q_r < \infty$.

Proof. If $\limsup_r q_r < \infty$, then there is $H > 0$ such that $q_r < H$ for all r. Assume that $\{T_k\} \in WS_{\theta}^{(\phi,\omega)}$ and $WS_{\theta}^{(\phi,\omega)} - \lim T_k = T$. For $\gamma \in (0,1)$ and $p > 0$, let

$$N_r = \left| \left\{ \begin{array}{c} k \in I_r : \left| \phi\left(x, T_k, p\right) - \phi\left(x, T, p\right) \right| \leq 1 - \gamma \\ \text{or } \left| \omega\left(x, T_k, p\right) - \omega\left(x, T, p\right) \right| \geq \gamma \end{array} \right\} \right|$$

Then for $\varepsilon > 0$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{N_r}{h_r} < \varepsilon \text{ for all } r > r_0. \tag{3}$$

Take $K = \max\{N_r : 1 \leq r \leq r_0\}$ and select n such that $k_{r-1} < n \leq k_r$. Then, we obtain

$$\begin{aligned}
&\frac{1}{n} \left| \left\{ \begin{array}{c} k \leq n : \left| \phi\left(x, T_k, p\right) - \phi\left(x, T, p\right) \right| \leq 1 - \gamma \\ \text{or } \left| \omega\left(x, T_k, p\right) - \omega\left(x, T, p\right) \right| \geq \gamma \end{array} \right\} \right| \\
&\leq \frac{1}{k_{r-1}} \left| \left\{ \begin{array}{c} k \leq k_r : \left| \phi\left(x, T_k, p\right) - \phi\left(x, T, p\right) \right| \leq 1 - \gamma \\ \text{or } \left| \omega\left(x, T_k, p\right) - \omega\left(x, T, p\right) \right| \geq \gamma \end{array} \right\} \right| \\
&= \frac{1}{k_{r-1}} \left\{ N_1 + N_2 + ... + N_{r_0} + N_{r_0+1} + ... + N_r \right\} \\
&\leq \tfrac{K}{k_{r-1}} r_0 + \frac{1}{k_{r-1}} \left\{ h_{r_0+1} \tfrac{N_{r_0+1}}{h_{r_0+1}} + ... + h_r \tfrac{N_r}{h_r} \right\} \\
&\leq \tfrac{r_0 K}{k_{r-1}} + \varepsilon q_r \leq \tfrac{r_0 K}{k_{r-1}} + \varepsilon H.
\end{aligned}$$

To prove the converse, assume that $\limsup_r q_r = \infty$. Since θ is a lacunary sequence, we can choose a subsequence $\left\{k_{r(j)}\right\}$ of θ such that $q_{r(j)} > j$. Let $\{k\} (\neq \{0\}) \in \mathcal{X}$. We define a sequence $\{T_k\}$ as follows:

$$T_k = \left\{ \begin{array}{ll} \{k\}, & \text{if } k_{r(j)-1} < k < 2k_{r(j)-1} \text{ for some } j = 1, 2, ... \\ \{0\}, & \text{otherwise.} \end{array} \right.$$

Since $\{k\} (\neq \{0\})$, we can select $\gamma \in (0,1)$ and $p > 0$ such that $\{k\} \notin B^y(0, \gamma, p)$. Now for $j > 1$,

$$\frac{1}{h_{r(j)}} \left| \left\{ \begin{array}{c} k \leq k_{r(j)} : \left| \phi\left(x, T_k, p\right) - \phi\left(x, \{0\}, p\right) \right| \leq 1 - \gamma \\ \text{or } \left| \omega\left(x, T_k, p\right) - \omega\left(x, \{0\}, p\right) \right| \geq \gamma \end{array} \right\} \right| < \frac{1}{j-1}.$$

Thus $\{T_k\} \in WS_{\theta}^{(\phi,\omega)}$. But $\{T_k\} \notin WS^{(\phi,\omega)}$. For

$$\frac{1}{2k_{r(j)-1}} \left| \left\{ \begin{array}{c} k \leq 2k_{r(j)-1} : \left| \phi\left(x, T_k, p\right) - \phi\left(x, \{0\}, p\right) \right| \leq 1 - \gamma \\ \text{or } \left| \omega\left(x, T_k, p\right) - \omega\left(x, \{0\}, p\right) \right| \geq \gamma \end{array} \right\} \right| > \frac{1}{2}$$

and

$$\frac{1}{k_{r(j)}}\left|\left\{\begin{array}{c} k\le k_{r(j)}:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\le 1-\gamma \\ \text{or }\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\ge\gamma \end{array}\right\}\right|>1-\frac{2}{j},$$

which is a contradiction. ☐

Theorem 2.8. *For any lacunary sequence* θ, $WS^{(\phi,\omega)}\subseteq WS_\theta^{(\phi,\omega)}$ *iff* $\liminf_r q_r>1$.

Proof. Suppose first that $\liminf_r q_r>1$, then there is a $\alpha>0$ such that $q_r\ge 1+\alpha$ for sufficiently large r, which implies that $\frac{h_r}{k_r}\ge\frac{\alpha}{1+\alpha}$.

Let $WS^{(\phi,\omega)}-\lim T_k=T$. Then, for each $\gamma\in(0,1)$ and $p>0$ and sufficiently large r, we get

$$\frac{1}{k_r}\left|\left\{\begin{array}{c} k\le k_r:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\le 1-\gamma \\ \text{or }\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\ge\gamma \end{array}\right\}\right|$$

$$\ge\frac{\alpha}{1+\alpha}\frac{1}{h_r}\left|\left\{\begin{array}{c} k\in I_r:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\le 1-\gamma \\ \text{or }\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\ge\gamma \end{array}\right\}\right|.$$

Thus $WS_\theta^{(\phi,\omega)}-\lim T_k=T$.

Conversely, suppose that $\liminf_r q_r=1$. Since θ is a lacunary sequence, we can choose a subsequence $\left\{k_{r(j)}\right\}$ of θ such that

$$\frac{k_{r(j)}}{k_{r(j)-1}}<1+\frac{1}{j}\text{ and }\frac{k_{r(j)}-1}{k_{r(j-1)}}>j$$

where $r(j)>r(j-1)+2$. Let $\{k\}(\neq\{0\})\in\mathcal{X}$. We define a sequence $\{T_k\}$ as follows:

$$T_k=\left\{\begin{array}{ll} \{k\} & k\in I_{r(j)}\text{ for some } j=1,2,\ \ldots \\ \{0\}, & \text{otherwise.} \end{array}\right.$$

We denote that $\{T_k\}\in WS^{(\phi,\omega)}$. Let $\gamma\in(0,1)$ and $p>0$. Choose $p_1>0$ and $\gamma_1\in(0,1)$ such that $B(0,\gamma_1,p_1)\subset B(0,\gamma,p)$ and $\{k\}\notin B(0,\gamma_1,p_1)$. Also, for each $n\in\mathbb{N}$ we can find $j_n>0$ such that $k_{r(j_n)-1}<n\le k_{r(j_n)}$. Then, for each $n\in\mathbb{N}$, we get

$$\frac{1}{n}\left|\left\{\begin{array}{c} k\le n:\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\le 1-\gamma \\ \text{or }\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\ge\gamma \end{array}\right\}\right|$$

$$\le\frac{k_{r(j_n-1)}+h_{r(j_n)}}{k_{r(j_n)-1}}\le\frac{2}{j_n}.$$

Thus $WS^{(\phi,\omega)}-\lim T_k=T$. Next we show that $\{T_k\}\notin WS_\theta^{(\phi,\omega)}$. Since $\{k\}(\neq\{0\})\in\mathcal{X}$, so we can select $p>0$ and $\gamma\in(0,1)$ such that $\{k\}\notin B(0,\gamma,p)$. Thus

$$\lim_{j\to\infty}\frac{1}{h_{r(j)}}\left|\left\{\begin{array}{c} k\in I_{r(j)}:\left|\phi\left(x,T_k,p\right)-\phi\left(x,\{0\},p\right)\right|\le 1-\gamma \\ \text{or }\left|\omega\left(x,T_k,p\right)-\omega\left(x,\{0\},p\right)\right|\ge\gamma \end{array}\right\}\right|=1$$

and for $r \neq r_j$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ \begin{array}{c} k \in I_r : |\phi(x, T_k, p) - \phi(x, T, p)| \leq 1 - \gamma \\ \text{or } |\omega(x, T_k, p) - \omega(x, T, p)| \geq \gamma \end{array} \right\} \right| = 1.$$

Hence $\{T_k\} \notin WS_\theta^{(\phi,\omega)}$, a contradiction. □

Corollary 2.9. *Let* $(\mathcal{X}, \phi, \omega, *, \Diamond)$ *be an IFMS. For any lacunary sequence* θ, $WS^{(\phi,\omega)} = WS_\theta^{(\phi,\omega)}$ *iff* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$.

Definition 2.10. Let $(\mathcal{X}, \phi, \omega, *, \Diamond)$ be an IFMS and θ be a lacunary sequence. A sequence $\{T_k\}$ in $\mathcal{X}$ is said to be Wijsman lacunary summable to T with regards to the IFM (ϕ, ω) if, for every $x \in \mathcal{X}$ and $p > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \phi(x, T_k, p) = \phi(x, T, p)$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \omega(x, T_k, p) = \omega(x, T, p).$$

In this case we write $WN_\theta^{(\phi,\omega)} - \lim T_k = T$ or $T_k \to T\left(WN_\theta^{(\phi,\omega)}\right)$.

Example 2.11. Let $(\mathcal{X}$,$\phi, \omega, *, \Diamond)$ be an IFMS and T, $\{T_k\}$ be any nonempty closed subsets of $\mathcal{X}$, Assume $\mathcal{X} = \mathbb{R}$ and $\{T_k\}$ are sequence defined by

$$T_k = \begin{cases} \{x \in \mathbb{R}, 2 \leq x \leq k_r - k_{r-1}\}, & \text{if } k \geq 2 \text{ and } k \text{ is square integer,} \\ \{1\}, & \text{otherwise.} \end{cases}$$

This sequence is not Wijsman lacunary summable in IFMS with regards to the IFM (ϕ, ω). But since

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ \begin{array}{c} |\phi(x, T_k, p) - \phi(x, \{1\}, p)| \leq 1 - \varepsilon \\ \text{or } |\omega(x, T_k, p) - \omega(x, \{1\}, p)| \geq \varepsilon \end{array} \right\} \right| = 0,$$

this sequence is Wijsman lacunary statistically convergent to the set $T = \{1\}$ in IFMS with regards to the IFM (ϕ, ω).

Definition 2.12. Let $(\mathcal{X}, \phi, \omega, *, \Diamond)$ be an IFMS and θ be a lacunary sequence. A sequence $\{T_k\}$ in $\mathcal{X}$ is said to be Wijsman lacunary strongly convergent to T with respect to the IFM (ϕ, ω) if, for every $\varepsilon \in (0, 1)$ and $p > 0$, there exist $r_0 \in \mathbb{N}$

$$\frac{1}{h_r} \sum_{k\in I_r} |\phi(x, T_k, p) - \phi(x, T, p)| > 1 - \varepsilon$$

and

$$\frac{1}{h_r} \sum_{k\in I_r} |\omega(x, T_k, p) - \omega(x, T, p)| < \varepsilon$$

for all $r > r_0$. In this case we write $\left[WN_\theta^{(\phi,\omega)}\right] - \lim T_k = T$ or $T_k \to T\left(\left[WN_\theta^{(\phi,\omega)}\right]\right)$.

Definition 2.13. Let $(\mathcal{X}, \phi, \omega, *, \Diamond)$ be an IFMS and θ be a lacunary sequence. A sequence $\{T_k\}$ in $\mathcal{X}$ is strongly Cesàro summable to T with respect to the IFM (ϕ, ω) if, for every $\varepsilon \in (0,1)$ and $p > 0$, there exist $n_0 \in \mathbb{N}$

$$\frac{1}{n}\sum_{k=1}^{n} |\phi(x, T_k, p) - \phi(x, T, p)| > 1 - \varepsilon$$

and

$$\frac{1}{n}\sum_{k=1}^{n} |\omega(x, T_k, p) - \omega(x, T, p)| < \varepsilon$$

for all $n > n_0$. In this case we write $|W\sigma|^{(\phi,\omega)} - \lim T_k = T$ or $T_k \to T \left(|W\sigma|^{(\phi,\omega)}\right)$.

Theorem 2.14. *Let $(\mathcal{X}, \phi, \omega, *, \Diamond)$ be an IFMS and θ be a lacunary sequence. Then, $|W\sigma|^{(\phi,\omega)} \subseteq \left[WN_\theta^{(\phi,\omega)}\right]$ if $\liminf_r q_r > 1$.*

Proof. Let $\liminf_r q_r > 1$ and $\{T_k\} \in |W\sigma|^{(\phi,\omega)}$. Then there exists $\delta > 0$ such that $q_r > 1 + \delta$ for all $r \geq 1$. Then

$$\frac{1}{h_r}\sum_{k \in I_r} |\phi(x, T_k, p) - \phi(x, T, p)| - 1$$

$$= \frac{1}{h_r}\sum_{k=1}^{k_r} |\phi(x, T_k, p) - \phi(x, T, p)|$$

$$-\frac{1}{h_r}\sum_{k=1}^{k_{r-1}} |\phi(x, T_k, p) - \phi(x, T, p)| - 1$$

$$= \frac{k_r}{h_r}\left[\frac{1}{k_r}\sum_{k=1}^{k_r} |\phi(x, T_k, p) - \phi(x, T, p)| - 1\right]$$

$$-\frac{k_{r-1}}{h_r}\left[\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}} |\phi(x, T_k, p) - \phi(x, T, p)| - 1\right].$$

Since $h_r = k_r - k_{r-1}$, $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$, $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$.

Also the terms

$$\frac{1}{k_r}\sum_{k=1}^{k_r} |\phi(x, T_k, p) - \phi(x, T, p)| - 1$$

and

$$\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}} |\phi(x, T_k, p) - \phi(x, T, p)| - 1$$

both converges to zero. So,

$$\frac{1}{h_r}\sum_{k \in I_r} |\phi(x, T_k, p) - \phi(x, T, p)| \to 1.$$

Similarly

$$\frac{1}{h_r}\sum_{k\in I_r}\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\to 0.$$

So, $\{T_k\}\in\left[WN_\theta^{(\phi,\omega)}\right]$. ☐

Theorem 2.15. *Let $(\mathcal{X},\phi,\omega,*,\Diamond)$ be an IFMS and θ be a lacunary sequence. Then, $WN_\theta^{(\phi,\omega)}\subseteq|W\sigma|^{(\phi,\omega)}$ if $\liminf_r q_r=1$.*

Proof. Let $\liminf_r q_r=1$ and $\{T_k\}\in\left[WN_\theta^{(\phi,\omega)}\right]$. Then for $p>0$ we have

$$H_r=\frac{1}{h_r}\sum_{k\in I_r}\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\to 1$$

and

$$H_r'=\frac{1}{h_r}\sum_{k\in I_r}\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\to 0$$

as $r\to\infty$. Then for $\varepsilon>0$, there exists $r_0\in\mathbb{N}$ such that $H_r<1+\varepsilon$ for all $r>r_0$. Also we can find $K>0$ such that $H_r<K$ and $H_r'<K$, $r=1,2,....$ Let n be an integer with $k_{r-1}<n\leq k_r$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|$$

$$\leq\frac{1}{k_{r-1}}\sum_{k=1}^{k_r}\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|$$

$$=\frac{1}{k_{r-1}}\left[\sum_{I_1}\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|+...+\sum_{I_r}\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\right]$$

$$=\sup\nolimits_{1\leq r\leq r_0}H_r\frac{k_{r_0}}{k_{r-1}}+\frac{h_{r_0+1}}{k_{r-1}}H_{r_0+1}+...+\frac{h_r}{k_{r-1}}H_r$$

$$<K\frac{k_{r_0}}{k_{r-1}}+(1+\varepsilon)\frac{k_r-k_{r_0}}{k_{r-1}}.$$

Since $k_{r-1}\to\infty$ as $n\to\infty$, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\left|\phi\left(x,T_k,p\right)-\phi\left(x,T,p\right)\right|\to 1.$$

Similarly we can show that

$$\frac{1}{n}\sum_{k=1}^{n}\left|\omega\left(x,T_k,p\right)-\omega\left(x,T,p\right)\right|\to 0.$$

Hence $\{T_k\}\in|W\sigma|^{(\phi,\omega)}$. ☐

Theorem 2.16. *If* $\{T_k\} \in \left[WN_\theta^{(\phi,\omega)}\right] \cap |W\sigma|^{(\phi,\omega)}$, *then* $\left[WN_\theta^{(\phi,\omega)}\right] - \lim T_k = |W\sigma|^{(\phi,\omega)} - \lim T_k$.

Proof. Let $\left[WN_\theta^{(\phi,\omega)}\right] - \lim T_k = T_1$ and $|W\sigma|^{(\phi,\omega)} - \lim T_k = T_2$. Given $\varepsilon > 0$, select $\gamma > 0$ such that $(1-\gamma) * (1-\gamma) > 1 - \varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. Then, for any $p > 0$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} \left| \phi\left(x, T_k, \tfrac{p}{2}\right) - \phi\left(x, T_1, \tfrac{p}{2}\right) \right| > 1 - \gamma$$

$$\text{and } \frac{1}{h_r} \sum_{k \in I_r} \left| \omega\left(x, T_k, \tfrac{p}{2}\right) - \omega\left(x, T_1, \tfrac{p}{2}\right) \right| < \gamma$$

for all $r > r_0$. Also, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n} \sum_{k=1}^{n} \left| \phi\left(x, T_k, \tfrac{p}{2}\right) - \phi\left(x, T_2, \tfrac{p}{2}\right) \right| > 1 - \gamma$$

$$\text{and } \frac{1}{n} \sum_{k=1}^{n} \left| \omega\left(x, T_k, \tfrac{p}{2}\right) - \omega\left(x, T_2, \tfrac{p}{2}\right) \right| < \gamma$$

for all $n > n_0$. Consider $r_1 = \max\{r_0, n_0\}$. Then we will get a $l \in \mathbb{N}$ such that

$$\begin{aligned} &\left| \phi\left(x, T_l, \tfrac{p}{2}\right) - \phi\left(x, T_1, \tfrac{p}{2}\right) \right| \\ &\geq \frac{1}{h_r} \sum_{k \in I_r} \left| \phi\left(x, T_k, \tfrac{p}{2}\right) - \phi\left(x, T_1, \tfrac{p}{2}\right) \right| > 1 - \gamma \end{aligned}$$

and

$$\begin{aligned} &\left| \phi\left(x, T_l, \tfrac{p}{2}\right) - \phi\left(x, T_2, \tfrac{p}{2}\right) \right| \\ &\geq \frac{1}{n} \sum_{k=1}^{n} \left| \phi\left(x, T_k, \tfrac{p}{2}\right) - \phi\left(x, T_2, \tfrac{p}{2}\right) \right| > 1 - \gamma. \end{aligned}$$

Therefore

$$\phi(T_1 - T_2, p) \geq (1-\gamma) * (1-\gamma) > 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\phi(T_1 - T_2, p) = 1$ for all $p > 0$, which yields $T_1 = T_2$. □

The following theorem can be proved using the standard techniques , so we state without proof.

Theorem 2.17. *Let* $(\mathcal{X}, \phi, \omega, *, \Diamond)$ *be an IFMS and* θ *be a lacunary sequence. Then*

(*a*) $\left[WN_\theta^{(\phi,\omega)}\right] - \lim T_k = T$ implies $WS_\theta^{(\phi,\omega)} - \lim T_k = T$.

(*b*) $\{T_k\} \in l_\infty^{(\phi,\omega)}$ and $WS_\theta^{(\phi,\omega)} - \lim T_k = T$ implies $\left[WN_\theta^{(\phi,\omega)}\right] - \lim T_k = T$.

(*c*) $l_\infty^{(\phi,\omega)} \cap WS_\theta^{(\phi,\omega)} = l_\infty^{(\phi,\omega)} \cap \left[WN_\theta^{(\phi,\omega)}\right]$.

Definition 2.18. Let $(\mathcal{X}, \phi, \omega, *, \Diamond)$ be an IFMS. A sequence $\{T_k\}$ is said to be Wijsman lacunary statistically Cauchy (or $WS_\theta^{(\phi,\omega)}$ -Cauchy) with regards to the IFM (ϕ, ω) if there is a subsequence $\left\{T_{k'(r)}\right\} \in I_r$ for each r, $(\phi,\omega) - \lim T_{k'(r)} = T$ and for each $\varepsilon \in (0,1)$ and $p > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ \begin{array}{l} k \in I_r : \left|\phi\left(x, T_k, p\right) - \phi\left(x, T_{k'(r)}, p\right)\right| \leq 1 - \varepsilon \\ \text{or } \left|\omega\left(x, T_k, p\right) - \omega\left(x, T_{k'(r)}, p\right)\right| \geq \varepsilon \end{array} \right\} \right| = 0.$$

Theorem 2.19. *If a sequence $\{T_k\}$ is Wijsman lacunary statistically convergent with regards to the IFM (ϕ, ω), then it is Wijsman lacunary statistically Cauchy with regards to the IFM (ϕ, ω).*

Proof. Let $WS_\theta^{(\phi,\omega)} - \lim T_k = T$ and for each n we write

$$K_n = \left\{ \begin{array}{c} k \in \mathbb{N} : \left|\phi\left(x, T_k, p\right) - \phi\left(x, T, p\right)\right| > 1 - \frac{1}{n} \\ \text{and } \left|\omega\left(x, T_k, p\right) - \omega\left(x, T, p\right)\right| < \frac{1}{n} \end{array} \right\}.$$

Then $K_{n+1} \subseteq K_n$ for each n and $\lim_{r\to\infty} \frac{|K_n \cap I_r|}{h_r} = 1$. So there is m_1 such that $r \geq m_1$ and $\frac{|K_1 \cap I_r|}{h_r} > 0$ i.e. $K_1 \cap I_r \neq \emptyset$. We next select $m_2 > m_1$ such that $r \geq m_2$ implies $K_2 \cap I_r \neq \emptyset$. Then, for each r satisfying $m_1 \leq r \leq m_2$ we select $k'(r) \in I_r$ such that $k'(r) \in K_1 \cap I_r$. In general we choose $m_{n+1} > m_n$ such that $r \geq m_{n+1}$ implies $k'(r) \in K_n \cap I_r$. Thus, $k'(r) \in I_r$ for each r and

$$\left|\phi\left(x, T_{k'(r)}, p\right) - \phi\left(x, T, p\right)\right| > 1 - \tfrac{1}{n}$$

$$\text{and } \left|\omega\left(x, T_{k'(r)}, p\right) - \omega\left(x, T, p\right)\right| < \tfrac{1}{n}.$$

Hence, $(\phi, \omega) - \lim T_{k'(r)} = T$.

Using Theorem 2.5, it is easily seen that

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ \begin{array}{l} k \in I_r : \left|\phi\left(x, T_k, p\right) - \phi\left(x, T_{k'(r)}, p\right)\right| \leq 1 - \varepsilon \\ \qquad \text{or } \left|\omega\left(x, T_k, p\right) - \omega\left(x, T_{k'(r)}, p\right)\right| \geq \varepsilon \end{array} \right\} \right| = 0.$$

Conversely, assume that $\{T_k\}$ is Wijsman lacunary statistically Cauchy sequence with regards to the IFM (ϕ, ω). For $\varepsilon > 0$ choose $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) > 1 - \varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. Hence, for any $p > 0$, take

$$K_{\phi,1} = \left\{ k \in \mathbb{N} : \left| \phi\left(x, T_k, \frac{p}{2}\right) - \phi\left(x, T_{k'(r)}, \frac{p}{2}\right) \right| > 1 - \gamma \right\}$$

and

$$K_{\phi,2} = \left\{ k \in \mathbb{N} : \left| \phi\left(x, T_{k'(r)}, \frac{p}{2}\right) - \phi\left(x, T, \frac{p}{2}\right) \right| > 1 - \gamma \right\}$$

Let $K_\phi = K_{\phi,1} \cap K_{\phi,2}$. Then $\delta_\theta\left(K_\phi\right) = 1$ and for $k \in K_\phi$,

$$\begin{aligned} \left|\phi\left(x, T_k, p\right) - \phi\left(x, T, p\right)\right| \geq & \left|\phi\left(x, T_k, \tfrac{p}{2}\right) - \phi\left(x, T_{k'(r)}, \tfrac{p}{2}\right)\right| \\ & * \left|\phi\left(x, T_{k'(r)}, \tfrac{p}{2}\right) - \phi\left(x, T, \tfrac{p}{2}\right)\right| > 1 - \varepsilon \end{aligned}$$

Similarly, if we take

$$K_{\omega,1} = \left\{ k \in \mathbb{N} : \left| \omega\left(x, T_k, \frac{p}{2}\right) - \omega\left(x, T_{k'(r)}, \frac{p}{2}\right) \right| < \gamma \right\}$$

and

$$K_{\omega,2} = \left\{ k \in \mathbb{N} : \left| \omega \left(x, T_{k'(r)}, \frac{p}{2} \right) - \omega \left(x, T, \frac{p}{2} \right) \right| < \gamma \right\}$$

Let $K_{\omega} = K_{\omega,1} \cap K_{\omega,2}$. Then $\delta_{\theta}(K_{\omega}) = 1$ and for $k \in K_{\phi}$,

$$|\phi(x, T_k, p) - \phi(x, T, p)| < \varepsilon.$$

Therefore

$$\lim_{r \to \infty} \frac{1}{h_r} \left| \left\{ \begin{array}{c} k \in I_r : |\phi(x, T_k, p) - \phi(x, T, p)| > 1 - \varepsilon \\ \text{and } |\omega(x, T_k, p) - \omega(x, T, p)| < \varepsilon \end{array} \right\} \right| = 1.$$

Hence, $\{T_k\}$ is Wijsman lacunary statistically convergent with regards to the IFM (ϕ, ω). □

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