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CERTAIN SUBCLASSES OF ALPHA-CONVEX FUNCTIONS WITH FIXED POINT

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Abstract. The present investigation is concerned with certain subclasses of alpha-convex functions with fixed point and defined with subordination in the unit disc $E = \{z :| z |< 1\}$. The estimates of the first four coefficients for the functions in these classes are obtained. The results due to various authors follow as special cases.

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1. Introduction

Let $\mathcal{U}$ be the class of Schwarzian functions of the form

$$u(z) = \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in the unit disc $E = \{z :| z |< 1\}$ and satisfying the conditions $u(0) = 0$ and $| u(z) |< 1$.

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in E of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Further, let $\mathcal{S}$ be the class of functions $f(z) \in \mathcal{A}$ and univalent in E.

Firstly, let us recall the following well known classes of univalent functions:
$\mathcal{S}^* = \left\{ f(z) \in \mathcal{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in E \right\}$, the class of starlike functions.

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$\mathcal{S}^c = \left\{ f(z) \in \mathcal{A} : Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0,\ z \in E \right\}$, the class of convex functions.
$\mathcal{M}_\alpha = \left\{ f(z) \in \mathcal{A} : Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right] > 0, 0 \leq \alpha \leq 1, z \in E \right\}$, the class of alpha-convex functions introduced by Mocanu [4].

Kanas and Ronning [3] introduced an interesting class $\mathcal{A}(w)$ of analytic functions of the form

$$f(z) = (z-w) + \sum_{k=2}^{\infty} a_k (z-w)^k \tag{2}$$

and normalized by the conditions $f(w) = 0, f'(w) = 1$, where w is a fixed point in E.

Accordingly the following classes were also defined in [3].

$\mathcal{S}^*(w) = \left\{ f \in \mathcal{A}(w) : Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0, z \in E \right\}$, known as the class of w-starlike functions.
$\mathcal{S}^c(w) = \left\{ f \in \mathcal{A}(w) : 1 + Re\left(\frac{(z-w)f''(z)}{f'(z)}\right) > 0, z \in E \right\}$, known as the class of w-convex functions.
The class $\mathcal{S}^*(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $\mathcal{S}^c(w)$ is defined by the property that the image of any circular arc centered at w is convex.
Also it is obvious that $f(z) \in \mathcal{S}^c(w)$ if and only if $(z-w)f'(z) \in \mathcal{S}^*(w)$.
Many researchers including Al-Hawary et al. [2], Acu and Owa [1], Olatunji and Oladipo [5] and others have worked on these classes.

For $w = 0$, the classes $\mathcal{S}^*(w)$ and $\mathcal{S}^c(w)$ reduces to the well known classes of starlike and convex functions, respectively.

Further the class $\mathcal{M}_\alpha(w)$ of w-α-convex functions was defined in [1] as follows:

$$\mathcal{M}_\alpha(w) = \left\{ f \in \mathcal{A}(w) : Re\left[(1-\alpha)\frac{(z-w)f'(z)}{f(z)} + \alpha\left(1 + \frac{(z-w)f''(z)}{f'(z)}\right)\right] > 0, \right.$$
$$\left. 0 \leq \alpha \leq 1, z \in E \right\}.$$

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a bounded function $u(z) \in U$ such that $f(z) = g(u(z))$. This result is known as principle of subordination.

Motivated by the above defined classes, now we introduce the following subclasses of w-α-convex functions with subordination:

Definition 1.1. A function $f(z) \in \mathcal{A}(w)$ is said to be in the class $\mathcal{M}_\alpha(w; A, B)$ if

$$(1-\alpha)\frac{(z-w)f'(z)}{f(z)} + \alpha\left[1 + \frac{(z-w)f''(z)}{f'(z)}\right] \prec \frac{1+A(z-w)}{1+B(z-w)}, \tag{3}$$

for $0 \leq \alpha \leq 1, -1 \leq B < A \leq 1$ and $z \in E$.

The following observations are obvious:
(i) $\mathcal{M}_\alpha(w; 1, -1) \equiv \mathcal{M}_\alpha(w)$.
(ii) $\mathcal{M}_0(w; A, B) \equiv \mathcal{S}^*(w; A, B)$, the subclass of w-starlike functions.
(iii) $\mathcal{M}_1(w; A, B) \equiv \mathcal{S}^c(w; A, B)$, the subclass of w-convex functions.
(iv) $\mathcal{M}_0(w; 1, -1) \equiv \mathcal{S}^*(w)$.
(v) $\mathcal{M}_1(w; 1, -1) \equiv \mathcal{S}^c(w)$.

Definition 1.2. A function $f(z) \in \mathcal{A}(w)$ is said to be in the class $\mathcal{M}^\alpha(w; A, B)$ if

$$\frac{(z-w)f'(z) + \alpha(z-w)^2 f''(z)}{(1-\alpha)f(z) + \alpha(z-w)f'(z)} \prec \frac{1+A(z-w)}{1+B(z-w)}, \tag{4}$$

where $0 \leq \alpha \leq 1, -1 \leq B < A \leq 1$ and $z \in E$.

In particular:
(i) $\mathcal{M}^\alpha(w; 1, -1) \equiv \mathcal{M}^\alpha(w)$.
(ii) $\mathcal{M}^0(w; A, B) \equiv \mathcal{S}^*(w; A, B)$.
(iii) $\mathcal{M}^1(w; A, B) \equiv \mathcal{S}^c(w; A, B)$.
(iv) $\mathcal{M}^0(w; 1, -1) \equiv \mathcal{S}^*(w)$.
(v) $\mathcal{M}^1(w; 1, -1) \equiv \mathcal{S}^c(w)$.

For deriving our main results, we need to the following lemma:

Lemma 1.3. [5] *For* $u(z) = \sum_{k=1}^{\infty} c_k (z-w)^k$, *if* $p(z) = \dfrac{1+Au(z)}{1+Bu(z)} = 1 + \sum_{k=1}^{\infty} p_k (z-w)^k$,
then

$$|p_n| \leq \frac{(A-B)}{(1+d)(1-d)^n}, n \geq 1, |w| = d.$$

In this paper, we obtain the upper bounds of $|a_2|, |a_3|, |a_4|$ and $|a_5|$ for the functions in the classes $\mathcal{M}_\alpha(w; A, B)$ and $\mathcal{M}^\alpha(w; A, B)$.

2. Main results

Theorem 2.1. *If* $f \in \mathcal{M}_\alpha(w; A, B)$, *then*

$$|a_2| \leq \frac{(A-B)}{(1-d^2)(1+\alpha)}, \tag{5}$$

$$|a_3| \le \frac{(A-B)}{2(1+\alpha)^2(1+2\alpha)(1-d^2)^2}[(1+d)(1+\alpha)^2+(1+3\alpha)(A-B)], \quad (6)$$

$$\begin{aligned}|a_4| \quad \le \quad & \frac{(A-B)}{6(1+\alpha)^3(1+2\alpha)(1+3\alpha)(1-d^2)^3}[3(1+5\alpha)(1+\alpha)^2(A-B)(1+d)\\ & +2(1+2\alpha)(1+\alpha)^3(1+d)^2+(A-B)^2(17\alpha^2+6\alpha+1)] \quad (7)\end{aligned}$$

and

$$\begin{aligned}|a_5| \le & \frac{(A-B)}{24(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+4\alpha)(1-d^2)^4}\\ & \times \big[6(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+d)^3\\ & +(1+\alpha)^3(1+d)^2(A-B)(11+124\alpha+361\alpha^2+296\alpha^3)\\ & +(A-B)^3(1+15\alpha+55\alpha^2+201\alpha^3+304\alpha^4)\\ & +6(1+\alpha)^2(1+d)(A-B)^2(1+10\alpha+53\alpha^2+80\alpha^3)\big]. \quad (8)\end{aligned}$$

Proof. From Definition 1.1, by principle of subordination, we have

$$\begin{aligned}(1-\alpha)\frac{(z-w)f'(z)}{f(z)}+\alpha\left[1+\frac{(z-w)f''(z)}{f'(z)}\right] &= p(z)=\frac{1+Au(z)}{1+Bu(z)}\\ &= 1+\sum_{k=1}^{\infty}p_k(z-w)^k, \quad (9)\end{aligned}$$

where $u(z)=\sum_{k=1}^{\infty}c_k(z-w)^k$.
On expanding, (9) yields

$$\begin{aligned}&1+(4+\alpha)a_2(z-w)+[4a_2^2+2(3+2\alpha)a_3](z-w)^2+[(12+\alpha)a_2a_3\\ &+(8+9\alpha)a_4](z-w)^3+[9a_3^2+4(4+\alpha)a_2a_4+2(5+8\alpha)a_5](z-w)^4+...\\ &=1+[3a_2+p_1](z-w)+[4a_3+2a_2^2+p_2+3a_2p_1](z-w)^2\\ &+[5a_4+5a_2a_3+p_3+4a_3p_1+2a_2^2p_1+3a_2p_2](z-w)^3\\ &+[6a_5+6a_2a_4+3a_3^2+p_4+3a_2p_3+5a_4p_1+5a_2a_3p_1+4a_3p_2+2a_2^2p_2](z-w)^4\\ &+... \quad (10)\end{aligned}$$

Equating the coefficients of $(z-w),(z-w)^2,(z-w)^3$ and $(z-w)^4$ on both sides of (10), we obtain

$$(1+\alpha)a_2=p_1, \quad (11)$$

$$2(1+2\alpha)a_3=p_2+3a_2p_1-2a_2^2, \quad (12)$$

$$3(1+3\alpha)a_4=-(7+\alpha)a_2a_3+p_3+4a_3p_1+2a_2^2p_1+3a_2p_2 \quad (13)$$

and

$$\begin{aligned}&4(1+4\alpha)a_5\\ &=-6a_3^2-2(5+2\alpha)a_2a_4+p_4+3a_2p_3+5a_4p_1+5a_2a_3p_1+4a_3p_2+2a_2^2p_2. \quad (14)\end{aligned}$$

By taking modulus and on applying Lemma 1.1 in (11), the result (5) is obvious. Using (11), taking modulus and applying triangle inequality in (12), it yields

$$|a_3| \leq \frac{1}{2(1+2\alpha)(1+\alpha)^2}[(1+\alpha)^2|p_2| + (1+3\alpha)|p_1|^2]. \tag{15}$$

On applying Lemma 1.1 in (15), the result (6) can be easily obtained.
Again using (11) and (12) in (13), taking modulus and applying triangle inequality, it yields

$$\begin{aligned}|a_4| \leq & \frac{1}{6(1+2\alpha)(1+3\alpha)(1+\alpha)^3} \\ & \times [3(1+\alpha)^2(1+5\alpha)|p_1||p_2| + 2(1+\alpha)^3(1+2\alpha)|p_3| \\ & \quad + (17\alpha^2+6\alpha+1)|p_1|^3].\end{aligned} \tag{16}$$

Using Lemma 1.1 in (16), the result (7) is obvious.
Further using (11), (12) and (13) in (14), taking modulus and applying triangle inequality, it takes the form

$$\begin{aligned}|a_5| \quad \leq \quad & \frac{1}{24(1+2\alpha)^2(1+3\alpha)(1+\alpha)^4(1+4\alpha)}[3(1+\alpha)^4(1+3\alpha)(1+8\alpha)|p_2|^2 \\ & +6(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)|p_4| + 8(1+\alpha)^3(1+2\alpha)^2(1+7\alpha)|p_1||p_3| \\ & +(1+15\alpha+55\alpha^2+201\alpha^3+304\alpha^4)|p_1|^4 \\ & +6(1+\alpha)^2(1+10\alpha+53\alpha^2+80\alpha^3)|p_2||p_1|^2].\end{aligned} \tag{17}$$

Using Lemma 1.1 in (17), the result (8) can be easily obtained.

□

For $A = 1, B = -1$, Theorem 2.1 gives the following result:

Corollary 2.2. *If $f(z) \in \mathcal{M}_\alpha(w)$, then*

$$|a_2| \leq \frac{2}{(1-d^2)(1+\alpha)},$$

$$|a_3| \leq \frac{1}{(1+\alpha)^2(1+2\alpha)(1-d^2)^2}[(1+d)(1+\alpha)^2 + 2(1+3\alpha)],$$

$|a_4| \leq \frac{2}{3(1+\alpha)^3(1+2\alpha)(1+3\alpha)(1-d^2)^3}[3(1+5\alpha)(1+\alpha)^2(1+d)+(1+2\alpha)(1+\alpha)^3(1+d)^2$
$+2(17\alpha^2+6\alpha+1)]$
and
$|a_5| \leq \frac{1}{6(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+4\alpha)(1-d^2)^4}[3(1+\alpha)^4(1+2\alpha)^2(1+3\alpha)(1+d)^3$
$+(1+\alpha)^3(1+d)^2(11+124\alpha+361\alpha^2+296\alpha^3)+4(1+15\alpha+55\alpha^2+201\alpha^3+304\alpha^4)$
$+12(1+\alpha)^2(1+d)(1+10\alpha+53\alpha^2+80\alpha^3)]$.

For $\alpha = 0$, Theorem 2.1 agrees with the following result:

Corollary 2.3. *If $f(z) \in \mathcal{S}^*(w; A, B)$, then*

$$|a_2| \leq \frac{(A-B)}{1-d^2},$$

$$|a_3| \leq \frac{(A-B)}{2(1-d^2)^2}[(1+d)+(A-B)],$$

$$|a_4| \leq \frac{(A-B)}{6(1-d^2)^3}[3(A-B)(1+d)+2(1+d)^2+(A-B)^2]$$

and

$$|a_5| \leq \frac{(A-B)}{24(1-d^2)^4}[6(1+d)^3+11(1+d)^2(A-B)+(A-B)^3+6(A-B)^2(1+d)].$$

For $\alpha = 1$, Theorem 2.1 follows:

Corollary 2.4. *If $f(z) \in \mathcal{S}^c(w; A, B)$, then*

$$|a_2| \leq \frac{(A-B)}{2(1-d^2)},$$

$$|a_3| \leq \frac{(A-B)}{6(1-d^2)^2}[(1+d)+(A-B)],$$

$$|a_4| \leq \frac{(A-B)}{24(1-d^2)^3}[3(A-B)(1+d)+2(1+d)^2+(A-B)^2]$$

and

$$|a_5| \leq \frac{(A-B)}{120(1-d^2)^4}[6(1+d)^3+11(1+d)^2(A-B)+(A-B)^3+6(A-B)^2(1+d)].$$

For $\alpha = 0, A = 1, B = -1$, Theorem 2.1 gives the following result:

Corollary 2.5. [1] *If $f(z) \in \mathcal{S}^*(w)$, then*

$$|a_2| \leq \frac{2}{1-d^2},$$

$$|a_3| \leq \frac{3+d}{(1-d^2)^2},$$

$$|a_4| \leq \frac{2(2+d)(3+d)}{3(1-d^2)^3}$$

and

$$|a_5| \leq \frac{(2+d)(3+d)(3d+5)}{6(1-d^2)^4}.$$

On putting $\alpha = 1, A = 1, B = -1$ in Theorem 2.1, it gives the following result:

Corollary 2.6. *If $f(z) \in \mathcal{S}^c(w)$, then*

$$|a_2| \leq \frac{1}{1-d^2},$$

$$|a_3| \leq \frac{3+d}{3(1-d^2)^2},$$

$$|a_4| \leq \frac{(2+d)(3+d)}{6(1-d^2)^3}$$

and

$$|a_5| \leq \frac{(2+d)(3+d)(3d+5)}{30(1-d^2)^4}.$$

Theorem 2.7. *If $f \in \mathcal{M}^\alpha(w; A, B)$, then*

$$|a_2| \leq \frac{(A-B)}{(1-d^2)(1+\alpha)}, \tag{18}$$

$$|a_3| \leq \frac{(A-B)}{2(1+2\alpha)(1-d^2)^2}[(1+d)+(A-B)], \tag{19}$$

$$|a_4| \leq \frac{(A-B)}{6(1+3\alpha)(1-d^2)^3}[3(A-B)(1+d)+2(1+d)^2+(A-B)^2] \tag{20}$$

and

$$|a_5| \leq \frac{(A-B)}{24(1+4\alpha)(1-d^2)^4}[6(1+d)^3+8(A-B)(1+d)^2 +6(A-B)^2(1+d)+3(A-B)(1+d)^2+(A-B)^3]. \tag{21}$$

Proof. From Definition 1.2, by principle of subordination, we have

$$\frac{(z-w)f'(z)+\alpha(z-w)^2f''(z)}{(1-\alpha)f(z)+\alpha(z-w)f'(z)} = p(z) = \frac{1+Au(z)}{1+Bu(z)} = 1+\sum_{k=1}^{\infty} p_k(z-w)^k \tag{22}$$

where $u(z) = \sum_{k=1}^{\infty} c_k(z-w)^k$.
Then proceeding as in Theorem 2.1, the results (18), (19), (20) and (21) can be easily obtained. □

For $A = 1, B = -1$, Theorem 3.1 gives the following result:

Corollary 2.8. *If $f(z) \in \mathcal{M}^\alpha(w)$, then*

$$|a_2| \leq \frac{2}{(1-d^2)(1+\alpha)},$$

$$|a_3| \leq \frac{1}{(1+2\alpha)(1-d^2)^2}(3+d),$$

$$|a_4| \leq \frac{2}{3(1+3\alpha)(1-d^2)^3}[(2+d)(3+d)]$$

and

$$|a_5| \leq \frac{1}{6(1+4\alpha)(1-d^2)^4}[(2+d)(3+d)(5+3d)].$$

For $\alpha = 0$ and $\alpha = 1$, Theorem 2.7 agrees with Corollary 2.3 and Corollary 2.4 respectively. Also for $\alpha = 0, A = 1, B = -1$ and $\alpha = 1, A = 1, B = -1$, Theorem 2.7 agrees with Corollary 2.5 and Corollary 2.6 respectively.

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