KYUNGPOOK Math. J. 62(2022), 673-687
https://doi.org/10.5666/KMJ.2022.62.4.673
pISSN 1225-6951 eISSN 0454-8124

Accelerated Tseng's Technique to Solve Cayley Inclusion Problem in Hilbert Spaces

Shamshad Husain
Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202002, India
e-mail : s_husain68@yahoo.com

Uqba Rafat*
Department of Applied Mathematics, Aligarh Muslim University, Aligarh 202002, India
e-mail : uqba.1709@gmail.com

Abstract. In this study, we solve the Cayley inclusion problem and the fixed point problem in real Hilbert space using Tseng's technique with inertial extrapolation in order to obtain more efficient results. We provide a strong convergence theorem to approximate a common solution to the Cayley inclusion problem and the fixed point problem under some appropriate assumptions. Finally, we present a numerical example that satisfies the problem and shows the computational performance of our suggested technique.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. A "Zero-Point Problem" **(ZPP)** for monotone operators is defined as follows: find $x^* \in \mathcal{H}$ such that

$$0 \in \mathcal{T}x^*, \tag{1.1}$$

where $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is a monotone operator. Several authors have focused on the convergence of iterative methods in order to locate a zero-point for monotone operators in Hilbert spaces. Martinet [9] constructed the "Proximal Point Algorithm" (**PPA**) to solve problem (1.1). The **PPA** is depicted as:

$$x_{n+1} = (\mathtt{I} + \lambda_n\mathcal{T})^{-1}x_n, \quad \forall n \geq 1, \tag{1.2}$$

* Corresponding Author.
Received December 2, 2021; revised April 9, 2022; accepted April 8, 2022.
2020 Mathematics Subject Classification: 47J22, 47J25, 49J45, 47H04.
Keywords and phrases: Tseng's technique, Cayley inclusion problem, Cayley operator, fixed point problem.

where $\mathtt{I}$ is the identity mapping, and $\{\lambda_n\}$ is a sequence of positive real numbers. After Martinet [9], several algorithms were developed to solve **(ZPP)**. The reader can see [10] - [14] and for more information, refer to the citations provided in this article.

The "Monotone Inclusion Problem" **(MIP)** is to obtain $x^* \in \mathcal{H}$ such that

$$0 \in (\mathfrak{D} + \mathcal{M})x^*. \tag{1.3}$$

where $\mathfrak{D} : \mathcal{H} \to \mathcal{H}$ is a single-valued mapping and $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ is a multi-valued mapping. The **MIP** (1.3) can be written as the **ZPP** (1.1) by setting $\mathcal{T} := \mathfrak{D} + \mathcal{M}$.

According to Lions and Mercier [6], the most common way to solve the problem (1.3) is to use the forward-backwrad splitting method, defined as follows:

$$x_{n+1} = (\mathtt{I} + \lambda\mathcal{M})^{-1}(\mathtt{I} - \lambda\mathfrak{D})x_n, \quad \forall n \geq 1, \tag{1.4}$$

where $x_1 \in \mathcal{H}$ is arbitrarily chosen and $\lambda > 0$. In algorithm (1.4), the operator $\mathfrak{D}$ is called a forward operator and $\mathcal{M}$ is called a backward operator. For more details about the forward-backward splitting method used to solve the **MIP** (1.3), the reader is directed to see [5, 7, 14].

Polyak [11] introduced the inertial extrapolation method to speed up the rate of convergence of the iteration. This is sometimes called the heavy ball method. Many scholars have exploited this notion to combine algorithms with inertial terms to speed up the rate of convergence.

In 2015, Lorenz and Pock [7] studied the **MIP** (1.3) and proposed the inertial forward-backward algorithm for monotone operators, which combines the heavy ball method and forward-backward method. The algorithm is defined as:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (\mathtt{I} + \lambda\mathcal{M})^{-1}(\mathtt{I} - \lambda\mathfrak{D})w_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $x_0, x_1 \in \mathcal{H}$ are arbitrarily chosen, and $\mathfrak{D} : \mathcal{H} \to \mathcal{H}$ and $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ are single and multi-valued mappings, respectively. The sequence $\{x_n\}$ generated by algorithm (1.5) converges weakly to a solution of **MIP** (1.3) with some suitable assumptions.

There are many ways to solve the **MIP** other than using an algorithm combined with the heavy ball idea. Tseng [15] introduced the modified forward-backward splitting method, a powerful iterative method for solving the monotone inclusion problem (1.3). In short, it is known as Tseng's splitting algorithm. Let C be a closed and convex subset of a real Hilbert space $\mathcal{H}$. Tseng's splitting algorithm is defined as:

$$\begin{cases} y_n = (\mathtt{I} + \lambda_n\mathcal{M})^{-1}(\mathtt{I} - \lambda_n\mathfrak{D})x_n \\ x_{n+1} = \mathcal{P}_C(y_n - \lambda_n(\mathfrak{D}y_n - \mathfrak{D}x_n)), \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $x_1 \in \mathcal{H}$ is arbitrarily chosen, λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, ..\}$ satisfying $\lambda\|\mathfrak{D}y_n - \mathfrak{D}x_n\| \leq \mu\|x_n - y_n\|$ where $\delta > 0, l \in (0, 1), \mu \in (0, 1)$ and $\mathcal{P}_C$ is the projection onto a closed convex subset C of $\mathcal{H}$.

Kitkaun and Kumam [5] combined the forward -backward splitting method with the viscosity approximation method for solving **MIP** (1.3). It is called the inertial viscosity forward-backward splitting algorithm, which is defined as:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \alpha_n \delta f(x_n) + (1-\alpha_n)(\mathtt{I} - \lambda_n \mathcal{M})^{-1}(\mathtt{I} - \lambda_n \mathfrak{D}) w_n, \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where $x_0, x_1 \in \mathcal{H}$ are arbitrarily chosen, $f : \mathcal{H} \to \mathbb{R}$ is a differentiable function such that its gradient δf is a contraction with constant $\rho \in (0,1)$ and $\mathfrak{D} : \mathcal{H} \to \mathcal{H}$ and $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ are an inverse strongly monotone and a maximal monotone operator, respectively. The sequence generated by the algorirthm (1.7) converges strongly to a solution of **MIP** (1.3) under suitable conditions.

Now we define a new type of monotone inclusion problem known as “Cayley's Inclusion Problem” (**CIP**). Find $x^* \in \mathcal{H}$ such that

$$0 \in (\mathfrak{C}_\lambda^{\mathcal{M}} + \mathcal{M})x^* \tag{1.8}$$

where $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ is a multi-valued maximal monotone mapping, $\mathfrak{C}_\lambda^{\mathcal{M}} := 2\mathfrak{J}_\lambda^{\mathcal{M}} - \mathtt{I}$ is the single valued mapping and known as “Cayley operator” and $\mathfrak{J}_\lambda^{\mathcal{M}} := (\mathtt{I} + \lambda\mathcal{M})^{-1}$ is the “resolvent operator” associated with maximal monotone mapping $\mathcal{M}$ with $\lambda > 0$. The solution set of **CIP** is denoted as:

$$\Omega := \{x \in \mathcal{H} : 0 \in \mathfrak{C}_\lambda^{\mathcal{M}}(x) + \mathcal{M}(x)\}.$$

Let $S : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. A fixed point of S is a point $x \in \mathcal{H}$ such that

$$\text{S(x)} = \text{x}. \tag{1.9}$$

The set of all fixed point of S is denoted by $\mathbf{Fix}(S)$=$\{x \in \mathcal{H} : S(x) = x\}$.

The objective of this article is to propose an algorithm consisting of Tseng's technique with inertial extrapolation and applying the viscosity iterative method in order to obtain the common solution of **CIP** and $\mathbf{Fix}(S)$ in the framework of real Hilbert space. Moreover, we show that the sequences generated by the algorithm are strongly convergent to the point in the solution set $\Sigma := \Omega \cap \text{Fix}(S)$. In particular, we give numerical illustrations of the algorithm.

2. Mathematical Preliminaries

Let $\mathcal{H}$ be a real Hilbert space and C be a nonempty closed convex subset of $\mathcal{H}$. The weak convergence of $\{x_n\}_{n=1}^\infty$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^\infty$ to x is written as $x_n \to x$ as $n \to \infty$. Now assume that x, y, z $\in \mathcal{H}$ the following relations are valid for inner product spaces,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.1}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{2.2}$$

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 &= \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ &\quad - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2\end{aligned} \tag{2.3}$$

for any $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha+\beta+\gamma = 1$.

The following definitions are required to obtain the desired results.

Definition 2.1. Let $\mathfrak{D} : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping, then it is said to be
(i) *nonexpansive* if for all $x, y \in \mathcal{H}$,

$$\|\mathfrak{D}(x) - \mathfrak{D}(y)\| \leq \|x-y\|,$$

(ii) *firmly nonexpansive* if for all $x, y \in \mathcal{H}$,

$$\|\mathfrak{D}(x) - \mathfrak{D}(y)\|^2 \leq \langle \mathfrak{D}x - \mathfrak{D}y, x-y\rangle,$$

(iii) *L-Lipschitz continuous* if for all $x, y \in \mathcal{H}$, there exists $L > 0$ such that

$$\|\mathfrak{D}(x) - \mathfrak{D}(y)\| \leq L\|x-y\|,$$

(iv) *α-strongly monotone* if for all $x, y \in \mathcal{H}$, there exists a constant $\alpha > 0$ such that

$$\langle \mathfrak{D}(x) - \mathfrak{D}(y), x-y\rangle \geq \alpha\|x-y\|^2,$$

(v) *μ- inverse strongly monotone* if for all $x, y \in \mathcal{H}$, there exists a constant $\mu > 0$ such that

$$\langle \mathfrak{D}(x) - \mathfrak{D}(y), x-y\rangle \geq \mu\|\mathfrak{D}(x) - \mathfrak{D}(y)\|^2.$$

Definition 2.2. Let $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ be a multi-valued mapping, then it is said to be
(i) *monotone* if for all $x, y \in \mathcal{H}, u \in \mathcal{M}(x), v \in \mathcal{M}(y)$ such that

$$\langle x-y, u-v\rangle \geq 0,$$

(ii) *strongly monotone* if for all $x, y \in \mathcal{H}, u \in \mathcal{M}(x), v \in \mathcal{M}(y)$, there exist $\theta > 0$ such that

$$\langle x-y, u-v\rangle \geq \theta\|x-y\|^2,$$

(iii) *maximal monotone* if $\mathcal{M}$ is monotone and $(\mathtt{I}+\lambda\mathcal{M})(\mathcal{H})=\mathcal{H}$ for all $\lambda > 0$, where $\mathtt{I}$ is the identity mapping on $\mathcal{H}$.

Definition 2.3 Let $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ be a multi-valued mapping, then the resolvent operator associated with $\mathcal{M}$ is defined as:

$$\mathfrak{J}_\lambda^{\mathcal{M}}(x) = [\mathtt{I} + \lambda\mathcal{M}]^{-1}(x), \forall x \in \mathcal{H}.$$

Here $\lambda > 0$ and $\mathtt{I}$ is the identity mapping.

Remark 2.3. The resolvent operator $\mathfrak{J}_\lambda^{\mathcal{M}}$ has the following properties:
(i) it is single-valued and nonexpansive, i.e.,

$$\|\mathfrak{J}_\lambda^{\mathcal{M}}(x) - \mathfrak{J}_\lambda^{\mathcal{M}}(y)\| \leq \|x - y\|, \forall x, y \in \mathcal{H},$$

(ii) it is 1-inverse strongly monotone, i.e,

$$\|\mathfrak{J}_\lambda^{\mathcal{M}}(x) - \mathfrak{J}_\lambda^{\mathcal{M}}(y)\|^2 \leq \langle x - y, \mathfrak{J}_\lambda^{\mathcal{M}}(x) - \mathfrak{J}_\lambda^{\mathcal{M}}(y)\rangle, \forall x, y \in \mathcal{H}.$$

Definition 2.4. Let $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ be a multi-valued mapping and $\mathfrak{J}_\lambda^{\mathcal{M}}$ be the resolvent operator associated with $\mathcal{M}$, then the Cayley operator $\mathfrak{C}_\lambda^{\mathcal{M}}$ is defined as:

$$\mathfrak{C}_\lambda^{\mathcal{M}}(x) = [2\mathfrak{J}_\lambda^{\mathcal{M}}(x) - I], \forall x \in \mathcal{H}. \tag{2.4}$$

Remark 2.5. Using Remark 2.3.(i), it can be easily seen that the Cayley operator $\mathfrak{C}_\lambda^{\mathcal{M}}$ is 3-Lipschitz continuous. Now, for the sake of convenience we shall denote $\mathfrak{C}_\lambda^{\mathcal{M}}$ by $\mathfrak{C}$ throughout the paper.

Lemma 2.6. *Let $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximal monotone mapping and $\mathcal{B} : \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous mapping. Then a mapping $\mathcal{B} + \mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximal monotone mapping.*

Lemma 2.7. *Let $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued maximal monotone mapping and $\lambda > 0$. Then the following statements hold:*

1. *$\mathfrak{J}_\lambda^{\mathcal{M}}$ is a single-valued and firmly nonexpansive mapping;*
2. *Fix($\mathfrak{J}_\lambda^{\mathcal{M}}$)= $\mathcal{M}^{-1}(0)$;*
3. $\|x - \mathfrak{J}_\lambda^{\mathcal{M}}\| \leq 2\|x - \mathfrak{J}_\gamma^{\mathcal{M}}\|$, $0 < \lambda \leq \gamma, \forall x \in \mathcal{H}$;
4. $(\mathtt{I} - \mathfrak{J}_\lambda^{\mathcal{M}})$ *is firmly nonexapansive mapping;*
5. *Suppose that $\mathcal{M}^{-1}(0) \neq \phi$. Then*
   $\|\mathfrak{J}_\lambda^{\mathcal{M}}(x) - z\|^2 \leq \|x - z\|^2 - \|\mathfrak{J}_\lambda^{\mathcal{M}}(x) - x\|^2$ *for all* $x \in \mathcal{H}$ *and* $z \in \mathcal{M}^{-1}(0)$
   and
   $\langle x - \mathfrak{J}_\lambda^{\mathcal{M}}, \mathfrak{J}_\lambda^{\mathcal{M}} - z\rangle \geq 0$ *for all* $x \in \mathcal{H}$ *and* $z \in \mathcal{M}^{-1}(0)$.

Lemma 2.8.([16])*Let $\{a_n\}$ and $\{c_n\}$ be nonnegative sequences of real numbers such that $\sum_{n=1}^{\infty} c_n < \infty$, and let $\{b_n\}$ be a sequence of real numbers such that* $\limsup_{n\to\infty} b_n \leq 0$.*If there exists $n_0 \in \mathbb{N}$ such that,for any $n \geq n_0$,*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n + c_n,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9.([4]) *Let C be a closed convex subset of $\mathcal{H}$ and $S : C \to C$ a nonexapansive mapping with Fix(S) $\neq \phi$. If there exists $\{x_n\}$ in C satisfying $x_n \rightharpoonup z$ and $\|x_n - Sx_n\| \to 0$, then $z = Sz$.*

Lemma 2.10.([8]) *Let $\{\gamma_n\}$ is a sequence of real numbers. Suppose that there is subsequence $\{\gamma_{n_j}\}_{j\geq 0}$ of $\{\gamma_n\}$ satisfying $\gamma_{n_j} \leq \gamma_{n_j+1}$ for each $j \geq 0$. Let $\{\phi(n)\}_{n\geq n^*}$ be a sequence of integers defined by*

$$\phi(n) := \max\{\mathrm{k} \leq \mathrm{n} : \gamma_{\mathrm{k}} < \gamma_{\mathrm{k+1}}\}. \tag{2.5}$$

Then $\{\phi(n)\}_{n\geq n^*}$ is a nondecreasing with $\lim_{n\to\infty} \phi(n) = \infty$. Moreover, for each $n \geq n^*$, we have $\gamma_{\phi(n)} \leq \gamma_{\phi(n)+1}$ and $\gamma_n \leq \gamma_{\phi(n)+1}$.

3. Main Result

In this section, we present an inertial Tseng type algorithm for solving our problem. For the convergence analysis of the proposed method, we consider the following assumptions in order to accomplish our goal.

Assumption 1. $\mathcal{H}$ is a real Hilbert space, $\mathfrak{C} : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous and monotone, and $\mathcal{M} : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximal monotone operator.

Assumption 2. $\Sigma := \Omega \cap F(S)$ is nonempty.

Assumption 3. $\{\theta_n\} \subset [0,\theta), \{\beta_n\} \subset (\beta^*, \beta^{'}) \subset (0, 1-\alpha_n)$ for some $\theta > 0, \beta^* > 0, \beta^{'} > 0$, and $\{\alpha_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Assumption 4. $f : \mathcal{H} \to \mathcal{H}$ is a ρ-contractive mapping.

Next the algorithm is presented.

Lemma 3.1. *Assume that Assumptions $1-4$ hold, then any sequence $\{\lambda_n\}$ in Algorithm is nonincreasing and converges to λ such that* $\min\{\lambda_1, \frac{\mu}{\mathrm{L}}\} \leq \lambda$.

Proof. See [17, Lemma 3.1]. □

Algorithm 3.2. Inertial-viscosity Tseng type algorithm

Initialization: Given $\lambda_1 > 0$ and $\mu \in (0,1)$. Select arbitrary elements $x_0, x_1 \in \mathcal{H}$ and set $n := 1$.
Iterative Steps: Construct $\{x_n\}$ by using the following steps:
Step 1. Set

$$w_n = x_n + \theta_n(x_n - x_{n-1})$$

and compute,

$$y_n = \mathfrak{J}^{\mathcal{M}}_{\lambda_n}(\mathtt{I} - \lambda_n \mathfrak{C})w_n.$$

If $w_n = y_n$, then stop and $w_n \in \Sigma$. Otherwise
Step 2. Compute

$$z_n = y_n - \lambda_n(\mathfrak{C}y_n - \mathfrak{C}w_n)$$

and
Step 3. Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n - \beta_n)x_n + \beta_n S(z_n)$$

and update,

$$\lambda_{n+1} = \begin{cases} \min\Big\{\mu \frac{\|w_n - y_n\|}{\|\mathfrak{C}w_n - \mathfrak{C}y_n\|}, \lambda_n\Big\} & \text{if } \mathfrak{C}w_n \neq \mathfrak{C}y_n; \\ \lambda_n & \text{otherwise.} \end{cases}$$

Replace n by $n+1$ and then repeat *Step*1.

Lemma 3.3. *Let $q \in \Sigma$. As given in the algorithm together with all four Assumptions, the following inequalities are true.*

$$\|z_n - q\|^2 \leq \|w_n - q\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 \tag{3.1}$$

and

$$\|z_n - y_n\| \leq \mu \frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|. \tag{3.2}$$

Proof. In the same manner as [3, Lemma 6], we obtain that inequalities (3.1) and (3.2) hold. □

Lemma 3.4. *Suppose that* $\lim_{n\to\infty} \|w_n - y_n\| = 0$.*If there exists a weakly convergent subsequences* $\{w_{n_j}\}$ *of* $\{w_n\}$*, then under Assumptions* $1-4$*, we have that the* *limit of $\{w_{n_j}\}$ belongs to Σ.*

Proof The proof is similar to the proof of [3, Lemma7]. $\square$

With the above results we are now ready for the main convergence theorem.

Theorem 3.5. *Suppose that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, then under Assumptions 1-4, we have $x_n \to q$ as $n \to \infty$, where $q = P_\Sigma \circ f(q)$.*

Proof. For the sake of simplicity, we divide the proof into four claims.

Claim 1. *$\{x_n\}$ is bounded sequence.*

We observe from (3.1) that $\lim_{n\to\infty}\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1-\mu^2 > 0$. Let $q' \in \Sigma$ and indeed, thanks to Lemma (3.2), we get

$$\|z_n - q'\| \le \|w_n - q'\|. \tag{3.3}$$

Also, from the definition of $\{y_n\}$ and nonexansiveness of $\mathfrak{J}_\lambda^{\mathcal{M}}$, we have

$$\begin{aligned}\|y_n - q'\| &\le \|\mathfrak{J}_{\lambda_n}^{\mathcal{M}}(\mathtt{I} - \lambda_n\mathfrak{C})w_n - q'\| \\ &\le \|w_n - q'\|\end{aligned} \tag{3.4}$$

By the sequence $\{\dfrac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\}$ converges to 0, we have that there exists a constant M_1 such that, for all $n \in \mathbb{N}$,

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1.$$

From the definition of w_n and (3.3), we obtain

$$\begin{aligned}\|z_n - q'\| \le \|w_n - q'\| &= \|x_n + \theta_n(x_n - x_{n-1})\| \\ &\le \|x_n - q'\| + \theta_n\|x_n - x_{n-1}\| \\ &\le \|x_n - q'\| + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\alpha_n \\ &\le \|x_n - q'\| + \alpha_n M_1.\end{aligned} \tag{3.5}$$

By Assumption 4, nonexpansiveness of S and using (3.10), the following relation is obtained:

$$\begin{aligned}
\|x_{n+1} - q'\| &= \|\alpha_n f(x_n) + (1 - \alpha_n - \beta_n)x_n + \beta_n S(z_n) - q'\| \\
&\le \|\alpha_n(f(x_n) - q') + (1 - \alpha_n - \beta_n)(x_n - q') + \beta_n(Sz_n - q')\| \\
&\le \alpha_n\|f(x_n) - q'\| + (1 - \alpha_n - \beta_n)\|x_n - q'\| + \beta_n\|Sz_n - q'\| \\
&\le \alpha_n\|f(x_n) - q'\| + (1 - \alpha_n - \beta_n)\|x_n - q'\| + \beta_n\|z_n - q'\| \\
&\le \alpha_n\|f(x_n) - f(q')\| + \alpha_n\|f(q') - q'\| + (1 - \alpha_n)\|x_n - q'\| + \alpha_n\beta_n M_1 \\
&\le [1 - \alpha_n(1 - \rho)]\|x_n - q'\| + \alpha_n(\|f(q') - q'\| + M_1) \\
&= [1 - \alpha_n(1 - \rho)]\|x_n - q'\| + \alpha_n(1 - \rho)\frac{\|f(q') - q'\| + M_1}{1 - \rho} \\
&\le \max\left\{\|\mathrm{x_n} - \mathrm{x}\|, \frac{\|\mathrm{f(q')} - \mathrm{q'}\| + \mathrm{M_1}}{1 - \rho}\right\} \\
&\le \max\left\{\|\mathrm{x_{n-1}} - \mathrm{x}\|, \frac{\|\mathrm{f(q')} - \mathrm{q'}\| + \mathrm{M_1}}{1 - \rho}\right\} \\
&\vdots \\
&\le \max\left\{\|\mathrm{x_0} - \mathrm{x}\|, \frac{\|\mathrm{f(q')} - \mathrm{q'}\| + \mathrm{M_1}}{1 - \rho}\right\}.
\end{aligned}$$

This leads to a conclusion that $\|x_{n+1} - q'\| \le \max\{\|\mathrm{x_0} - \mathrm{x}\|, \frac{\|\mathrm{f(q')} - \mathrm{q'}\| + \mathrm{M_1}}{1-\rho}\}$. Consequently, the sequence $\{x_n\}$ is bounded. In addition, $\{f(x_n)\}$ is also bounded. Since Σ is closed and convex set, $P_\Sigma \circ f$ is a $\rho-$ contractive mapping. Now, we can uniquely find $q \in \Sigma$ with $q = P_\Sigma \circ f(q)$ duq' to the Banach fixed point theorem. We also get, that for any $q' \in \Sigma$,

$$\langle f(q) - q, q' - q\rangle \le 0.$$

Now, for each $n \in \mathbb{N}$, set $\gamma_n := \|x_n - q\|^2$.

Claim 2. *There is* $M_0 > 0$ *such that*

$$\beta_n(1 - \alpha_n - \beta_n)\|x_n - Sz_n\|^2 \le \gamma_n - \gamma_{n+1} + \alpha_n(\|f(x_n) - q\|^2 + M_0).$$

Applying (3.5), we have

$$\begin{aligned}
\|z_n - q\|^2 &\le (\|x_n - q\| + \alpha_n M_1)^2 \\
&= \gamma_n + \alpha_n(2M_1\|x_n - q\| + \alpha_n M_1^2) \\
&\le \gamma_n + \alpha_n M_0
\end{aligned} \tag{3.6}$$

for some $M_0 > 0$. It follows from the assumption on f, (2.3) and (3.6) that

$$\begin{aligned}
\gamma_{n+1} &= \|\alpha_n(f(x_n)-q)+(1-\alpha_n-\beta_n)(x_n-q)+\beta_n(Sz_n-q)\|^2\\
&= \alpha_n\|f(x_n)-q\|^2+(1-\alpha_n-\beta_n)\gamma_n+\beta_n\|Sz_n-q\|^2-\alpha_n(1-\alpha_n-\beta_n)\|f(x_n)-x_n\|^2\\
&- \beta_n(1-\alpha_n-\beta_n)\|x_n-Sz_n\|^2-\alpha_n\beta_n\|f(x_n)-Sz_n\|^2\\
&\le \alpha_n\|f(x_n)-q\|^2+(1-\alpha_n-\beta_n)\gamma_n+\beta_n\|z_n-q\|^2-\beta_n(1-\alpha_n-\beta_n)\|x_n-Sz_n\|^2\\
&\le \alpha_n\|f(x_n)-q\|^2+(1-\alpha_n)\gamma_n+\alpha_n\beta_nM_0-\beta_n(1-\alpha_n-\beta_n)\|x_n-Sz_n\|^2\\
&\le \gamma_n+\alpha_n(\|f(x_n)-q\|^2+M_0)-\beta_n(1-\alpha_n-\beta_n)\|x_n-Sz_n\|^2.
\end{aligned}$$

Therefore, Claim 2 is obtained

Claim 3. *There is $M > 0$ such that*

$$\begin{aligned}
\gamma_{n+1} &\le [1-\alpha_n(1-\rho)]\gamma_n+\alpha_n(1-\rho)\left[\frac{3M}{1-\rho}\cdot\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\right]\\
&+\alpha_n(1-\rho)\left[\frac{2M}{1-\rho}\|x_n-Sz_n\|+\frac{2}{1-\rho}\langle f(q)-q,x_{n+1}-q\rangle\right]
\end{aligned}\tag{3.7}$$

Indeed, setting $t_n := (1-\beta_n)x_n+\beta_nSz_n$. From inequality (3.3), nonexpansiveness of S, and the definition of w_n, we get

$$\begin{aligned}
\|t_n-q\| &\le (1-\beta_n)\|x_n-q\|+\beta_n\|Sz_n-q\|\\
&\le (1-\beta_n)\|x_n-q\|+\beta_n\|z_n-q\|\\
&\le (1-\beta_n)\|x_n-q\|+\beta_n\|w_n-q\|\\
&\le \|x_n-q\|+\beta_n\theta_n\|x_n-x_{n-1}\|
\end{aligned}\tag{3.8}$$

and

$$\|x_n-t_n\| = \beta_n\|x_n-Sz_n\|. \tag{3.9}$$

Hence, from the assumption on f, and (3.1), (3.8), and (3.9), we obtain,

$$\begin{aligned}\gamma_{n+1} &= \|(1-\alpha_n)(t_n-q)+\alpha_n(f(x_n)-f(q))-\alpha_n(x_n-t_n)-\alpha_n(q-f(q))\|^2\\ &\le \|(1-\alpha_n)(t_n-q)+\alpha_n(f(x_n)-f(q))\|^2-2\alpha_n\langle x_n-t_n+q-f(q),x_{n+1}-q\rangle\\ &\le (1-\alpha_n)\|t_n-q\|^2+\alpha_n\|f(x_n)-f(q)\|^2+2\alpha_n\langle t_n-x_n,x_{n+1}-q\rangle\\ &+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle\\ &\le (1-\alpha_n)(\|x_n-q\|+\beta_n\theta_n\|x_n-x_{n-1}\|)^2+\alpha_n\rho^2\|x_n-q\|^2\\ &+2\alpha_n\|t_n-x_n\|\|x_{n+1}-q\|+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle\\ &\le (1-\alpha_n)\gamma_n+2\theta_n\|x_n-q\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2+\alpha_n\rho\gamma_n\\ &+2\alpha_n\beta_n\|x_n-Sz_n\|\|x_{n+1}-q\|++2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle\\ &\le [1-\alpha_n(1-\rho)]\gamma_n+\theta_n\|x_n-x_{n-1}\|(2\|x_n-x\|+\theta_n\|x_n-x_{n-1}\|)\\ &+2\alpha_n\beta_n\|x_n-Sz_n\|\|x_{n+1}-q\|+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle\\ &\le [1-\alpha_n(1-\rho)]\gamma_n+3M\theta_n\|x_n-x_{n-1}\|+2M\alpha_n\beta_n\|x_n-Sz_n\|\\ &+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle\\ &\le [1-\alpha_n(1-\rho)]\gamma_n+\alpha_n(1-\rho)\left[\frac{3M}{1-\rho}\cdot\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\right]\\ &+\alpha_n(1-\rho)\left[\frac{2M}{1-\rho}\|x_n-Sz_n\|+\frac{2}{1-\rho}\langle f(q)-q,x_{n+1}-q\rangle\right]\end{aligned}$$

for $M := \sup_{n\in\mathbb{N}}\{\|x_n-q\|,\theta\|x_n-x_{n-1}\|\}>0$. Recall that our task is to show that $x_n\to p$ which is now equivalent to show that $\gamma_n\to 0$ as $n\to\infty$.

Claim 4. $\gamma_n\to 0$ as $n\to\infty$.

Consider the following two cases:

Case (i). We can find $N\in\mathbb{N}$ satisfying $\gamma_{n+1}\le\gamma_n$ for each $n\ge N$.

Since each term γ_n is nonnegative, it is convergent. Due to the fact that $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\beta_n\in(0,1)$, and by Claim 2,

$$\lim_{n\to\infty}\|x_n-Sz_n\|=0 \tag{3.10}$$

Indeed, we immediately get

$$\lim_{n\to\infty}\|x_n-\theta_n\|=\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\alpha_n=0 \tag{3.11}$$

In addition, from the definition of z_n and by using the triangle inequlity, we obtained the following inequalities:

$$\begin{aligned}\|z_n-w_n\| &= \|z_n-x_n+x_n-w_n\|\\ &\le \|z_n-x_n\|+\|x_n-w_n\|\end{aligned}$$

and

$$\|w_n-y_n\|\le\|w_n-z_n\|+\|z_n-y_n\|$$

It follows from inequality (3.2) that

$$(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|w_n-y_n\| \le \|z_n-x_n\|+\|x_n-w_n\|$$

since $\lim_{n\to\infty}[1-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})^2]=1-\mu^2>0$, (3.10) and (3.11),

$$\lim_{n\to\infty}\|w_n-y_n\|=0. \tag{3.12}$$

Note that, for each $n\in\mathbb{N}$

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|x_{n+1}-z_n\|+\|z_n-x_n\|\\ &\le \alpha_n\|f(x_n)-x_n\|+(2-\beta_n)\|x_n-z_n\|.\end{aligned} \tag{3.13}$$

Consequently, since $\lim_{n\to\infty}\alpha_n=0$ and by (3.13), $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Next observe that, for the reason that $\{x_n\}$ is bounded, there is $z\in\mathcal{H}$ so that $x_{n_k}\rightharpoonup z$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then Lemma 3.3 together with (3.12) implies that $z\in\Sigma$. As a result, by the definition of q, it is straightforward to show that

$$\limsup_{n\to\infty}\langle f(q)-q,x_n-q\rangle=\lim_{k\to\infty}\langle f(q)-q,x_{n_k}-q\rangle=\langle f(q)-q,z-q\rangle\le 0.$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, the following result obtained:

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q)-q,x_{n+1}-q\rangle &\le \limsup_{n\to\infty}\langle f(q)-q,x_{n+1}-x_n\rangle+\limsup_{n\to\infty}\langle f(q)-q,x_n-q\rangle\\ &\le 0.\end{aligned}$$

Applying Lemma 2.8. to the inequality from Claim 3, we can conclude that $\lim_{n\to\infty}\gamma_n=0$.

Case (ii). We can find $n_j\in\mathbb{N}$ satisfying $n_j\ge j$ and $\gamma_{n_j}<\gamma_{n_j+1}$ for all $j\in\mathbb{N}$.

According to Lemma 2.10., the inequality $\gamma_{\phi(n)}\le\gamma_{\phi(n)+1}$ is obtained, where $\phi:\mathbb{N}\to\mathbb{N}$ is defined by (2.5). This implies, by Claim 2, that

$$\begin{aligned}&\beta_{\phi(n)}(1-\alpha_{\phi(n)}-\beta_{\phi(n)})\|x_{\phi(n)}-Sz_{\phi(n)}\|^2\\ &\qquad\le \gamma_{\phi(n)}-\gamma_{\phi(n+1)}+\alpha_{\phi(n)}\big(\|f(x_{\phi(n)})-q\|^2+M_0\big).\end{aligned}$$

Similar to case (i), since $\alpha_n\to 0$ as $n\to\infty$, we obtain

$\lim_{n\to\infty}\|x_{\phi(n)}-z_{\phi(n)}\|=0.$

Furthermore, an argument similar in case (i) shows that

$$\limsup_{n\to\infty}\langle f(q)-q,x_{\phi(n)+1}-q\rangle\le 0. \tag{3.14}$$

Finally, from the inquality $\gamma_{\phi(n)} \leq \gamma_{\phi(n)+1}$ and by Claim 3, we obtain

$$\gamma_{\phi(n)+1} \leq [1-\alpha_{\phi(n)}(1-\rho)]\gamma_{\phi(n)} + \alpha_{\phi(n)}(1-\rho)\left[\frac{3M}{1-\rho}\cdot\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\|x_{\phi(n)} - x_{\phi(n)-1}\|\right] + \alpha_{\phi(n)}(1-\rho)\left[\frac{2M}{1-\rho}\|x_{\phi(n)} - Sz_{\phi(n)}\| + \frac{2}{1-\rho}\langle f(q)-q, x_{\phi(n)+1}-q\rangle\right].$$

Some simple calculations yield

$$\gamma_{\phi(n)+1} \leq \frac{3M}{1-\rho}\cdot\frac{\theta_{\phi(n)}}{\alpha_{\phi(n)}}\|x_{\phi(n)} - x_{\phi(n)-1}\| + \frac{2M}{1-\rho}\|x_\phi(n) - Sz_\phi(n)\| + \frac{2}{1-\rho}\langle f(q)-q, x_{\phi(n)+1}-q\rangle.$$

From this it follows that $\limsup_{n\to\infty}\gamma_{\phi(n)+1} \leq 0$. Thus, $\lim_{n\to\infty}\gamma_{\phi(n)+1} = 0$. In addition by Lemma (2.5),

$$\lim_{n\to\infty}\gamma_n \leq \lim_{n\to\infty}\gamma_{\phi(n)+1} = 0$$

Hence, x_n converges strongly to q. This proves our theorem. □

4. Numerical Illustration

Example 4.1. Let $\mathcal{H} = \mathbb{R}$, the set of real numbers and $f : \mathbb{R} \to \mathbb{R}$ be contraction mapping and $\mathcal{M} : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued map. Let $f(x) = \frac{x}{10}$ and $\mathcal{M}=\{\frac{x}{5}\}$ $\forall x \in \mathbb{R}$, then we calculate resolvent operator $\mathfrak{J}_\lambda^{\mathcal{M}}$ and Cayley operator $\mathfrak{C}_\lambda^{\mathcal{M}}$ for $\lambda = 1$ as

$$\mathfrak{J}_\lambda^{\mathcal{M}}(x) = [I + \lambda\mathcal{M}]^{-1}(x) = \frac{5x}{6}.$$
$$\mathfrak{C}_\lambda^{\mathcal{M}}(x) = [2\mathfrak{J}_\lambda^{\mathcal{M}}(x) - I] = \frac{2x}{3},$$

let $S : \mathbb{R} \to \mathbb{R}$ be defined as $S(x) = x$ and $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{2n}$, $\lambda_n = \frac{1}{n+3}$ and $\theta_n = \frac{1}{(n+1)^2}$.

All the assumptions of Theorem 3.1 are satisfied and algorithm 3.1 reduces to

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}) \\ y_n &= \mathfrak{J}_{\lambda_n}^{\mathcal{M}}(I - \lambda_n\mathfrak{C})w_n \\ z_n &= y_n - \lambda_n(\mathfrak{C}y_n - \mathfrak{C}w_n) \\ x_{n+1} &= \frac{1}{n}f(x_n) - \frac{2n-3}{2n}x_n + \frac{1}{2n}S(z_n). \end{aligned}$$

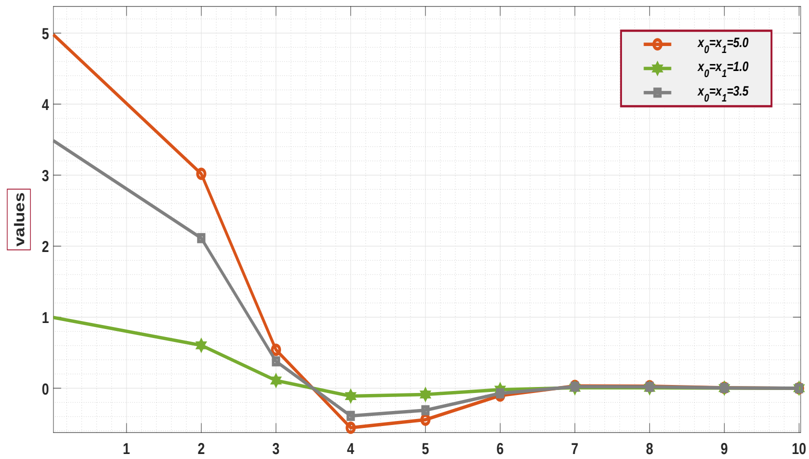

Figure 1: x_n converges to $q = 0$ for different initial values of x_0 and x_1

The iterative sequence $\{x_n\}$ generated in the above algorithm is converges strongly to $q = 0$.

All of the codes have been developed in MATLAB R2021a for simplicity. We've tried for different initial points $x_0 = x_1 = 1, 3.5, 5.0$, and found that the sequence $\{x_n\}$ converges to the solution of the problem in each case. Graph of convergence is depicted in the fig.1.

5. Conclusion

We conclude that by combining the Tseng's technique with inertial extrapolation, the algorithm generated for Cayley inclusion problem and fixed point problem is feasible and converges to some point in the solution set Σ of our problem in the framework of real Hilbert space. The numerical illustration ensures that the rate of converges of the suggested algorithm is effective and faster as compare it to the previously known algorithms in [2].

References

[1] P. Cholamjiak and S. Suantai, *Viscosity approximation methods for a nonexpansive semigroup in Banach spaces with gauge functions*, J. Glob. Optim., **54(1)**(2012), 185–197.

[2] A. H. Dar, M. K. Ahmad, J. Iqbal and W. A. Mir, *Algorithm of Common Solutions to the Cayley Inclusion and Fixed Point Problems*, Kyungpook Math. J., **61(2)**(2021), 257–267.

[3] A. Gibali and D. V. Thong, *Tseng type methods for solving inclusion problems and its applications*, Calcolo, **55**(2018), 1–22.

[4] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, UK(1990).

[5] D. Kitkuan, P. Kumam, J. Martnez-Moreno and K. Sitthithakerngkiet, *Inertial viscosity forwardbackward splitting algorithm for monotone inclusions and its application to image restoration problems*, Int. J. Comput. Math., **97(1-2)**(2020), 482–497.

[6] P. L. Lions and B. Mercier, *Splitting algorithms for the sum of two nonlinear operators*, SIAM J. Numer. Anal., **16**(1979), 964–979.

[7] D. A. Lorenz and T. Pock, *An inertial forward-backward algorithm for monotone inclusions*, J. Math. Imaging Vis., **51**(2015), 311–325.

[8] P. E. Maing, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Anal., **16**(2008), 899–912.

[9] B. Martinet, *Rgularisation dinquations variationnelles par approximations successives*, Rev. Franaise Informat. Recherche Oprationnelle, **4**(1970), 154–158.

[10] O. Nevanlinna and S. Reich, *Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces*, Israel J. Math., **32**(1979), 44–58.

[11] B. T. Polyak, *Some methods of speeding up the convergence of iteration methods*, Ussr Comput. Math. Math. Phys., **4**(1964), 1–17.

[12] S. Reich, *Extension problems for accretive sets in Banach spaces*, J. Functional Analysis, **26(4)**(1977), 378–395.

[13] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim, **14(5)**(1976), 877–898.

[14] D. V. Thong and P. Cholamjiak, *Strong convergence of a forwardbackward splitting method with a new step size for solving monotone inclusions*, Comput. Appl. Math., **38(2)**(2019), 1–16.

[15] P. Tseng, *A modified forward-backward splitting method for maximal monotone mapping*, SIAM J. Control Optim, **38(2)**(2000), 431–446.

[16] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc.(2), **66(1)**(2002), 240–256.

[17] J. Yang and H. Liu, *Strong convergence result for solving monotone variational inequalities in Hilbert space*, Numer. Algorithms, **80(3)**(2019), 741–752.