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On Injectivity of Modules via Semisimplicity

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Abstract. A right R-module N is called pseudo semisimple-M-injective if for any monomorphism from every semisimple submodule of M to N, can be extended to a homomorphism from M to N. In this paper, we study some properties of pseudo semisimple-injective modules. Moreover, some results of pseudo semisimple-injective modules over formal triangular matrix rings are obtained.

1. Introduction

Throughout the paper, R represents an associative ring with identity $1 \neq 0$ and all modules are unitary R-modules. We write M_R (resp., ${}_RM$) to indicate that M is a right (resp., left) R-module. We also write J (resp., Z_r, S_r) for the Jacobson radical (resp., the right singular ideal, the right socle) of R and $E(M_R)$ for the injective hull of M_R. If X is a subset of R, the right (resp., left) annihilator of X in R is denoted by $r_R(X)$ (resp., $l_R(X)$) or simply $r(X)$ (resp., $l(X)$) if no confusion appears. If N is a submodule of M (resp., proper submodule) we denote by $N \leq M$ (resp., $N < M$). Moreover, we write $N \leq^e M$, $N \ll M$, $N \leq^{\oplus} M$ and $N \leq^{max} M$ to indicate that N is an essential submodule, a small submodule, a direct summand and a maximal submodule of M, respectively. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M.

Recently, some authors considered some generalizations of quasi-injective modules and automorphism-invariant modules (pseudo-injective modules)(see [1, 6, 9,

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10, 12, 14, 15, 16, 17]). Some properties of automorphism-invariant modules and the structure of rings via the class of automorphism-invariant modules are studied (see [3, 8, 11, 18, 19]). In 2005, Hai Quang Dinh studied a generalization of the M-injective module that is pseudo M-injective. A module N is called pseudo M-injective if for any submodule A of M and every monomorphism from A to N, can be extended to a homomorphism from M to N. A module M is called pseudo-injective if M is pseudo M-injective.

A generalization of M-injective modules, Amin-Yousif-Zeyada ([4]) introduced the soc M-injective. A right R-module N is called soc-M-injective if for any homomorphism $Soc(M) \to N$, can be extended to a homomorphism from M to N. A module M is called soc-quasi-injective if M is soc-M-injective.

The purpose of this paper, we consider a generalization of soc-M-injective and pseudo M-injective modules, that is pseudo semisimple-M-injective. We call that a module N is pseudo semisimple-M-injective if for any monomorphism from every semisimple submodule of M to N, can be extended to a homomorphism from M to N. A module M is called pseudo semisimple-injective if M is pseudo semisimple-M-injective. In this paper, we will give some properties of pseudo semisimple-injective modules and structure of rings via these modules.

In Section 2, we give some basic properties of pseudo semisimple-injective modules and relatively pseudo semisimple-injective modules. It is well known that a module pseudo-injective is direct-injective (C2-module) (see [6, Theorem 2.6]). We study this result for pseudo semisimple-injective modules. We prove in Proposition 2.4 that pseudo semisimple-injective modules are semisimple-direct-injective. On the other hand, we show that if $M = \oplus_{i \in I} M_i$ is a direct sum of uniform submodules M_i, then M is soc-quasi-injective if and only if M is pseudo semisimple-injective (see Theorem 2.12). Next, we consider the projectivity of socles of modules via the pseudo semisimple-injectivity and we obtain in Theorem 2.13 that; if M is a projective module, then $Soc(M)$ is projective iff every quotient module of a pseudo semisimple-M-injective module is pseudo semisimple-M-injective, iff every quotient module of a semisimple-M-injective module is pseudo semisimple-M-injective, iff every quotient module of an injective module is pseudo semisimple-M-injective. From the definition of pseudo semisimple-injective module, we study structure of rings in which every semisimple right module is pseudo semisimple-M-injective for every cyclic rightmodule M. We show that a ring R is a right Noetherian right V-ring iff every semisimple right R-module is pseudo semisimple-M-injective for every cyclic right R-module M, iff every right R-module is pseudo semisimple-M-injective for every cyclic right R-module M (see Theorem 2.23). Some other properties are studied and extended. Finally, we study the pseudo semisimple-injectivity of modules over formal triangular matrix rings.

2. On pseudo semisimple-injective modules

Definition 2.1. A right R-module N is called pseudo semisimple-M-injective if for any semisimple submodule A of M, any monomorphism $f : A \to N$ extends to a homomorphism from M to N. A module M is called pseudo semisimple-injective if M is pseudo semisimple-M-injective.

A right R-module N is called *soc-M-injective* if for any homomorphism from $Soc(M)$ to N, can be extended to a homomorphism from M to N. A module M is called *soc-quasi-injective* if M is soc-M-injective (see [4]).

All soc-M-injective modules are pseudo semisimple-M-injective. But the converse is not true in general.

Example 2.2. Assume that a right R-module M has only five submodules $0, M_1, M_2, M_1 \oplus M_2, M$, which $M_1 \not\simeq M_2$ and $End(M_i) \simeq \mathbb{Z}_2$ (see Hallett's example and Teply's example). Then M is pseudo semisimple-M-injective. Note that $Soc(M) = M_1 \oplus M_2$ and the projection of $Soc(M)$ to M_1 cannot be extended to a homomorphism from M to M. It follows that M is not soc-M-injective.

Lemma 2.3. *Let M and N be two modules.*

(1) *If N is pseudo semisimple-M-injective and A is a direct summand of N, then A is pseudo semisimple-M-injective.*

(2) *If N is pseudo semisimple-M-injective and B is a closed submodule of M, then N is pseudo semisimple-B-injective.*

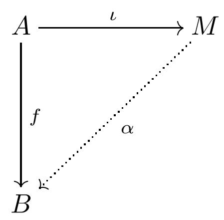
(3) *If M is pseudo semisimple-injective, then A is pseudo semisimple-injective for all fully invariant closed submodule A of M.*

Proof. It is obvious. □

A module M is called *semisimple-direct-injective* if for any semisimple submodules A, B of M with $A \cong B$ and B a direct summand of M, A is a summand of M (see [2]).

Proposition 2.4. *Every pseudo semisimple-injective module is semisimple-direct-injective.*

Proof. Assume that M is a pseudo semisimple-injective module. Let B be a direct summand of M and A be a semisimple submodule of M with $A \simeq B$. We show that B is a direct summand of M. Let $f : A \to B$ be an isomorphism. We have that B is a direct summand of M and obtain that B is pseudo semisimple-M-injective by Lemma 2.3. There exists a homomorphism $\alpha : M \to B$ that is an extension of f.



That is $\alpha\iota = f$ with the inclusion map $\iota : A \to M$. We deduce that ι splits and so A is a direct summand of M. □

Corollary 2.5. *Let M be a pseudo semisimple-injective module. If $M = A_1 \oplus A_2$ where A_1 is semisimple and $f : A_1 \to A_2$ is a homomorphism, then* $\mathrm{Im}(f)$ *is a direct summand of A_2.*

Theorem 2.6. *Let R and S be Morita-equivalent rings with the category equivalence $\mathcal{F} : Mod - R \to Mod - S$. Let M, N and K be right R-modules and $f : H \to L$ be a homomorphism of right R-modules. Then:*

(1) *K_R is semisimple if and only if $\mathcal{F}(K)_S$ is semisimple.*

(2) *f is a monomorphism if and only if $\mathcal{F}(f)$ is a monomorphism.*

(3) *M_R is pseudo semisimple-N-injective if and only if $\mathcal{F}(M)_S$ is pseudo semisimple-$\mathcal{F}(N)_S$-injective.*

Proof. (1) and (2) by [5, Proposition 21.4, 21.8].
(3) is followed from (1) and (2). □

A ring R is called *right pseudo semisimple-injective* if R_R is pseudo semisimple-injective.

Corollary 2.7. *Right pseudo semisimple-injectivity is a Morita invariant property of rings.*

Proposition 2.8. *Let M and N be modules and $X = M \oplus N$. The following conditions are equivalent:*

(1) *N is soc-M-injective.*

(2) *For each semisimple submodule K of X, where $K \cap N = 0$, there exists $C \leq X$ such that $K \leq C$ and $N \oplus C = X$.*

Proof. (1) $\Rightarrow$ (2). Let K be a semisimple submodule of X, with $K \cap N = 0$, $\pi_M : M \oplus N \to M$ and $\pi_N : M \oplus N \to N$ the canonical projections. We can check that $N \oplus K = N \oplus \pi_M(K)$ and $\pi_M(K)$ is a semisimple submodule of M. Let $\varphi : \pi_M(K) \to \pi_N(K)$ be a homomorphism defined as follows: for $k = m + n \in K$ (with $m \in M, n \in N$), $\varphi(m) = n$. It is easy to see that φ is a monomorphism. Since N is pseudo semisimple-M-injective, there is a homomorphism $\bar{\varphi} : M \to N$, which extends φ. Let $C = \{m - \bar{\varphi}(m) |\ m \in M\}$ be a submodule of X. Then $X = N \oplus C$ and K is contained in C.

(2) $\Rightarrow$ (1). Let A be a semisimple submodule of M and $\varphi : A \to N$ be a homomorphism. Put $K = \{a - \varphi(a) |\ a \in A\}$ be a submodule of X. It follows that $K \leq A \oplus \varphi(A)$. Then $\pi_M(K) = A$, $N \oplus K = N \oplus \pi_M(K) = N \oplus A$ and K is a semisimple submodule of X. By assumption, there exists a submodule C of X containing K with $N \oplus C = X$. Let $\pi : N \oplus C \to N$ be the natural projection. Then the restriction $\pi|_M$ extends φ, proving (1). ☐

Similarly, we have a result for pseudo semisimple-M-injective modules.

Proposition 2.9. *Let M and N be modules and $X = M \oplus N$. The following conditions are equivalent:*

(1) *N is pseudo semisimple-M-injective.*

(2) *For each semisimple submodule K of X, where $K \cap M = K \cap N = 0$, there exists $C \leq X$ such that $K \leq C$ and $N \oplus C = X$.*

Theorem 2.10. *If $M \oplus N$ is a pseudo semisimple-injective module, then N is soc-M-injective.*

Proof. Assume that $M \oplus N$ is pseudo semisimple-injective, and $f : Soc(M) \to N$ is a homomorphism. Define $g : Soc(M) \to M \oplus N$ by $g(m) = (m, f(m))$ (for all $m \in Soc(M)$). Clearly, g is a monomorphism. By Lemma 2.3, $M \oplus N$ is pseudo semisimple-M-injective, whence g extends to a homomorphism $g^* : M \to M \oplus N$. Let $\pi : M \oplus N \to N$ be the natural projection. Then πg^* is a homomorphism extending f. Consequently, N is soc-M-injective. ☐

Corollary 2.11. *For any integer $n \geq 2$, M^n is pseudo semisimple-injective if and only if M is soc-quasi-injective.*

Theorem 2.12. *Let $M = \oplus_{i \in I} M_i$ be a direct sum of uniform submodules M_i. Then M is soc-quasi-injective if and only if M is pseudo semisimple-injective.*

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$) First let M be a uniform pseudo semisimple-injective module. Let $f : Soc(M) \to M$ be a homomorphism. If $Ker f = 0$, then f can be extended to an endomorphism of M. Otherwise, $Ker f \neq 0$. Let $g = \iota - f$, where $\iota : Soc(M) \to M$ is the inclusion homomorphism. Since $Ker f \neq 0$ and M is uniform, $Ker g = 0$. Then, by the pseudo semisimple-injectivity, g can be extended to some $h \in End(M)$. Now $1_M - h \in End(M)$ is an extension of f. Thus M is soc-quasi-injective.

Now let M be a pseudo semisimple-injective module and $M = \oplus_{i \in I} M_i$. For all $j \in I$, we have $\oplus_{i \in I \setminus \{j\}} M_i$ is pseudo semisimple-M_j-injective by Theorem 2.10. Since direct summands of pseudo semisimple-injective are obviously pseudo semisimple-injective and by the remark above, each M_j is soc-quasi-injective. Therefore, M is soc-quasi-injective □

Theorem 2.13. *The following conditions are equivalent for a projective module M:*

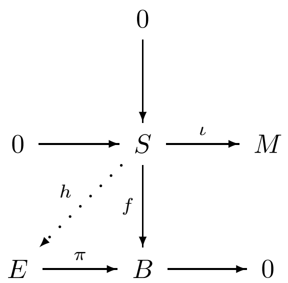
(1) *$Soc(M)$ is projective.*

(2) *Every quotient module of a pseudo semisimple-M-injective module is pseudo semisimple-M-injective.*

(3) *Every quotient module of a soc-M-injective module is pseudo semisimple-M-injective.*

(4) *Every quotient module of an injective module is pseudo semisimple-M-injective.*

Proof. (1) ⇒ (2). Assume that E_R is pseudo semisimple-M-injective and $\pi : E \longrightarrow B$ is an epimorphism. Let $f : S \longrightarrow B$ be a monomorphism with S a semismple submodule of M.



where ι is the inclusion.

By (1), $Soc(M)$ is projective, and so S is projective. Therefore, there exists an R-homomorphism $h : S \longrightarrow E$ such that $\pi h = f$. Since f is monomorphism, h is too. Now since E is pseudo semisimple-M-injective, there is an R-homomorphism $h' : M \longrightarrow E$ such that $h' \iota = h$. Let $h'' = \pi h' : M \longrightarrow B$, then $h'' \iota = f$. This means B is pseudo semisimple-M-injective.

(2) ⇒ (3) ⇒ (4) is obvious.

(4) ⇒ (1). We consider the epimorphism $h : A \longrightarrow B$ and an R-homomorphism $\alpha : Soc(M) \longrightarrow B$.

Since $B = h(A) \overset{\psi}{\cong} A/Kerh \overset{\iota_1}{\hookrightarrow} E(A)/Kerh$, where ι_1 is the inclusion and $\psi(h(a)) = a + Kerh$, for all $a \in A$. Then let $j = \iota_1\psi$. We consider the following diagram:

$$\begin{array}{ccccc} & & Soc(M) & \overset{\iota}{\hookrightarrow} & M \\ & \overset{\varphi}{\swarrow} & \overset{\alpha}{\to}\downarrow & & \\ A & \overset{h}{\longrightarrow} & B & \longrightarrow & 0 \\ & & \overset{j}{\to}\downarrow & & \\ E(A) & \overset{p}{\longrightarrow} & E(A)/Kerh & \longrightarrow & 0 \end{array}$$

where ι is the inclusion and p is the natural epimorphism.

By (4), $E(A)/Kerh$ is pseudo semisimple-M-injective and then there exists an R-homomorphism $\alpha' : M \longrightarrow E(A)/Kerh$ such that $\alpha'\iota = j\alpha$. Since M is projective, there is an R-homomorphism $\alpha'' : M \longrightarrow E(A)$ such that $p\alpha'' = \alpha'$. Let $h' = \alpha''\iota : Soc(M) \longrightarrow E(A)$. It is easy to see that $h'(Soc(M)) \leq A$, so there exists an R-homomorphism $\varphi : Soc(M) \longrightarrow A$ such that $\varphi(x) = h'(x)$, for all $x \in Soc(M)$.

Now we claim that $h\varphi = \alpha$. In fact, for each $x \in Soc(M)$ we have

$$j(\alpha(x)) = \alpha'(\iota(x)) = \alpha'(x) = p(\alpha''(x)) = p(h'(x)) = p(\varphi(x)).$$

Since α is an epimorphism, $\alpha(x) = h(a)$ for some $a \in A$. Therefore $j(\alpha(x)) = j(h(a)) = a + Kerh$, and so $a + Kerh = \varphi(x) + Kerh$, $h(a - \varphi(x)) = 0$. Hence $h\varphi(x) = h(a) = \alpha(x)$. Thus $Soc(M)$ is projective. □

Corollary 2.14. *The following conditions are equivalent:*

(1) *$Soc(R_R)$ is projective.*

(2) *Every quotient module of a pseudo semisimple-R_R-injective module is pseudo semisimple-R_R-injective.*

(3) *Every quotient module of a soc-R_R-injective module is pseudo semisimple-R_R-injective.*

(4) *Every quotient module of an injective module is pseudo semisimple-R_R-injective.*

Proposition 2.15. *Let M be a finitely generated module. If every direct sum of pseudo semisimple-M-injective modules is pseudo semisimple-M-injective, then $Soc(M)$ is finitely generated.*

Proof. Assume that $Soc(M) = \oplus_I S_i$ with S_i simple. Let $i : Soc(M) \to \oplus_I E(S_i)$ be the inclusion monomorphism. Since $\oplus_I E(S_i)$ is pseudo semisimple-M-injective, there exists a homomorphism $g : M \to \oplus_I E(S_i)$ such that g is an extension of i. Since M is finitely generated, $i(Soc(M)) = g(Soc(M)) \leq \oplus_K E(S_i)$ for some finite subset K of I. Moreover, $Soc(\oplus_K E(S_i))$ is finitely generated and so $Soc(M)$ is finitely generated. □

Proposition 2.16. *For a right R-module M, the following conditions are equivalent:*

(1) *M is soc-$E(M)$-injective.*

(2) *M is pseudo semisimple-N-injective for all right R-modules N.*

Proof. (1) ⇒ (2) by [4, Theorem 3.1].

(2) ⇒ (1). By [4, Theorem 3.1], we only prove $M = E \oplus T$ with E injective and $Soc(T) = 0$. If $Soc(M) = 0$, we are done. Otherwise, we have that M is pseudo semisimple-$E(Soc(M))$-injective and obtain that there exists a homomorphism $f : E(Soc(M)) \to M$ such that $f(x) = x$ for all $x \in Soc(M)$. Since $Soc(M) \leq^e E(Soc(M))$, f is a monomorphism. That means f is a splitting monomorphism. Thus, $M = E \oplus T$ with E injective and $Soc(T) = 0$. □

Corollary 2.17. *The following conditions on a ring R are equivalent:*

(1) *R is right Noetherian.*

(2) *If $S_1, S_2, \ldots, S_n \ldots$ are simple right R-modules, $\oplus_{i=1}^{\infty} E(S_i)$ is pseudo semisimple-N-injective for all right R-modules N.*

Lemma 2.18. *The following conditions are equivalent for a right R-module M:*

(1) *Every right R-module is pseudo semisimple-M-injective.*

(2) *Every semisimple right R-module is pseudo semisimple-M-injective.*

(3) *$Soc(M)$ is a direct summand of M.*

Proof. (1) ⇒ (2) and (3) ⇒ (1) are obvious.

(2) ⇒ (3). Assume that every semisimple right R-module is pseudo semisimple-M-injective. Then, $Soc(M)$ is pseudo semisimple-M-injective. It follows that $Soc(M)$ is a direct summand of M.

□

A ring R is called a *right V-ring* if every simple right R-module is injective.

Proposition 2.19. *The following conditions are equivalent for a ring R:*

(1) *R is a right V-ring.*

(2) *Every finitely cogenerated right R-module is a pseudo semisimple-injective right R-module.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let S be a simple right R-module. Then, $S \oplus E(S)$ is a finitely cogenerated R-module. Take $\iota : S \to E(S)$ the inclusion map. It follows that $S = \iota(S)$ is a direct summand of $E(S)$ by Corollary 2.5. We deduce that $E = E(S)$ is injective. □

Corollary 2.20. *The following conditions are equivalent for a ring R:*

(1) *R is a right Noetherian right V-ring.*

(2) $S \oplus E(S)$ *is a pseudo semisimple-injective right R-module for all simple right R-module S.*

Similarly, we also have the following result for Noetherian V-rings.

Proposition 2.21. *The following conditions are equivalent for a ring R:*

(1) *R is a right Noetherian right V-ring.*

(2) *Every right R-module with essential socle is a pseudo semisimple-injective right R-module.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let $\{S_i\}_{i \in I}$ be a family of simple modules. Then, $(\oplus_{i \in I} S_i) \oplus E(\oplus_{i \in I} S_i)$ is a right R-module with essential socle, and so it is a semisimple-injective right R-module. It follows that $\oplus_{i \in I} S_i$ is injective. □

Corollary 2.22. *The following conditions are equivalent for a ring R:*

(1) *R is a right Noetherian right V-ring.*

(2) $S \oplus E(S)$ *is a pseudo semisimple-injective right R-module for all semisimple right R-module S.*

Theorem 2.23. *The following conditions are equivalent for a ring R:*

(1) *R is a right Noetherian right V-ring.*

(2) *Every semisimple right R-module is pseudo semisimple-M-injective for every cyclic right R-module M.*

(3) *Every right R-module is pseudo semisimple-M-injective for every cyclic right R-module M.*

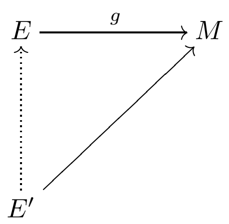
Proof. (1) $\Rightarrow$ (2). Since R is a right Noetherian right V-ring, every semisimple right R-module is injective, and hence every semisimple right R-module is pseudo semisimple-M-injective for every cyclic right R-module M.

(2) $\Rightarrow$ (3). Assume that every semisimple right R-module is pseudo semisimple-C-injective for every cyclic right R-module C. Let M be a cyclic right R-module. Then, $Soc(M)$ is a direct summand of M. We deduce that every right R-module is pseudo semisimple-M-injective by Lemma 2.18.

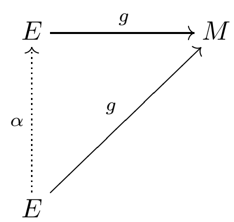
(3) $\Rightarrow$ (1) We show that every semisimple right R-module is injective. Let S be a semisimple right R-module and N be a cyclic right R-module. Then, every right R-module is pseudo semisimple-N-injective by (3). It follows that $Soc(N)$ is a direct summand of N by Lemma 2.18. This implies that S is semisimple-N-injective. We deduce that S is injective by [4, Lemma 3.11]. □

Enochs [7] introduced the injective cover notion which is the dual to the injective envelope, and showed that a ring R is a right Noetherian ring if and only if every right R-module has an injective cover. Now, we introduce the pseudo semisimple-injective cover notion.

Definition 2.24. An R-homomorphism $g : E \to M$ is called a *psi-cover* of a right R-module M if E is a pseudo semisimple-injective module such that any diagram



with E' a pseudo semisimple-injective module can be completed; and the diagram



can be completed only by an automorphism α.

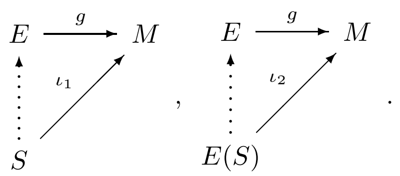
Now, we prove in Theorem 2.25 that a ring R is a right Noetherian right V-ring if and only if every right R-modules with essential socle has a psi-cover.

Theorem 2.25. *The following are equivalent for a ring R:*

(1) *R is a right Noetherian right V-ring.*

(2) *Every right R-modules with essential socle has a psi-cover.*

Proof. (1) $\Rightarrow$ (2). It is obvious.
(2) $\Rightarrow$ (1) Let S be a semisimple right R-module and let $M = S \oplus E(S)$. We show that M is pseudo semisimple-injective. Call $g : E \to M$ a psi-cover of M. Consider the following diagrams:



where $\iota_1 : M \to S$ and $\iota_2 : M \to E(S)$ are the canonical injections. Note that all modules S and $E(S)$ are pseudo semisimple-injective modules. By the definition of psi-cover, there exist homomorphisms $\alpha_1 : S \to E$ and $\alpha_2 : E(S) \to E$ such that $g\alpha_i = \iota_i$ for $i = 1, 2$. Define $\alpha : M \to E$ by $\alpha(x_1 + x_2) = \alpha_1(x_1) + \alpha_2(x_2)$ for all $x_1 \in S$ and $x_2 \in E(S)$. It can easily be checked that α is well-defined and we have

$$g\alpha(x_1 + x_2) = g\alpha_1(x_1) + g\alpha_2(x_2) = \iota_1(x_1) + \iota_2(x_2) = x_1 + x_2.$$

Thus, $g\alpha = 1_M$, and $\alpha : M \to E$ is a split monomorphism. Then M is isomorphic to a direct summand of E. Since a direct summand of a pseudo semisimple-injective module is again a pseudo semisimple-injective module, M is a pseudo semisimple-injective module. By Corollary 2.22, R is a right Noetherian V-ring.

□

Let R and S be two rings and M be an $R - S$-bimodule (left R-module and right S-module). Take

$$K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \mid r \in R, s \in S, m \in M \right\}$$

a ring with the addition and multiplication as follows:

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} + \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} r + r' & m + m' \\ 0 & s + s' \end{pmatrix}$$

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rm' + ms \\ 0 & ss' \end{pmatrix}$$

The ring K is also called a formal triangular matrix ring (see [13]). It is well-known that the category of right K-module Mod-K is equivalent to the category $\mathbb{T}$ of triples (X, Y, f), where X is a right R-module, Y is a right S-module and $f : X \otimes_R M \to Y$ is a homomorphism of right S-modules. The right K-module (X, Y, f) is the additive group $X \oplus Y$ with right K-action given by

$$(x\ y)\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = (xr, f(x \otimes m) + ys)$$

Next, we consider homomorphisms of K-modules. Let (X_1, Y_1, f_1) and (X_2, Y_2, f_2) be right K-modules. A right K-homomorphism $\varphi : (X_1, Y_1, f_1) \to (X_2, Y_2, f_2)$ is a pair (φ_1, φ_2) where $\varphi_1 : X_1 \to X_2$ is a homomorphism of right R-modules and $\varphi_2 : Y_1 \to Y_2$ is a homomorphism of right S-modules such that the following diagram is commutative

$$\begin{array}{ccc} X_1 \otimes_R M & \xrightarrow{f_1} & Y_1 \\ \Big\downarrow{\scriptstyle \varphi_1 \otimes 1_M} & & \Big\downarrow{\scriptstyle \varphi_2} \\ X_2 \otimes_R M & \xrightarrow{f_2} & Y_2 \end{array}$$

Note that a K-homomorphism $\varphi = (\varphi_1, \varphi_2) : (X_1, Y_1, f_1) \to (X_2, Y_2, f_2)$ is a monomorphism (epimorphism) if and only if φ_1 and φ_2 are monomorphisms (epimorphisms).

A submodule of a right K-module (X, Y, f) is a triple (X_0, Y_0, f_0), where $X_0 \leq X_R$, $Y_0 \leq Y_S$ such that the following diagram is commutative.

$$\begin{array}{ccc} X_0 \otimes_R M & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle \iota_1 \otimes 1_M}\Big\downarrow & & \Big\downarrow{\scriptstyle \iota_2} \\ X \otimes_R M & \xrightarrow{f} & Y \end{array}$$

with $\iota_1 : X_0 \to X$, $\iota_2 : Y_0 \to Y$ the inclusion maps.

Proposition 2.26. *Let* $K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *and* (X, Y, f) *be a right* K*-module. If* (X, Y, f) *is a pseudo semisimple-injective right* K*-module then*

(1) Y *is a pseudo semisimple-injective right* S*-module.*

(2) *$H = \{x \in X \mid f(x \otimes m) = 0$ for all $m \in M\}$ is a pseudo semisimple-injective right R-module.*

Proof. (1) Let Y_0 be a semisimple submodule of Y and $\varphi : Y_0 \to Y$ is an S-monomorphism. Then, $(0, Y_0, 0)$ is a semisimple submodule of K-module (X, Y, f) and $\gamma = (0, \varphi) : (0, Y_0, 0) \to (X, Y, f)$ is a K-homomorphism. By our assumption, $(0, \varphi)$ is a K-monomorphism, and so there exists an endomorphism $\theta = (\theta_1, \theta_2)$ of (X, Y, f) such that θ is an extension of γ. It follows that $\theta_2 : Y \to Y$ is an extension of φ. Hence Y is a pseudo semisimple-injective module.

(2) Let X_0 be a semisimple submodule of H and $\beta : X_0 \to H$ is an R-monomorphism. Then, $(X_0, 0, 0)$ is a semisimple submodule of K-module (X, Y, f) and $\delta = (\beta, 0) : (X_0, 0, 0) \to (X, Y, f)$ is a K-monomorphism, and so there exists an endomorphism $\omega = (\omega_1, \omega_2)$ of (X, Y, f) such that ω is an extension of δ. It means that the following is commutative

$$\begin{array}{ccc} X \otimes_R M & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \omega_1 \otimes 1_M} & & \downarrow{\scriptstyle \omega_2} \\ X \otimes_R M & \xrightarrow{f} & Y \end{array}$$

and so, $\omega_2 \circ f = f \circ (\omega_1 \otimes 1_M)$. We deduce that $\omega_1(H) \leq H$. Then, $\omega_1|_H : H \to H$ is an extension of β. It shows that H is a pseudo semisimple-injective module. □

Proposition 2.27. *Let $K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ and (X, Y, f) be a right K-module. If*

(1) *Y is a pseudo semisimple-injective right S-module and*

(2) *$H = \{x \in X \mid f(x \otimes m) = 0$ for all $m \in M\}$ is a pseudo semisimple-injective right R-module.*

then $(H, Y, 0)$ is a pseudo semisimple-injective right K-module.

Proof. Let (X_0, Y_0, f_0) be a semisimple submodule of $(H, Y, 0)$ and $\alpha = (\alpha_1, \alpha_2) : (X_0, Y_0, f_0) \to (H, Y, 0)$ is a K-monomorphism. Then, $f_0 = 0$ and $\alpha_1 : X_0 \to H$, $\alpha_2 : Y_0 \to Y$ are monomorphisms. Note that X_0 is a semisimple submodule of H and Y_0 is a semisimple submodule of Y. Since H and Y are pseudo semisimple-injective, there exist an endomorphism β_1 of H and β_2 of Y such that β_1 is an extension of α_1 and β_2 is an extension of α_2. One can check that $\beta = (\beta_1, \beta_2)$ is an endomorphism of $(H, Y, 0)$ and it is an extension of α. □

Let (X, Y, f) be a right K-module. Then, we have the following R-homomorphism

$$\begin{aligned} \tilde{f} : X &\longrightarrow Hom_S(M, Y) \\ x &\longmapsto \tilde{f}(x) : M \to Y \\ &\qquad\qquad m \mapsto \tilde{f}(x)(m) = f(x \otimes m) \end{aligned}$$

Proposition 2.28. *Let* $K = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *and* (X, Y, f) *be a right* K*-module. If*

(1) Y *is a pseudo semisimple-injective right* S*-module and*

(2) $\tilde{f}$ *is an isomorphism of right* R*-module.*

then (X, Y, f) *is a pseudo semisimple-injective right* K*-module.*

Proof. Let (X_0, Y_0, f_0) be a semisimple submodule of (X, Y, f) and $\alpha = (\alpha_1, \alpha_2) : (X_0, Y_0, f_0) \to (X, Y, f)$ is a K-monomorphism. Then, $\alpha_1 : X_0 \to X$ and $\alpha_2 : Y_0 \to Y$ are monomorphisms with $\alpha_2 \circ f_0 = f \circ (\alpha_1 \otimes 1_M)$. Note that Y_0 is a semisimple submodule of Y. Since Y is a pseudo semisimple-injective module, there exists an endomorphism β_2 of Y such that β_2 is an extension of α_2.

Fix $x \in X$. For any $m \in M$, set $\theta(m) = \beta_2(f(x \otimes m))$. It follows that $\theta : M \to Y$ is an S-homomorphism. By assumption there exists a unique element $x' \in X$ such that $\tilde{f}(x') = \theta$. Then, for all $m \in M$ we have

$$f(x' \otimes m) = \tilde{f}(x')(m) = \theta(m) = \beta_2(f(x \otimes m))$$

We define $\beta_1 : X \to X$ via $\beta_1(x) = x'$. One can check that β_1 is an R-homomorphism and satisfies $f \circ (\beta_1 \otimes 1_M) = \beta_2 \circ f$. This means that $\beta = (\beta_1, \beta_2) : (X, Y, f) \to (X, Y, f)$ is a K-homomorphism. Next, we show that β_1 extends α_1. In fact, for any $x_0 \in X_0$ and for all $m \in M$, we have $(\alpha_2 \circ f_0)(x_0 \otimes m) = f \circ (\alpha_1 \otimes 1_M)(x_0 \otimes m)$ or $\beta_2 \circ f(x_0 \otimes m) = f(\alpha_1(x_0) \otimes m)$. It follows that $f(\beta_1(x_0) \otimes m) = f(\alpha_1(x_0) \otimes m)$ or $\tilde{f}(\beta_1(x_0)) = \tilde{f}(\alpha_1(x_0))$. Since $\tilde{f}$ is an isomorphism, $\beta_1(x_0) = \alpha_1(x_0)$. We deduce that β extends α and so, (X, Y, f) is pseudo semisimple-injective. ☐

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