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Some Geometric Properties of η_*-Ricci Solitons on α-Lorentzian Sasakian Manifolds

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Abstract. We investigate the geometric properties of η_*-Ricci solitons on α-Lorentzian Sasakian (α-LS) manifolds, and show that a Ricci semisymmetric η_*-Ricci soliton on an α-LS manifold is an η_*-Einstein manifold. Further, we study φ_*-symmetric η_*-Ricci solitons on such manifolds. We prove that φ_*-Ricci symmetric η_*-Ricci solitons on an α-LS manifold are also η_*-Einstein manifolds and provide an example of a 3-dimensional α-LS manifold for the existence of such solitons.

1. Introduction

The Ricci flow, which is used to compute the canonical metric based on the smooth manifold, was proposed by Hamilton [19] in 1982. The Ricci flow provides an evolution expression of metrics for a Riemannian manifold as follows:

$$\frac{\partial}{\partial t} g_{*ij}(t) = -2R_{ij.} \tag{1.1}$$

The Einstein metric can be naturally generalized to Ricci solitons which are defined on the Riemannian manifold (M, g_*) [6]. The triplet (g_*, V, ω_1) is a Ricci soliton where g_*, V are the Riemannian metric or the pseudo Riemannian metric,

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and vector field (potential vector field), respectively. The real scalar ω_1 is expressed in terms of a Ricci tensor S, and a Lie derivative operator $\pounds_V$, as

$$\pounds_V g_* + 2S + 2\omega_1 g_* = 0, \tag{1.2}$$

If $\omega_1 > 0$, the Ricci solitons is called expanding while it is called shrinking if $\omega_1 < 0$. The case $\omega_1 = 0$ represents the steady Ricci soliton [20]. The Einstein equation can be recovered from Ricci solitons for $V = 0$. The metric expressed by (2) is generally known as quasi-Einstein [7], [8] and is used frequently in physical systems. The fixed Ricci flow points for $\frac{\partial}{\partial t} g_* = -2S$ which are projected from metrics space onto its diffeomorphic quotient, modulo scaling, are referred to as the compact Ricci solitons. These solitons frequently arise in Ricci flow on compact manifolds for larger limiting cases. Ricci solitons play an interesting role in string theory, the initial aspects of which were discussed by Friedman [17]. The similarity solution of Ricci flow in Riemannian geometry was introduced by [19] as it explores the concept of a singularity. Several authors have studied the geometric properties of Ricci solitons over different manifolds, for instance see [11], [12], [14] and [15],[23].

Cho and Kimura [9] proposed the concept of η_*-Ricci solitons as type of generalized Ricci solitons. Calin and Crasmareanu [5], [6] extended this concept for Hopf hypersurfaces in complex space. The tuple $(g_*, V, \omega_1, \omega_2)$ with constants ω_1 and ω_2 denote the η_*-Ricci solitons with the condition

$$\pounds_V g_* + 2S + 2\omega_1 g_* + 2\omega_2 \eta_* \otimes \eta_* = 0, \tag{1.3}$$

In the current scenario, η-Ricci solitons are studied by various researchers have considered such η-Ricci solitons, and have found interesting geometric properties in many contexts: on Lorentzian para-Sasakian manifolds [2], [28], gradient η-Ricci solitons [3], on ϵ-para Sasakian manifolds [21] and [4], quasi-Sasakian 3-manifolds [22], 3-dimensional Kenmotsu manifolds [24], Sasakian 3-manifolds [25], para-Sasakian manifolds [26] and para Kenmotsu manifolds [29] and studied Lorentzian Sasakian manifold [27] etc.

The structure of the paper is as follows. The neccessary basic theory about α-LS manifolds is given in Section 2. In Section 3, the geometric properties of η_*-Ricci solitons on α-LS manifolds are investigated. In Section 4, we show that Ricci semisymmetric η_*-Ricci solitons on α-LS manifolds reduce to an η_*-Einstein manifold. In Section 5, we study φ_*-symmetric η_*-Ricci solitons on α-LS manifolds. In Section 6, we show that a φ_*-Ricci symmetric η_*-Ricci soliton on such a manifold is also an η_*-Einstein manifold. Finally, we provide an example of a 3-dimensional α-LS manifold for the existence of such solitons.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M is said to be an α-LS manifold if it admits a $(1,1)$-tensor field φ_*, a vector field ζ and 1-form η_* and Lorentzian metric g_* which satisfy the following conditions:

$$\varphi_*^2 = I + \eta_* \otimes \zeta, \ \ \eta_*(\zeta) = -1, \tag{2.1}$$

$$\varphi_* \zeta = 0, \ \ \eta_* \circ \varphi_* = 0, \tag{2.2}$$

$$g_*(\varphi_* E, \varphi_* F) = g_*(E, F) + \eta_*(E)\eta_*(F), \tag{2.3}$$

$$g_*(E, \zeta) = \eta_*(E), \tag{2.4}$$

for any vector fields E, F on M.

Also, α-LS manifolds satisfy [16],

$$\nabla_E \zeta = \alpha \varphi_* E, \tag{2.5}$$

$$(\nabla_E \eta_*)F = \alpha g_*(\varphi_* E, F), \tag{2.6}$$

$$(\nabla_E \varphi_*)F = \alpha g_*(E, F)\zeta - \alpha \eta_*(F)E, \tag{2.7}$$

where ∇ has the usual meaning.

Moreover, on α-LS manifolds the following relations hold (see [1]):

$$R(\zeta, E)F = \alpha^2[g_*(E, F)\zeta - \eta_*(F)E], \tag{2.8}$$

$$R(E, F)\zeta = \alpha^2[\eta_*(F)E - \eta_*(E)F], \tag{2.9}$$

$$R(\zeta, E)\zeta = \alpha^2[\eta_*(E)\zeta + E], \tag{2.10}$$

$$S(E, \zeta) = 2n\alpha^2 \eta_*(E), \tag{2.11}$$

$$Q\zeta = 2n\alpha^2 \zeta, \tag{2.12}$$

$$S(\zeta, \zeta) = -2n\alpha^2, \tag{2.13}$$

$$S(\varphi_* E, \varphi_* F) = S(E, F) + 2n\alpha^2 \eta_*(E)\eta_*(F), \tag{2.14}$$

where R, S, Q are the Riemannian curvature, Ricci tensor and Ricci operator, respectively while α is a constant. S and Q are related by $S(E, F) = g_*(QE, F)$ for all $E, F \in \chi(M)$.

As per the definition of the Lie derivative, we have

$$
\begin{aligned}
(\pounds_\zeta g_*)(E,F) &= (\nabla_\zeta g_*)(E,F) + g_*(\alpha\varphi_* E,F) + g_*(E,\alpha\varphi_* F) \\
&= 2\alpha g_*(\varphi_* E,F), \qquad (2.15)
\end{aligned}
$$

$$
\begin{aligned}
(\pounds_\zeta \varphi_*)E &= [\zeta,\varphi_* E] - \varphi_*([\zeta,E]) \\
&= \nabla_\zeta \varphi_* E - \nabla_{\varphi_* E}\zeta - \varphi_*(\nabla_\zeta E) + \varphi_*(\nabla_E \zeta) \\
&= \nabla_\zeta \varphi_* E - \varphi_*(\nabla_\zeta E) \\
&= (\nabla_\zeta \varphi_*)E \\
&= 0. \qquad (2.16)
\end{aligned}
$$

Definition 2.1. ([18]) A $(2n+1)$-dimensional α-LS manifold with constants a, b and vector fields E, F defined on M satisfying the condition

$$S(E,F) = ag_*(E,F) + b\eta_*(E)\eta_*(F),$$

is called an η_*-Einstein manifold.

Definition 2.2. An α-LS manifold M with vector fields E, F defined on M satisfying the condition

$$R(E,F)\cdot S = 0, \qquad (2.17)$$

is called Ricci semisymmetric.

Definition 2.3. ([13]) An α-LS manifold M with vector fields E, F defined on M satisfying the condition

$$\varphi_*^2((\nabla_E Q)(F)) = 0, \qquad (2.18)$$

is called φ_*-Ricci symmetric.

If E and F are orthogonal to ζ, then the manifold is said to be locally φ_*-Ricci symmetric.

Definition 2.4. ([30]) An α-LS manifold M is called φ_*-symmetric if

$$\varphi_*^2((\nabla_H R)(E,F)G) = 0, \qquad (2.19)$$

for all vector fields E, F, G, H on M.

3. Ricci and η_*-Ricci Solitons on α-LS Manifolds

Let $(M, \varphi_*, \zeta, \eta_*, g_*)$ be an α-LS manifold. The (potential) vector field V spans, and is orthogonal to, ζ, so we only consider the case $V = \zeta$. Using equation (3), we obtain

$$(\pounds_\zeta g_*)(E,F) + 2S(E,F) + 2\omega_1 g_*(E,F) + 2\omega_2 \eta_*(E)\eta_*(F) = 0, \qquad (3.1)$$

Because of (2.15), the (3.1) becomes

$$2\alpha g_*(\varphi_* E, F) + 2S(E,F) + 2\omega_1 g_*(E,F) + 2\omega_2 \eta_*(E)\eta_*(F) = 0,$$

for any $E, F \in \chi(M)$, or equivalently:

$$S(E,F) = -\alpha g_*(\varphi_* E, F) - \omega_1 g_*(E,F) - \omega_2 \eta_*(E)\eta_*(F), \tag{3.2}$$

for any $E, F \in \chi(M)$.

The data $(g_*, \zeta, \omega_1, \omega_2)$ which satisfy the equation (3.1) is said to be an η_*-Ricci soliton on M (see [9]), in particular, if $\omega_2 = 0$, (g_*, ζ, ω_1) is a Ricci soliton [20]. Ricci solitons is said to be expanding, shrinking and steady according as ω_1 is positive, negative or zero [10].

From (3.2), we get

$$S(\varphi_* E, \varphi_* F) = -\alpha g_*(\varphi_* E, F) - \omega_1 g_*(\varphi_* E, \varphi_* F). \tag{3.3}$$

Taking $F = \zeta$ in (3.2), we have

$$S(E, \zeta) = (\omega_2 - \omega_1)\eta_*(E). \tag{3.4}$$

From (2.11) and (3.4), we obtain

$$\omega_2 - \omega_1 = 2n\alpha^2. \tag{3.5}$$

4. Ricci Semisymmetric η_*-Ricci Solitons on α-LS Manifolds

In this section, we investigate Ricci semisymmetric α-LS manifolds on η_*-Ricci solitons . According to Definition 2.2, we get

$$R(E,F) \cdot S = 0,$$

which implies that

$$S(R(E,F)G, K) + S(G, R(E,F)K) = 0. \tag{4.1}$$

Taking $E = \zeta$ in (4.1), we obtain

$$S(R(\zeta,F)G, K) + S(G, R(\zeta,F)K) = 0. \tag{4.2}$$

Using (2.8) in (4.2), we infer

$$\begin{aligned} \alpha^2[g_*(F,G)S(\zeta,K) + \eta_*(G)S(F,K) \\ + g_*(F,K)S(G,\zeta) - \eta_*(K)S(G,F)] = 0. \end{aligned} \tag{4.3}$$

Since $\alpha \neq 0$, we obtain

$$g_*(F,G)S(\zeta,K) + \eta_*(G)S(F,K) + g_*(F,K)S(G,\zeta) - \eta_*(K)S(G,F) = 0. \tag{4.4}$$

Using (3.2) and (3.4) in (4.4), we infer

$$\begin{aligned}(\omega_2 - \omega_1)&[g_*(F,G)\eta_*(K) + g_*(F,K)\eta_*(G)]\\ &+ \alpha[g_*(\varphi_* F,K)\eta_*(G) + g_*(\varphi_* F,G)\eta_*(K)]\\ &+ \omega_1[g_*(F,K)\eta_*(G) + g_*(F,G)\eta_*(K)]\\ &+ 2\omega_2\eta_*(F)\eta_*(G)\eta_*(K)\\ &= 0.\end{aligned} \tag{4.5}$$

Putting $K = \zeta$ in (4.5) and using (2.2), we infer

$$\alpha g_*(\varphi_* F,G) + \omega_2[g_*(F,G) + \eta_*(F)\eta_*(G)] = 0, \tag{4.6}$$

which is equivalent to

$$\alpha g_*(\varphi_* F,G) + \omega_2 g_*(\varphi_* F,\varphi_* G) = 0, \tag{4.7}$$

Putting $G = \varphi_* G$, we obtain

$$\alpha g_*(\varphi_* F,\varphi_* G) + \omega_2 g_*(\varphi_* F,G) = 0. \tag{4.8}$$

Subtracting (4.7) and (4.8), we get

$$(\alpha - \omega_2)[g_*(\varphi_* F,G) - g_*(\varphi_* F,\varphi_* G)] = 0. \tag{4.9}$$

for any F, G on M and follows $\omega_2 = \alpha$. From the relation (3.5), we have $\omega_1 = \alpha - 2n\alpha^2$.

Now from above, we are able to state our results.

Theorem 4.1. *If $(M, \varphi_*, \zeta, \eta_*, g_*)$ is a Ricci semisymmetric α-LS manifold, $(g_*, \zeta, \omega_1, \omega_2)$ is an η_*-Ricci soliton on M, then $\omega_2 = \alpha$ and $\omega_1 = \alpha - 2n\alpha^2$.*

In case $\omega_2 = 0$, we derive the following.

Corollary 4.2. *If α-LS manifolds $(M, \varphi_*, \zeta, \eta_*, g_*)$ satisfy the condition $R(\zeta, F)\cdot S = 0$, then there does not exist Ricci solitons with potential vector field ζ.*

From (3.2), (3.5) and (4.7), we obtain

$$S = (\omega_2 - \omega_1)\{g_*(E,F) + \eta_*(E)\eta_*(F)\} = 2n\alpha^2\{g_*(E,F) + \eta_*(E)\eta_*(F)\}. \tag{4.10}$$

As a consequence, we can state following proposition.

Proposition 4.3. *Let $(M, \varphi_*, \zeta, \eta_*, g_*)$ be an α-LS manifold. If M is Ricci semisymmetric and $(g_*, \zeta, \omega_1, \omega_2)$ is an η_*-Ricci soliton on M, then the manifold is an η_*-Einstein manifold.*

5. φ_*-Symmetric η_*-Ricci Solitons on α-LS Manifolds

Consider φ_*-symmetric η_*-Ricci solitons on α-LS manifolds. Then from definition 2.3, we have

$$\varphi_*^2((\nabla_E Q)(F)) = 0. \tag{5.1}$$

Using (2.1) and (5.1), we get

$$(\nabla_E Q)(F) + \eta_*((\nabla_E Q)(F))\zeta = 0. \tag{5.2}$$

Taking inner product in (5.2) with G, we have

$$g_*((\nabla_E Q)(F), G) + \eta_*((\nabla_E Q)(F))\eta_*(G) = 0, \tag{5.3}$$

which implies

$$g_*(\nabla_E QF - Q(\nabla_E F), G) + \eta_*((\nabla_E Q)(F))\eta_*(G) = 0. \tag{5.4}$$

After simplification, we obtain

$$g_*(\nabla_E QF, G) - S(\nabla_E F, G) + \eta_*((\nabla_E Q)(F))\eta_*(G) = 0. \tag{5.5}$$

Putting $F = \zeta$ in (5.5), we have

$$g_*(\nabla_E Q\zeta, G) - S(\nabla_E \zeta, G) + \eta_*((\nabla_E Q)(\zeta))\eta_*(G) = 0. \tag{5.6}$$

Using (2.5) and (3.3) in (5.6), we infer

$$(\omega_2 - \omega_1)\alpha g_*(\varphi_* E, G) - \alpha S(\varphi_* E, G) + \eta_*((\nabla_E Q)(\zeta))\eta_*(G) = 0. \tag{5.7}$$

Taking $G = \varphi_* G$ in (5.7), we get

$$(\omega_2 - \omega_1)\alpha g_*(\varphi_* E, \varphi_* G) - \alpha S(\varphi_* E, \varphi_* G) = 0. \tag{5.8}$$

Since $\alpha \neq 0$, we get

$$(\omega_2 - \omega_1) g_*(\varphi_* E, \varphi_* G) - S(\varphi_* E, \varphi_* G) = 0. \tag{5.9}$$

Now, using (3.3) in (5.9), we infer

$$\omega_2 g_*(\varphi_* E, \varphi_* G) + \alpha g_*(E, \varphi_* G) = 0. \tag{5.10}$$

Taking $E = \varphi_* E$ in (5.10), we have

$$\omega_2 g_*(E, \varphi_* G) + \alpha g_*(\varphi_* E, \varphi_* G) = 0. \tag{5.11}$$

Subtracting (5.10) from (5.11), we obtain

$$(\omega_2 - \alpha)(g_*(\varphi_* E, \varphi_* G) - g_*(E, \varphi_* G)) = 0, \tag{5.12}$$

for any E, G and follows $\omega_2 = \alpha$. From the relation (3.5), we obtain $\omega_1 = \alpha - 2n\alpha^2$.

Now, we are ready to state the following results.

Theorem 5.1. *If $(M, \varphi_*, \zeta, \eta_*, g_*)$ is φ_*-symmetric on α-LS manifolds, $(g_*, \zeta, \omega_1, \omega_2)$ is an η_*-Ricci soliton on M, then $\omega_2 = \alpha$ and $\omega_1 = \alpha - 2n\alpha^2$.*

In case $\omega_2 = 0$, we can state next result.

Corollary 5.2. *If α-LS manifolds $(M, \varphi_*, \zeta, \eta_*, g_*)$ satisfy the condition $\varphi_*^2((\nabla_E Q)(\zeta)) = 0$, then there does not exist Ricci solitons with potential vector field ζ.*

From (3.2), (3.4) and (5.10) we have

$$S = (\omega_2 - \omega_1)\{g_*(E, F) + \eta_*(E)\eta_*(F)\} = 2n\alpha^2\{g_*(E, F) + \eta_*(E)\eta_*(F)\}. \tag{5.13}$$

This leads to the following proposition.

Proposition 5.3. *Let $(M, \varphi_*, \zeta, \eta_*, g_*)$ be an α-LS manifold. If M is a φ_*-symmetric and $(g_*, \zeta, \omega_1, \omega_2)$ is an η_*-Ricci soliton on M, then the manifold is an η_*-Einstein manifold.*

6. φ_*-Ricci Symmetric η_*-Ricci Solitons on α-LS Manifolds

This section is devoted to the study of φ_*-Ricci symmetric η_*-Ricci solitons on α-LS manifolds.

Consider a φ_*-Ricci symmetric η_*-Ricci solitons on α-LS manifolds. Then from definition 2.4, we have

$$\varphi_*^2((\nabla_H R)(E, F)G) = 0. \tag{6.1}$$

Using (2.1), we infer

$$(\nabla_H R)(E, F)G - \eta_*((\nabla_H R)(E, F)G)\zeta = 0, \tag{6.2}$$

Taking inner product with U in (57), we obtain

$$g_*((\nabla_H R)(E, F)G, U) - \eta_*((\nabla_H R)(E, F)G)g_*(\zeta, U) = 0. \tag{6.3}$$

Let $\{\sigma_i\}$, $i = 1, 2, ..., n$, be an orthonormal basis of the tangent space at any point of the manifold. Then by putting $E = U = \sigma_i$ in (53) and taking summation over i , $1 \leq i \leq n$, we get

$$(\nabla_H S)(F, G) + \sum_{i=1}^{n} \eta_*((\nabla_H R)(\sigma_i, F)G)g_*(\zeta, \sigma_i) = 0. \tag{6.4}$$

Putting $G=\zeta$ in (6.4), we get

$$(\nabla_H S)(F,\zeta)+\sum_{i=1}^{n}\eta_*((\nabla_H R)(\sigma_i,F)\zeta)g_*(\zeta,\sigma_i)=0. \tag{6.5}$$

The second term of (6.5), takes the form

$$\begin{aligned}\eta_*((\nabla_H R)(\sigma_i,F)\zeta) \;=\;& g_*(\nabla_H R(\sigma_i,F)\zeta,\zeta)-g_*(R(\nabla_H\sigma_i,F)\zeta,\zeta)\\ &-g_*(R(\sigma_i,\nabla_H F)\zeta,\zeta)-g_*(R(\sigma_i,F)\nabla_H\zeta,\zeta),\end{aligned} \tag{6.6}$$

and we obtain

$$\eta_*((\nabla_H R)(\sigma_i,F)\zeta)=0. \tag{6.7}$$

The equations (6.5) and (6.7) imply that

$$(\nabla_H S)(F,\zeta)=0,$$

which gives

$$\nabla_H(S(F,\zeta))-S(\nabla_H F,\zeta)-S(F,\nabla_H\zeta)=0.$$

In view of (2.5) and (3.4), we have

$$(\omega_2-\omega_1)(\nabla_H\eta_*(F)-\eta_*(\nabla_H F))-\alpha S(F,\varphi_*H)=0. \tag{6.8}$$

Putting $F=\varphi_*F$ in (6.8), we infer

$$\alpha S(\varphi_*F,\varphi_*H)=(\omega_1-\omega_2)g_*((\nabla_H\varphi_*)F,\zeta). \tag{6.9}$$

Using (2.4), (2.7) and (3.3) in (6.9), we get

$$\alpha\omega_2 g_*(\varphi_*F,\varphi_*H)+\alpha^2 g_*(F,\varphi_*H)=0. \tag{6.10}$$

Since $\alpha\neq 0$, we infer

$$\omega_2 g_*(\varphi_*F,\varphi_*H)+\alpha g_*(F,\varphi_*H)=0. \tag{6.11}$$

Putting $F=\varphi_*F$, we have

$$\omega_2 g_*(F,\varphi_*H)+\alpha g_*(\varphi_*F,\varphi_*H)=0. \tag{6.12}$$

Subtracting (6.11) from (6.12), we get

$$(\omega_2-\alpha)(g_*(\varphi_*F,\varphi_*H)-g_*(F,\varphi_*H))=0, \tag{6.13}$$

for any F,H it follows $\omega_2=\alpha$. Using (3.5), we obtain $\omega_1=\alpha-2n\alpha^2$.

Hence, we can state the following results.

Theorem 6.1. *If $(M, \varphi_*, \zeta, \eta_*, g_*)$ is a φ_*-Ricci symmetric on an α-LS manifold and $(g_*, \zeta, \omega_1, \omega_2)$ is an η_*-Ricci soliton, then $\omega_2 = \alpha$ and $\omega_1 = \alpha - 2n\alpha^2$.*

In case $\omega_2 = 0$, we deduce

Corollary 6.2. *If α-LS manifolds $(M, \varphi_*, \zeta, \eta_*, g_*)$ satisfy the condition $\varphi_*^2((\nabla_H R)(E, F)\zeta) = 0$, then there does not exist Ricci solitons with potential vector field ζ.*

Using (3.2), (3.5) and (6.11) we have

$$S = (\omega_2 - \omega_1)\{g_*(E, F) + \eta_*(E)\eta_*(F)\} = 2n\alpha^2\{g_*(E, F) + \eta_*(E)\eta_*(F)\}. \tag{6.14}$$

This leads to the following proposition.

Proposition 6.3. *Let $(M, \varphi_*, \zeta, \eta_*, g_*)$ be an α-LS manifold. If M is a φ_*-Ricci symmetric and $(g_*, \zeta, \omega_1, \omega_2)$ is an η_*-Ricci soliton on M, then the manifold is an η_*-Einstein manifold.*

Example 6.4. Now, we assume the 3-dimensional manifold

$$M = \{(p, q, r) \in R^3 : r \neq 0\}$$

where p, q, r are the standard coordinates in R^3.

The vector fields

$$\sigma_1 = e^r \frac{\partial}{\partial q}, \ \sigma_2 = e^r \left(\frac{\partial}{\partial p} + \frac{\partial}{\partial q} \right), \ \sigma_3 = \alpha \frac{\partial}{\partial r},$$

are linearly independent at each point of M and α is a constant.

Let g_* be the Lorentzian metric defined as

$$\begin{aligned} g_*(\sigma_1, \sigma_3) &= g_*(\sigma_2, \sigma_3) = g_*(\sigma_1, \sigma_2) = 0, \\ g_*(\sigma_1, \sigma_1) &= g_*(\sigma_2, \sigma_2) = 1, g_*(\sigma_3, \sigma_3) = -1. \end{aligned}$$

Let $\sigma_3 = \zeta$. Then Lorentzian metric on M is given below

$$g_* = \frac{1}{(e^r)^2}\{2(dp)^2 + (dq)^2 - 2dpdq\} - \frac{1}{(\alpha)^2}(dr)^2.$$

Let η_* be the 1-form defined as

$$\eta_*(G) = g_*(G, \sigma_3),$$

for any vector field G on M.

Let φ_* be the $(1,1)$-tensor field defined as

$$\varphi_*(\sigma_1) = -\sigma_1, \ \varphi_*(\sigma_2) = -\sigma_2, \ \varphi_*(\sigma_3) = 0.$$

Then, using the linearity of φ_* and g_*, we get

$$\begin{aligned}\eta_*(\sigma_3) &= -1,\ \varphi_*^2 G = G + \eta_*(G)\sigma_3,\\ g_*(\varphi_* G, \varphi_* H) &= g_*(G,H) + \eta_*(G)\eta_*(H),\end{aligned}$$

for any vector field G, H on M.

It is easy to observe

$$\eta_*(\sigma_1) = 0,\ \eta_*(\sigma_2) = 0,\ \eta_*(\sigma_3) = -1.$$

Thus for $\sigma_3 = \zeta$, the structure $(\varphi_*, \zeta, \eta_*, g_*)$ defines a Lorentzian almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g_*. Then we have

$$[\sigma_1, \sigma_2] = 0,\ [\sigma_1, \sigma_3] = -\alpha\sigma_1,\ [\sigma_2, \sigma_3] = -\alpha\sigma_2.$$

Using Koszul's formula

$$\begin{aligned}2g_*(\nabla_E F, G) &= E g_*(F,G) + F g_*(G,E) - G g_*(E,F)\\ &\quad -g_*(E,[F,G]) - g_*(F,[E,G])\\ &\quad +g_*(G,[E,F]),\end{aligned}$$

One can easily obtain

$$\begin{gathered}\nabla_{\sigma_1}\sigma_1 = -\alpha\sigma_3,\quad \nabla_{\sigma_1}\sigma_2 = 0,\quad \nabla_{\sigma_1}\sigma_3 = -\alpha\sigma_1,\\ \nabla_{\sigma_2}\sigma_1 = 0,\quad \nabla_{\sigma_2}\sigma_2 = -\alpha\sigma_3,\quad \nabla_{\sigma_2}\sigma_3 = -\alpha\sigma_2,\\ \nabla_{\sigma_3}\sigma_1 = 0,\quad \nabla_{\sigma_3}\sigma_2 = 0,\quad \nabla_{\sigma_3}\sigma_3 = 0.\end{gathered}$$

Now, we see that the manifold is an α-LS manifold.

Also, the Riemannian curvature tensor R is given by

$$R(E,F)G = \nabla_E\nabla_F G - \nabla_F\nabla_E G - \nabla_{[E,F]}G.$$

Then

$$R(\sigma_1,\sigma_2)\sigma_2 = \alpha^2\sigma_1,\ R(\sigma_1,\sigma_3)\sigma_3 = -\alpha^2\sigma_1,\ R(\sigma_2,\sigma_1)\sigma_1 = \alpha^2\sigma_2,$$

$$R(\sigma_2,\sigma_3)\sigma_3 = -\alpha^2\sigma_2,\ R(\sigma_3,\sigma_1)\sigma_1 = \alpha^2\sigma_3,\ R(\sigma_3,\sigma_2)\sigma_2 = \alpha^2\sigma_3.$$

Then, the Ricci tensor S is given by

$$S(\sigma_1,\sigma_1) = S(\sigma_2,\sigma_2) = 2\alpha^2,\quad S(\sigma_3,\sigma_3) = -2\alpha^2.$$

From (3.2), we obtain $S(\sigma_1,\sigma_1) = S(\sigma_2,\sigma_2) = \alpha - \omega_1$ and $S(\sigma_3,\sigma_3) = \omega_1 - \omega_2$, therefore $\omega_1 = \alpha - 2\alpha^2$ and $\omega_2 = \alpha$. The data $(g_*, \zeta, \omega_1, \omega_2)$ for $\omega_1 = \alpha - 2\alpha^2$ and $\omega_2 = \alpha$ provides an η_*-Ricci soliton on an α-LS manifold.

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