KYUNGPOOK Math. J. 62(2022), 689-697
https://doi.org/10.5666/KMJ.2022.62.4.689
pISSN 1225-6951 eISSN 0454-8124

Dual Integrable Representations of Hypergroups

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Abstract. In this paper, we study dual integrable representations of locally compact commutative hypergroups and give a sufficient and necessary condition for a dual integrable representation to have an admissible vector.

1. Introduction and Notations

Dual integrable representations of abelian locally compact groups were introduced in [6] (see also [1, 7]). Left regular representations are well known examples of such dual integrable representations. In this paper, we initiate the stucy of this concept for locally compact hypergroups and prove its basic properties. We find some some results which are novel even for the group case. In fact, we give an equivalence condition for a dual integrable representation to have an admissible vector, which is a version of our recent results in [9].

Throughout this paper K is a locally compact hypergroup. For the definitions and basic properties of hypergroups we refer to the monographs [8] (in which hypergroups are called "convo") and [2] (see also [5]). The space of all complex Radon measures on K is denoted by $M(K)$, and the space of positive elements of $M(K)$ is denoted by $M^+(K)$. $B(\mathcal{H})$ denotes the space of all bounded operators on a Hilbert space $\mathcal{H}$.

2. Main Results

In this section, first we recall the following definition from [8, 11.3].

* Corresponding Author.
Received April 18, 2021; revised January 18, 2022; accepted January 24, 2022.
2020 Mathematics Subject Classification: Primary 43A62; Secondary 43A65, 42C15.
Key words and phrases: locally compact hypergroup, locally compact group, dual integrable representation, left regular representation, frame.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space. A norm-decreasing $*$-homomorphism $\pi : M(K) \to B(\mathcal{H})$ is called a *representation* of K on $\mathcal{H}$ if $\pi(\delta_e) = \mathrm{Id}_{\mathcal{H}}$ and for each $\varphi, \psi \in \mathcal{H}$, the mapping $\mu \mapsto \langle \pi(\mu)\varphi, \psi\rangle$ from $M^+(K)$ to $\mathbb{C}$ is continuous, where $M^+(K)$ is equipped with the cone topology. For each $x \in K$, we denote $\pi_x := \pi(\delta_x)$. Let K be a commutative hypergroup with a left Haar measure m.

The left regular representation $\tau : M(K) \to B(L^2(K))$ is defined by $\tau(\mu)f := \mu * f$, where $f \in L^2(K)$ and $\mu \in M(K)$. In particular, let H be a compact subgroup of a locally compact group G and $G/\!/H$ be the double coset hypergroup introduced in [8, 8.2] with quotient mapping $q : G \to G/\!/H$ defined by $q(x) := HxH$. Then, $G/\!/H$ has a left Haar measure m [8, 8.2B], and for each $f \in L^2(G/\!/H, m)$ the left regular representation of $G/\!/H$ can be obtained by

$$\begin{aligned}\tau(\mu)f(q(x)) &= \int_{G/\!/H} f(q(y^{-1}) * q(x))d\mu(q(y))\\ &= \int_{G/\!/H}\int_{G/\!/H} f(q(z))d(\delta_{q(y^{-1})} * \delta_{q(x)})(q(z))d\mu(q(y))\\ &= \int_{G/\!/H}\int_{H} f(q(y^{-1}tx))d\sigma(t)d\mu(q(y)),\end{aligned}$$

where σ is the normalized Haar measure on H and $\mu \in M(G/\!/H)$.

Definition 2.2. Let K be a commutative hypergroup with a Haar measure ω on $\hat{K}$ [8, 7.3I]. A representation $\pi : M(K) \to B(\mathcal{H}_\pi)$ is called *dual integrable* if there exists a corresponding mapping $[\cdot,\cdot]_\pi : \mathcal{H}_\pi \times \mathcal{H}_\pi \to L^1(\hat{K}, \omega)$, called *bracket product*, such that for all $\varphi, \psi \in \mathcal{H}_\pi$ and $\mu \in M(K)$,

$$\langle \varphi, \pi(\mu)\psi\rangle = \int_{\hat{K}} [\varphi, \psi]_\pi(\xi)\overline{\hat{\mu}(\xi)}d\omega(\xi),$$

where $\hat{\mu}$ is the Fourier-Stieltjes transform of μ.

Remark 2.3. The bracket product of a dual integrable representation is conjugate symmetry and linear in the first argument. Also, for each $x \in K$, putting $\mu = \delta_x$, we have

$$\langle \varphi, \pi_x\psi\rangle = \int_{\hat{K}} [\varphi, \psi]_\pi(\xi)\xi(x)d\omega(\xi) = \widehat{[\varphi, \psi]}_\pi(x^-). \tag{2.1}$$

In particular,

$$\langle \varphi, \psi\rangle = \int_{\hat{K}} [\varphi, \psi]_\pi(\xi)d\omega(\xi). \tag{2.2}$$

For each $\varphi \in \mathcal{H}_\pi$, $[\varphi, \varphi] = 0$ a.e. if and only if $\varphi = 0$. In fact, if $\varphi = 0$, then by (2.1) for each $x \in K$ we have

$$0 = \langle \varphi, \pi_{x^-}\varphi\rangle = \widehat{[\varphi, \varphi]}_\pi(x),$$

and so, by [2, 2.1.6(vi)], $[\varphi, \varphi] = 0$ a.e. The converse follows from (2.2).

Example 2.4. The left regular representation τ of a commutative hypergroup K is dual integrable. In fact, for each $\varphi, \psi \in L^2(K)$ and $\mu \in M(K)$ by [2, 2.2.26 and 2.2.34],

$$\begin{aligned}\langle \varphi, \tau(\mu)\psi \rangle_{L^2(K)} &= \int_K \varphi(x)\overline{(\mu * \psi)(x)}dm(x)\\ &= \int_{\hat{K}} \hat{\varphi}(\xi)\overline{\widehat{(\mu * \psi)}(\xi)}d\omega(\xi)\\ &= \int_{\hat{K}} \hat{\varphi}(\xi)\overline{\hat{\mu}(\xi)}\overline{\hat{\psi}(\xi)}d\omega(\xi)\\ &= \int_{\hat{K}} [\varphi, \psi]_\tau \overline{\hat{\mu}(\xi)}d\omega(\xi),\end{aligned}$$

with $[\varphi, \psi]_\tau(\xi) := \hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)}$.

Remark 2.5. Same as the group case, for each representations $(\pi, \mathcal{H}_\pi)$ and $(\pi', \mathcal{H}_{\pi'})$ of a hypergroup K, if π and π' are equivalence via a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ and π is dual integrable, then π' is also dual integrable. In fact,

$$\begin{aligned}\langle \varphi', \pi'(\mu)\psi' \rangle &= \langle \varphi', U\pi(\mu)U^*\psi' \rangle\\ &= \langle U^*\varphi', \pi(\mu)U^*\psi' \rangle\\ &= \int_{\hat{K}} [U^*\varphi', U^*\psi']_\pi(\xi)\overline{\hat{\mu}(\xi)}d\omega(\xi),\end{aligned}$$

and so, $[\varphi', \psi']_{\pi'} = [U^*\varphi', U^*\psi']_\pi$ for all $\varphi', \psi' \in \mathcal{H}'_\pi$, where U^* is adjoint of the operator U.

Proposition 2.6. *Let G_1 and G_2 be abelian locally compact groups and π and σ be dual representations of G_1 and G_2 respectively. Then, $\pi \otimes \sigma$ is a dual integrable representation of $G_1 \times G_2$.*

Proof. For each $\varphi_1, \varphi_2 \in \mathcal{H}_\pi$ and $\psi_1, \psi_2 \in \mathcal{H}_\sigma$ we define

$$[\varphi_1 \otimes \psi_1, \varphi_2 \otimes \psi_2]_{\pi\otimes\sigma}(\xi, \eta) := [\varphi_1, \varphi_2]_\pi(\xi)\,[\psi_1, \psi_2]_\sigma(\eta), \qquad (\xi \in \widehat{G_1},\, \eta \in \widehat{G_2}).$$

Then, by [4, Proposition 4.6] and [4, Section 7.3], for each $x \in G_1$ and $y \in G_2$

we have

$$\begin{aligned}\langle\varphi_1\otimes\psi_1,(\pi\otimes\sigma)_{(x,y)}(\varphi_2\otimes\psi_2)\rangle &= \langle\varphi_1\otimes\psi_1,\pi_x\varphi_2\otimes\sigma_y\psi_2\rangle\\ &= \langle\varphi_1,\pi_x\varphi_2\rangle\,\langle\psi_1,\sigma_y\psi_2\rangle\\ &= \int_{\widehat{G_1}}[\varphi_1,\varphi_2]_\pi\,(\xi)\overline{\xi(x)}d\mu(\xi)\int_{\widehat{G_2}}[\psi_1,\psi_2]_\sigma\,(\eta)\overline{\eta(y)}d\nu(\eta)\\ &= \int_{\widehat{G_1}}\int_{\widehat{G_2}}[\varphi_1,\varphi_2]_\pi\,(\xi)\,[\psi_1,\psi_2]_\sigma\,\overline{\eta(y)}d\mu(\xi)d\nu(\eta)\\ &= \int_{\widehat{G_1\times G_2}}[\varphi_1\otimes\psi_1,\varphi_2\otimes\psi_2]_{\pi\otimes\sigma}\,(\xi,\eta)\overline{(\xi,\eta)(x,y)}d(\mu\times\nu)(\xi,\eta),\end{aligned}$$

and the proof is complete. □

Corollary 2.7. *Let π be a dual integrable representation of a locally compact abelian group G_1 and τ be the left regular representation of a locally compact abelian group G_2. Then, the induced representation $\mathrm{ind}_{G_1}^{G_1\times G_2}$ is dual integrable.*

Proof. The proof is obtained from Proposition , Example 2.3 and [4, Proposition 7.26]. □

In spite of the group case, for a commutative hypergroup K we only have

$$\Delta(L^1(K))=\mathfrak{X}_b(K):=\{\xi\in C_b(K):\xi(x*y)=\xi(x)\xi(y)\text{ for all }x,y\in K\},$$

while in general $\hat{K}\subseteq\mathfrak{X}_b(K)$.

There are hypergroups with $\hat{K}\neq\mathfrak{X}_b(K)$ (see [8, Example 9.5]). Although, for a large class of hypergroups, like orbit hypergroups, we have $\hat{K}=\mathfrak{X}_b(K)$ (for more details see [3]). The following theorem holds for hypergroups with $\hat{K}=\mathfrak{X}_b(K)$.

Theorem 2.8. *Let K be a commutative hypergroup with $\mathfrak{X}_b(K)=\hat{K}$ and π be a representation of K on $\mathcal{H}_\pi$. Then, there is a projection-valued Borel measure T on $\hat{K}$ such that*

$$\pi(\mu):=\int_{\hat{K}}\overline{\hat{\mu}(\xi)}dT(\xi),\qquad(\mu\in M(K)).$$

Also, for each $\varphi,\psi\in\mathcal{H}_\pi$, there is a complex Borel measure $\mu_{\varphi,\psi}$ on $\hat{K}$ such that

$$\mu_{\varphi,\psi}(E)=\langle T(E)\varphi,\psi\rangle=\langle\varphi,T(E)\psi\rangle=\langle T(E)\varphi,T(E)\psi\rangle$$

for all Borel set $E\subseteq\hat{K}$, and

$$\langle\varphi,\pi(\mu)\psi\rangle=\int_{\hat{K}}\overline{\hat{\mu}(\xi)}d\mu_{\varphi,\psi}(\xi)\qquad(\mu\in M(K)).$$

Proof. By [8, 6.1G], $M(K)$ is a commutative Banach $*-$algebra. Therefore, by [4, 1.54], there exists a unique regular projection-valued measure T on $\hat{K}$ such that

$$\pi(\mu) := \int_{\hat{K}} \overline{\hat{\mu}(\xi)} dT(\xi)$$

for each $\mu \in M(K)$. Then, for each $\varphi, \psi \in \mathcal{H}_\pi$,

$$\langle \varphi, \pi(\mu)\psi \rangle = \int_{\hat{K}} \overline{\hat{\mu}(\xi)} d \langle \varphi, T(\xi)\psi \rangle .$$

The mapping $\mu_{\varphi,\psi}$ defined by

$$\mu_{\varphi,\psi}(E) := \langle \varphi, T(E)\psi \rangle$$

for all Borel set $E \subseteq \hat{K}$, is a meaure on $\hat{K}$ and

$$\langle \varphi, \pi(\mu)\psi \rangle = \int_{\hat{K}} \overline{\hat{\mu}(\xi)} d\mu_{\varphi,\psi}(\xi) \qquad (\mu \in M(K)).$$

□

As the group case, by [2, 2.1.6(vi)], we can conclude:

Corollary 2.9. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of a commutative hypergroup K and ω be the Haar measure on $\hat{K}$. Then, the following statements are equivalent:*

(i) π *is dual integrable.*

(ii) *For each $\varphi, \psi \in \mathcal{H}_\pi$, the measure $\mu_{\varphi,\psi}$ is absolutely continuous with respect to ω.*

Corollary 2.10. *Let $(\pi, \mathcal{H}_\pi)$ be a dual integrable representation of a commutative hypergroup K. Then, for all $\varphi, \psi \in \mathcal{H}_\pi$:*

(i) $[\varphi, \varphi]_\pi \geq 0$ *a.e.*

(ii) $|[\varphi, \psi]_\pi| \leq [\varphi, \varphi]_\pi^{1/2} . [\psi, \psi]_\pi^{1/2}$ *a.e.*

(iii) $\| [\varphi, \psi]_\pi \|_1 \leq \|\varphi\| \, \|\psi\| .$

Corollary 2.11. *Let $(\pi, \mathcal{H}_\pi)$ be a dual integrable representation of K and $\varphi \in \mathcal{H}_\pi$. Put $A_\varphi := \{\xi \in \hat{K}; [\varphi, \varphi]_\pi (\xi) > 0\}$. Then, for all $\psi \in \mathcal{H}_\pi$, $[\varphi, \psi]_\pi = [\varphi, \psi]_\pi \chi_{A_\varphi}$ in $L^1(\hat{K})$.*

Proof. By the Cauchy-Schwartz inequality,

$$0 \leq \int_{\hat{K}-A_\varphi} |[\varphi, \psi]_\pi (\xi)| \, d\omega(\xi) \leq \int_{\hat{K}-A_\varphi} ([\varphi, \varphi]_\pi (\xi))^{1/2} ([\psi, \psi]_\pi (\xi))^{1/2} d\omega(\xi) = 0,$$

and the proof is complete. □

Corollary 2.12. *Let $(\pi, \mathcal{H}_\pi)$ be a dual integrable representation of a commutative hypergroup K. Then, for each $\mu \in M(K)$ and $\varphi, \psi \in \mathcal{H}_\pi$, we have*

$$[\pi(\mu)\varphi, \psi]_\pi = \left[\varphi, \pi(\mu^-)\psi\right]_\pi = \hat{\mu}\left([\varphi, \psi]_\pi\right).$$

Proof. By dual integrability of π, [2, 2.1.6(iv)] and [8, 12.1E], for all $\mu, \nu \in M(K)$ and $\varphi, \psi \in \mathcal{H}_\pi$ we have

$$\begin{aligned}\langle \pi(\mu)\varphi, \pi(\nu)\psi\rangle &= \langle \varphi, \pi(\mu)^*\pi(\nu)\psi\rangle \\ &= \langle \varphi, \pi(\mu^* * \nu)\psi\rangle \\ &= \int_{\hat{K}} [\varphi, \psi]_\pi (\xi)\overline{\widehat{(\mu^* * \nu)}}(\xi) d\omega(\xi) \\ &= \int_{\hat{K}} [\varphi, \psi]_\pi (\xi)\hat{\mu}(\xi)\overline{\hat{\nu}}(\xi) d\omega(\xi).\end{aligned}$$

On the other hand,

$$\langle \pi(\mu)\varphi, \pi(\nu)\psi\rangle = \int_{\hat{K}} [\pi(\mu)\varphi, \psi]_\pi (\xi)\overline{\hat{\nu}}(\xi) d\omega(\xi).$$

So,

$$\int_{\hat{K}} [\varphi, \psi]_\pi (\xi)\hat{\mu}(\xi)\overline{\hat{\nu}}(\xi) d\omega(\xi) = \int_{\hat{K}} [\pi(\mu)\varphi, \psi]_\pi (\xi)\overline{\hat{\nu}}(\xi) d\omega(\xi).$$

Finally, by uniqueness in Rieze Representation Theorem and [8, 7.3H],

$$[\pi(\mu)\varphi, \psi]_\pi = \hat{\mu}\left([\varphi, \psi]_\pi\right).$$

Similarly,

$$\left[\varphi, \pi(\mu^-)\psi\right]_\pi = \hat{\mu}\left([\varphi, \psi]_\pi\right).$$

□

Definition 2.13. Let π be a dual integrable representation of a commutative hypergroup K on $\mathcal{H}_\pi$. For each $\varphi, \psi \in \mathcal{H}_\pi$, we say φ and ψ are π-orthogonal, and write $\varphi \perp_\pi \psi$, if $[\varphi, \psi]_\pi = 0$ a.e. on $\hat{K}$. A subset $E \subseteq \mathcal{H}_\pi$ is called π-orthogonal if for each distinct $\varphi, \psi \in E$, $\varphi \perp_\pi \psi$.

Corollary 2.14. *Let $(\pi, \mathcal{H}_\pi)$ be a dual integrable representation of a Pontrjagin hypergroup K. Then, $\varphi, \psi \in \mathcal{H}_\pi$ are π-orthogonal if and only if $\varphi \perp \overline{\pi(M(K))\psi}$ in $\mathcal{H}_\pi$.*

Proof. Since π is dual integrable, $\varphi \perp \overline{\pi(M(K)\psi}$ if and only if

$$\int_{\hat{K}} [\varphi, \psi]_\pi (\xi)\overline{\hat{\mu}(\xi)} d\omega(\xi) = 0,$$

for all $\mu \in M(K)$. Thus, for each $x \in K$, setting $\mu = \delta_x$, we have

$$\widehat{[\varphi,\psi]_\pi}(x) = \int_{\hat{K}} [\varphi,\psi]_\pi(\xi)\overline{\xi(x)}d\omega(\xi) = 0. \tag{2.3}$$

This implies that $[\varphi,\psi]_\pi = 0$ a.e. in $\hat{K}$. Clearly, by (2.3), if $[\varphi,\psi]_\pi = 0$ a.e. in $\hat{K}$, then $\varphi \perp \overline{\pi(M(K)\psi}$. □

Proposition 2.15. Let π be a dual integrable representation of a commutative hypergroup K on $\mathcal{H}_\pi$. The subset $E \subseteq \mathcal{H}_\pi$ is π-orthogonal if and only if the set $\{\pi(\mu)\varphi : \varphi \in E, \mu \in M(K)\}$ is orthogonal.

Proof. For each $\varphi, \psi \in E$ and $\mu, \nu \in M(K)$, by Corollary we have

$$\langle \pi(\mu)\varphi, \pi(\nu)\psi\rangle = \int_{\hat{K}} [\pi(\mu)\varphi,\psi]_\pi(\xi)\overline{\hat{\nu}(\xi)}d\omega(\xi) \tag{4}$$

$$= \int_{\hat{K}} [\varphi,\psi]_\pi(\xi)\hat{\mu}(\xi)\overline{\hat{\nu}(\xi)}d\omega(\xi). \tag{5}$$

Hence, if E is π-orthogonal, then $\langle \pi(\mu)\varphi, \pi(\nu)\psi\rangle = 0$ for all distinct $\varphi, \psi \in \mathcal{H}_\pi$ and $\mu, \nu \in M(K)$.

Conversely, if $\{\pi(\mu)\varphi : \varphi \in E, \mu \in M(K)\}$ is orthogonal, then putting $\nu = \delta_e$ and $\mu = \delta_x$ $(x \in K)$ in (5), we have

$$\int_{\hat{K}} [\varphi,\psi]_\pi(\xi)\overline{\xi(x)}d\omega(\xi) = 0$$

for all distinct φ and $\psi \in \mathcal{H}_\pi$, and so $\widehat{[\varphi,\psi]_\pi} = 0$ a.e. Finally, by [2, 2.1.6(vi)] we have $[\varphi,\psi]_\pi = 0$ in $L^1(\hat{K})$, and the proof is complete. □

Corollary 2.16. *Let $(\pi, \mathcal{H}_\pi)$ be a dual integrable representation of a commutative hypergroup K and $\varphi, \psi \in \mathcal{H}_\pi$. The following statements are equivalent:*

(i) $\varphi \perp_\pi \psi$

(ii) $\varphi \perp \overline{\pi(M(K))\psi}$

(iii) $\overline{\pi(M(K))\varphi} \perp \overline{\pi(M(K))\psi}$.

Proposition 2.17. *Let $(\pi, \mathcal{H}_\pi)$ be a dual integrable representation of a strong commutative compact hypergroup K and $\varphi \in \mathcal{H}_\pi$. If $[\varphi,\varphi]_\pi = 1$ in $L^1(\hat{K})$, then the set $\{\pi_x\varphi : x \in K\}$ is an orthogonal system in $\mathcal{H}_\pi$.*

Proof. Since π is dual integrable, for each $x, y \in K$ we have

$$\langle \pi_x\varphi, \pi_y\varphi\rangle = \int_{\hat{K}} [\varphi,\varphi]_\pi(\xi)\xi(x * y^-)d\omega(\xi).$$

Therefore, if $[\varphi,\varphi]_\pi = 1$ in $L^1(\hat{K})$, then by [2, proof of Theorem 2.2.9(ii)] and [2, Theorem 2.4.3], $\langle \pi_x\varphi, \pi_y\varphi\rangle = 0$. □

We recall the next definition from [9].

Definition 2.18. Let K be a hypergroup with a (left) Haar measure m, H be a subhypergroup of K, and $\pi : M(K) \to B(\mathcal{H}_\pi)$ be a representation of K on a Hilbert space $\mathcal{H}_\pi$, and $V \subseteq \mathcal{H}_\pi$. A vector $h_0 \in \mathcal{H}_\pi$ is called a (π, V)*–admissible vector* with respect to H if there are constant numbers $A, B > 0$ such that for every $h \in V$,

$$A\|h\|^2 \leq \int_H |\langle \pi_x(h_0), h\rangle|^2 dm_H(x) \leq B\|h\|^2,$$

where m_H is a left Haar measure on H.

If $A = B = 1$, h_0 is called *Parseval* admissible.

Proposition 2.19. *Let K be a Pontrjagin hypergroup with a left Haar measure m, $\pi : M(K) \to B(\mathcal{H}_\pi)$ be a dual integrable representation of K corresponding to a Plancherel measure ω, and $V \subseteq \mathcal{H}_\pi$. A vector $h_0 \in \mathcal{H}_\pi$ is a Parseval (π, V)-admissible vector if and only if for every $h \in V$, $\|h\| = \| [h_0, h] \|_2$.*

Proof. Let $h_0 \in \mathcal{H}_\pi$. Since π is dual integrable, for every $h \in V$ we have

$$\langle \pi_x(h_0), h\rangle = \int_{\hat{K}} [h_0, h]_\pi (\xi)\overline{\hat{\delta}_{x^-}(\xi)} d\omega(\xi) = \int_{\hat{K}} [h_0, h]_\pi \overline{\xi(x)} d\omega(\xi) = \widehat{[h_0, h]}_\pi(x).$$

So by Plancherel Theorem, h_0 is a Parseval (π, V)-admissible vector if and only if for each $h \in V$,

$$\|h\|^2 = \int_K |\widehat{[h_0, h]}_\pi(x)|^2 dm_K(x) = \int_{\hat{K}} |[h_0, h]_\pi(\xi)|^2 d\omega(\xi) = \| [h_0, h] \|^2.$$

□

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