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Time-dependent Double Obstacle Problem Arising from European Option Pricing with Transaction Costs

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Abstract. In this paper, we investigate a time-dependent double obstacle problem associated with the model of European call option pricing with transaction costs. We prove the existence and uniqueness of a $W^{2,1}_{p,\mathrm{loc}}$ solution to the problem. We then characterize the behavior of the free boundaries in terms of continuity and values of limit points.

1. Introduction

This paper concerns a double obstacle problem arising from the model of European call option pricing with transaction costs. Since transaction costs have been generally reduced and will be reduced in many countries (see [8, page 187] and [10, page 535]), we substitute the time-dependent transaction costs $\lambda(t)$ and $\mu(t)$ for the usual λ and μ which are a constant fraction of the purchase price of the stock. To be specific, we consider the case that $\lambda(t)$ and $\mu(t)$ diminish over time and analyze the value $Q(y,S,t)$ satisfying

$$(1.1)\quad \begin{cases} \min\left\{\partial_y Q + \gamma(1+\bar{\lambda}(t))SQe^{r(T-t)}, -\left(\partial_y Q + \gamma(1-\bar{\mu}(t))SQe^{r(T-t)}\right),\right. \\ \qquad \left.\partial_t Q + \frac{\sigma^2}{2}S^2\partial_{SS}Q + \alpha S\partial_S Q\right\} = 0, \quad y\in\mathbb{R}, \quad S>0, \quad 0\le t<T, \\ Q(y,S,T) = \exp\{-\gamma c(y,S)\}, \end{cases}$$

where

$$c(y,S) = \begin{cases} (1+\lambda_0)yS, & \text{if } y<0, \\ (1-\mu_0)yS, & \text{if } y\ge 0, \end{cases}$$

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and $\sigma > 0, \alpha > r \geq 0, \gamma > 0$ are constants. Also, we define

$$\bar{\lambda}(t) := (\lambda_0 + 1)e^{q(T-t)} - 1, \qquad \bar{\mu}(t) := (\mu_0 + 1)e^{\bar{q}(T-t)} - 1,$$

where $0 \leq \bar{\lambda}(t), \bar{\mu}(t) < 1$ for all $0 \leq t \leq T$, $\lambda_0 := \bar{\lambda}(T)$, $\mu_0 := \bar{\mu}(T)$, and the range of q and $\bar{q}$ is to be chosen later. For the double obstacle problem arising from European option pricing with constant transaction costs, we refer the reader to [4].

Since (1.1) is a backward parabolic problem, we transform it to a forward parabolic problem. Letting $\tau = T - t$, we have

$$\text{(1.2)} \quad \begin{cases} \max\left\{ -\left(\partial_y Q + \gamma(1+\lambda(\tau))SQe^{r\tau}\right), \partial_y Q + \gamma(1-\mu(\tau))SQe^{r\tau}\right., \\ \qquad \left.\partial_\tau Q - \frac{\sigma^2}{2}S^2\partial_{SS}Q - \alpha S\partial_S Q\right\} = 0, \quad y \in \mathbb{R}, \quad S > 0, \quad 0 < \tau \leqslant T, \\ Q(y,S,0) = \exp\{-\gamma c(y,S)\}, \end{cases}$$

where $\lambda(\tau) = (\lambda_0 + 1)e^{q\tau} - 1, \quad \mu(\tau) = (\mu_0 + 1)e^{\bar{q}\tau} - 1.$

Using the transformation described in Section 2, we obtain the time-dependent double obstacle problem:

$$\text{(1.3)} \quad \begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, & \text{if } 1 - \mu(\tau) < V < 1 + \lambda(\tau); \\ \partial_\tau V - \mathcal{L}_z V \leq 0, & \text{if } \quad V = 1 + \lambda(\tau); \\ \partial_\tau V - \mathcal{L}_z V \geq 0, & \text{if } \quad V = 1 - \mu(\tau); \\ V(z,0) = \begin{cases} 1 + \lambda_0, & \text{if } z < 0, \\ 1 - \mu_0, & \text{if } z \geq 0, \end{cases} \end{cases}$$

where

$$\text{(1.4)} \quad \begin{aligned} \mathcal{L}_z V &:= -\frac{1}{\gamma}\frac{\partial}{\partial z}(LV^*) \\ &= \frac{\sigma^2}{2}z^2\partial_{zz}V + \left(\sigma^2 + \alpha - r\right)z\partial_z V + (\alpha - r)V - \gamma\sigma^2 zV\left(z\partial_z V + V\right). \end{aligned}$$

There are various studies on the double obstacle problems. Yang and Yi [13] studied the double obstacle problem associated with European option pricing with transaction costs. Dai and Yi [3] studied the free boundaries of the parabolic double obstacle problem arising from the optimal investment problem of a Constant Relative Risk Aversion (CRRA) investor who faces proportional transaction costs. Furthermore, Chen et al [2] analyzed the double obstacle problem related to a time-dependent Hamilton-Jacobi-Bellman equation with gradient constraints. However, few attempts have been made to analyze the time-dependent parabolic double obstacle problem employing the variational inequality approach. Besides the papers mentioned above, there is a vast literature related to the problem with transaction costs, see for instance, [1, 6] and the references therein.

The aim of the paper is to characterize the free boundaries of problem (1.3). Indeed, we obtain the existence and uniqueness of a $W^{2,1}_{p,\mathrm{loc}}$ solution for (1.3) using a penalty method. We also show the limits of the free boundaries and the continuity of the free boundaries, which is motivated by [12] and [13]. The main difficulty in carrying out this construction is that free boundaries are not always monotonic. To overcome this problem, we employ a transformation $y = x + k(\tau)$, $\bar{v}(y,\tau) = v(x,\tau)$, where $k(\tau)$ is chosen later. This guarantees that the free boundaries are monotonic. Hence, we can see the continuity of the free boundaries.

The present paper is organized as follows. In Section 2, we prove the existence and uniqueness of a $W^{2,1}_{p,\mathrm{loc}}$ solution for (1.3). In Section 3, we analyze the behavior of two free boundaries. In Section 4, we establish the equivalence between the double obstacle problem (1.3) and the original problem (1.2).

2. The Existence and Uniqueness of Solution to the Problem (1.3)

In this section, we show the existence and uniqueness of a $W^{2,1}_{p,\mathrm{loc}}$ solution to the problem (1.3). First, we prove that the problem (1.2) implies the problem (1.3). Since a positive value Q can be inferred from the reality of the setup, or from results in the paper, we let $U = \ln Q$. Then U satisfies the following equations:

$$(2.1)\quad \begin{cases} \max\Big\{-\left(\partial_y U + \gamma(1+\lambda(\tau))Se^{r\tau}\right), \quad \partial_y U + \gamma(1-\mu(\tau))Se^{r\tau}\ , \\ \qquad\qquad \partial_\tau U - \frac{\sigma^2}{2}S^2\partial_{SS}U - \alpha S\partial_S U - \frac{\sigma^2}{2}\left(S\partial_S U\right)^2\Big\} = 0, \\ \qquad\qquad\qquad\qquad y\in\mathbb{R}, \quad S>0, \quad 0<\tau\leqslant T, \\ U(y,S,0) = -\gamma c(y,S). \end{cases}$$

Also, letting $z = e^{r\tau}yS$ and $V^*(z,\tau) = U(y,S,\tau)$ shows the equalities:

$$(2.2)\qquad \partial_y U = e^{r\tau}S\partial_z V^*, \qquad \partial_S U = e^{r\tau}y\partial_z V^*,$$

$$(2.3)\qquad \partial_\tau U = \partial_\tau V^* + rz\partial_z V^*, \qquad \partial_{SS}U = e^{2r\tau}y^2\partial_{zz}V^*.$$

Using equalities (2.2) and (2.3), we have

$$\begin{aligned} &\partial_y U(y,S,\tau) + \gamma(1+\lambda(\tau))Se^{r\tau} = e^{r\tau}S\left[\partial_z V^*(z,\tau) + \gamma(1+\lambda(\tau))\right], \\ &\partial_y U(y,S,\tau) + \gamma(1-\mu(\tau))Se^{r\tau} = e^{r\tau}S\left[\partial_z V^*(z,\tau) + \gamma(1-\mu(\tau))\right], \\ &\partial_\tau U - \frac{\sigma^2}{2}S^2\partial_{SS}U - \alpha S\partial_S U - \frac{\sigma^2}{2}\left(S\partial_S U\right)^2 = \partial_\tau V^* - LV^*, \end{aligned}$$

where

$$(2.4)\qquad LV^* = \frac{\sigma^2}{2}z^2\partial_{zz}V^* + (\alpha-r)z\partial_z V^* + \frac{\sigma^2}{2}(z\partial_z V^*)^2.$$

Therefore, $V^* = V^*(z,\tau)$ satisfies

$$(2.5)\quad \begin{cases} \max\left\{-\left(\partial_z V^* + \gamma(1+\lambda(\tau))\right), \partial_z V^* + \gamma(1-\mu(\tau)), \partial_\tau V^* - LV^*\right\} = 0, \\ \qquad\qquad\qquad\qquad z\in\mathbb{R}, \quad 0<\tau\leqslant T, \\ V^*(z,0) = \begin{cases} -\gamma(1+\lambda_0)z, & z<0, \\ -\gamma(1-\mu_0)z, & z\geq 0. \end{cases} \end{cases}$$

Differentiating with respect to z in (2.4), we get

$$\begin{aligned}\frac{\partial}{\partial z}\left(LV^*\right) =& \frac{\sigma^2}{2} z^2 \partial_{zz}\left(\partial_z V^*\right) + \left(\alpha - r + \sigma^2\right) z \partial_z \left(\partial_z V^*\right) + (\alpha - r)\left(\partial_z V^*\right) \\ &+ \sigma^2 z \left(\partial_z V^*\right)\left(z \partial_z \left(\partial_z V^*\right) + \left(\partial_z V^*\right)\right).\end{aligned}$$

If we denote

$$V(z,\tau) = -\frac{1}{\gamma}\partial_z V^*(z,\tau), \tag{2.6}$$

then we have

$$\begin{aligned}\mathcal{L}_z V :=& -\frac{1}{\gamma}\frac{\partial}{\partial z}(LV^*) \\ =& \frac{\sigma^2}{2} z^2 \partial_{zz} V + \left(\sigma^2 + \alpha - r\right) z \partial_z V + (\alpha - r)V - \gamma\sigma^2 z V\left(z\partial_z V + V\right),\end{aligned}$$

which is equivalent to (1.4). The proof is completed by showing that (2.5) implies (1.3); we do this in Section 4.

Now, note that the operator $\mathcal{L}_z$ is degenerate at $z = 0$. Using the Fichera Theorem from [9], the problem (1.3) can be divided into two parts: the problems in the domains $\{z < 0\}$ and $\{z > 0\}$ independently. Furthermore, we see that $V(z,\tau) = 1 + \lambda(\tau)$ is the solution of problem (1.3) in the domain $\{z < 0\}$. Indeed,

$$\begin{aligned}(\partial_\tau - \mathcal{L}_z)[1+\lambda(\tau)] &= \lambda'(\tau) - (\alpha - r)(1+\lambda(\tau)) + \gamma\sigma^2 z(1+\lambda(\tau))^2 \\ &= (q - (\alpha - r))(1+\lambda(\tau)) + \gamma\sigma^2 z(1+\lambda(\tau))^2 \le 0,\end{aligned}$$

provided that $0 \le q \le \alpha - r$. From now on, we only consider the problem (1.3) in the domain $\{z > 0\}$. Let $z = e^x$ and $v(x,\tau) = V(z,\tau)$. Then $v(x,\tau)$ satisfies

$$\begin{cases}\partial_\tau v - \mathcal{L}_x v = 0, & \text{if } 1-\mu(\tau) < v < 1+\lambda(\tau), \quad x \in \mathbb{R}, \quad 0 < \tau \le T, \\ \partial_\tau v - \mathcal{L}_x v \ge 0, & \text{if } \quad v = 1-\mu(\tau), \quad x \in \mathbb{R}, \quad 0 < \tau \le T, \\ \partial_\tau v - \mathcal{L}_x v \le 0, & \text{if } \quad v = 1+\lambda(\tau), \quad x \in \mathbb{R}, \quad 0 < \tau \le T, \\ v(x,0) = 1-\mu_0, \quad x \in \mathbb{R}, \end{cases} \tag{2.7}$$

where

$$\mathcal{L}_x v = \frac{\sigma^2}{2}\partial_{xx} v + \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x v + (\alpha - r)v - \gamma\sigma^2 e^x v\left(\partial_x v + v\right). \tag{2.8}$$

Since the domain is unbounded, we confine our attention to (1.3) in a bounded domain $(-n,n)\times(0,T)$. In order to do so, set $\Omega_T = \mathbb{R}\times(0,T]$ and $\Omega_T^n = (-n,n)\times(0,T]$. Let us consider the following problem in Ω_T^n:

$$\begin{cases}\partial_\tau v_n - \mathcal{L}_x v_n = 0, & \text{if } 1-\mu(\tau) < v_n < 1+\lambda(\tau) \text{ and } (x,\tau) \in \Omega_T^n; \\ \partial_\tau v_n - \mathcal{L}_x v_n \ge 0, & \text{if } v_n = 1-\mu(\tau) \text{ and } (x,\tau) \in \Omega_T^n; \\ \partial_\tau v_n - \mathcal{L}_x v_n \le 0, & \text{if } v_n = 1+\lambda(\tau) \text{ and } (x,\tau) \in \Omega_T^n; \\ \partial_x v_n(x,\tau) = 0, & \quad x = \pm n, \quad 0 \le \tau \le T; \\ v_n(x,0) = 1-\mu_0, & -n \le x \le n. \end{cases} \tag{2.9}$$

Lemma 2.1. *For any fixed $n \in \mathbb{N}$, there exists a unique solution $v_n \in C(\bar{\Omega}_T^n) \cap W_p^{2,1}(\Omega_T^n)$ to the problem (2.9), where $1 < p < +\infty$. Moreover,*

$$-v_n \leq \partial_x v_n \leq 0 \qquad \text{and} \qquad \partial_\tau v_n \geq -\mu'(\tau) \ \text{a.e. in } \Omega_T^n. \tag{2.10}$$

Proof. We consider a penalty approximation of the problem (2.9):

$$\begin{cases} \partial_\tau v_{\varepsilon,n} - \mathcal{L}_x v_{\varepsilon,n} + \beta_\varepsilon\left(v_{\varepsilon,n} - (1-\mu(\tau))\right) \\ \qquad\qquad -\beta_\varepsilon\left(-v_{\varepsilon,n} + (1+\lambda(\tau))\right) = 0 \quad \text{in } \Omega_T^n, \\ \partial_x v_{\varepsilon,n}(x,\tau) = 0, \quad x = \pm n, \quad 0 \leq \tau \leq T, \\ v_{\varepsilon,n}(x,0) = 1-\mu_0, \quad -n \leq x \leq n, \end{cases} \tag{2.11}$$

where

$$\begin{aligned} &\beta_\varepsilon(t) \in C^2(-\infty,+\infty); \\ &\beta_\varepsilon(t) \leq 0, \quad \beta_\varepsilon'(t) \geq 0, \quad \beta_\varepsilon''(t) \leq 0, \quad \forall t \in \mathbb{R}; \\ &\beta_\varepsilon(0) = -C_0, \quad C_0 \geq \max\{\gamma\sigma^2 e^n(1-\mu_0)^2 + \lambda'(0), \ (\alpha - r)(1+\lambda(T))\}, \end{aligned} \tag{2.12}$$

and moreover,

$$\lim_{\varepsilon \to 0^+} \beta_\varepsilon(t) = \begin{cases} 0, & t > 0, \\ -\infty, & t < 0. \end{cases}$$

For simplicity, we let $\beta_\varepsilon(\cdot) := \beta_\varepsilon(v_{\varepsilon,n} - 1 + \mu(\tau))$ and $\beta_\varepsilon(\cdot\cdot) := \beta_\varepsilon(-v_{\varepsilon,n} + 1 + \lambda(\tau))$ when no confusion can arise.

Following standard procedure, we can use the Leray-Schauder fixed point theorem, we get the existence and uniqueness of the solution of (2.11). Next, we show that $1 - \mu(\tau) \leq v_{\varepsilon,n} \leq 1 + \lambda(\tau)$. We define the operator $\mathcal{T}$ by

$$\mathcal{T}v := \partial_\tau v - \mathcal{L}_x v + \beta_\varepsilon\left(v - (1-\mu(\tau))\right) - \beta_\varepsilon\left(-v + (1+\lambda(\tau))\right).$$

From the definition of C_0, we obtain

$$\begin{aligned} &\mathcal{T}[1-\mu(\tau)] \\ &= -\mu'(\tau) - (\alpha - r)(1-\mu(\tau)) + \gamma\sigma^2 e^x(1-\mu(\tau))^2 + \beta_\varepsilon(0) - \beta_\varepsilon(\mu(\tau) + \lambda(\tau)) \\ &= -\mu'(\tau) - (\alpha - r)(1-\mu(\tau)) + (\gamma\sigma^2 e^x(1-\mu(\tau))^2 - C_0) \\ &\leq -\mu'(0) - (\alpha - r)(1-\mu(T)) - \lambda'(0) \\ &\leq 0 \end{aligned}$$

for sufficiently small ε. Combining the above inequality and the initial and boundary conditions, we get $1 - \mu(\tau) \leq v_{\varepsilon,n}$ by the comparison principle. Similarly, from the definition of C_0, we have

$$\begin{aligned}
&\mathfrak{T}[1+\lambda(\tau)] \\
&= \lambda'(\tau) - (\alpha - r)(1+\lambda(\tau)) + \gamma\sigma^2 e^x (1+\lambda(\tau))^2 + \beta_\varepsilon(\mu(\tau)+\lambda(\tau)) - \beta_\varepsilon(0) \\
&= \lambda'(\tau) + \gamma\sigma^2 e^x (1+\lambda(\tau))^2 + (C_0 - (\alpha - r)(1+\lambda(\tau))) \\
&\geq \lambda'(0) + \gamma\sigma^2 e^{-n}(1+\lambda_0)^2 \\
&\geq 0
\end{aligned}$$

for sufficiently small ε, which proves $v_{\varepsilon,n} \leq 1+\lambda(\tau)$.

Next, we prove $-v_{\varepsilon,n} \leq \partial_x v_{\varepsilon,n} \leq 0$. To prove this, we differentiate $\mathfrak{T}v_{\varepsilon,n} = 0$ with respect to x, and let $W := \partial_x v_{\varepsilon,n}$. Then W satisfies

$$\text{(2.13)} \qquad \begin{cases} \partial_\tau W - \frac{\sigma^2}{2}\partial_{xx}W - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x W - (\alpha - r)W \\ \quad +\gamma\sigma^2 e^x \left[v_{\varepsilon,n}\partial_x W + 3v_{\varepsilon,n}W + W^2\right] + \beta'_\varepsilon(\cdot)W + \beta'_\varepsilon(\cdot\cdot)W \\ \qquad = -\gamma\sigma^2 e^x v_{\varepsilon,n}^2 \leq 0, \quad \text{in } \Omega_T^n, \\ W(x,\tau) = 0, \quad \text{on } \partial_p\Omega_T^n. \end{cases}$$

Using the maximum principle, we get $\partial_x v_{\varepsilon,n} \leq 0$ in Ω_T^n.

On the other hand, we define the operator $\hat{\mathfrak{T}}$ by

$$\begin{aligned}
\hat{\mathfrak{T}}w := \partial_\tau w - \frac{\sigma^2}{2}\partial_{xx}w - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x w - (\alpha - r)w& \\
+ \gamma\sigma^2 e^x \left[v_{\varepsilon,n}\partial_x w + 3vw + w^2\right] + \beta'_\varepsilon(\cdot)w + \beta'_\varepsilon(\cdot\cdot)w.&
\end{aligned}$$

Then, we have

$$\text{(2.14)} \qquad \hat{\mathfrak{T}}[W] = -\gamma\sigma^2 e^x v_{\varepsilon,n}^2.$$

To make calculations easier, we will use two properties related to the operator $\hat{\mathfrak{T}}$: For each $w_1 = w_1(x,\tau)$ and $w_2 = w_2(x,\tau)$,

1. $\hat{\mathfrak{T}}[w_1 + w_2] = \hat{\mathfrak{T}}[w_1] + \hat{\mathfrak{T}}[w_2] + 2\gamma\sigma^2 e^x w_1 w_2$.
2. $\hat{\mathfrak{T}}[kw_1] = k\hat{\mathfrak{T}}[w_1] + (k^2 - k)\gamma\sigma^2 e^x w_1^2 \quad$ for each $k \in \mathbb{R}$.

Set $\hat{W} := W + v_{\varepsilon,n}$. From the equation (2.14), we have

$$\hat{\mathfrak{T}}[\hat{W} - v_{\varepsilon,n}] = -\gamma\sigma^2 e^x v_{\varepsilon,n}^2.$$

Using the property 1 with respect to $\hat{\mathfrak{T}}$, we obtain

$$\hat{\mathfrak{T}}[\hat{W}] + \hat{\mathfrak{T}}[-v_{\varepsilon,n}] + 2\gamma\sigma^2 e^x \hat{W}(-v_{\varepsilon,n}) = -\gamma\sigma^2 e^x v_{\varepsilon,n}^2.$$

Using the property 2 with respect to $\hat{\mathfrak{T}}$, we get

$$\begin{aligned}
\hat{\mathfrak{T}}[\hat{W}] &= \hat{\mathfrak{T}}[v_{\varepsilon,n}] + 2\gamma\sigma^2 e^x \hat{W} v_{\varepsilon,n} - 3\gamma\sigma^2 e^x v_{\varepsilon,n}^2 \\
&= \partial_\tau v_{\varepsilon,n} - \frac{\sigma^2}{2}\partial_{xx} v_{\varepsilon,n} - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x v_{\varepsilon,n} - (\alpha - r) v_{\varepsilon,n} \\
&\quad + \gamma\sigma^2 e^x \left[v_{\varepsilon,n}\partial_x v_{\varepsilon,n} + 3v_{\varepsilon,n}^2 + v_{\varepsilon,n}^2\right] + \beta'_\varepsilon(\cdot) v_{\varepsilon,n} + \beta'_\varepsilon(\cdot\cdot) v_{\varepsilon,n} \\
&\quad + 2\gamma\sigma^2 e^x \hat{W} v_{\varepsilon,n} - 3\gamma\sigma^2 e^x v_{\varepsilon,n}^2 \\
&= ((\partial_\tau - \mathcal{L}_x)[v_{\varepsilon,n}] - \beta_\varepsilon(\cdot) + \beta_\varepsilon(\cdot\cdot)) + 3\gamma\sigma^2 e^x v_{\varepsilon,n}^2 + \beta'_\varepsilon(\cdot) v_{\varepsilon,n} + \beta'_\varepsilon(\cdot\cdot) v_{\varepsilon,n} \\
&\quad + 2\gamma\sigma^2 e^x \hat{W} v_{\varepsilon,n} - 3\gamma\sigma^2 e^x v_{\varepsilon,n}^2 \\
&= -\beta_\varepsilon(\cdot) + \beta_\varepsilon(\cdot\cdot) + \beta'_\varepsilon(\cdot) v_{\varepsilon,n} + \beta'_\varepsilon(\cdot\cdot) v_{\varepsilon,n} + 2\gamma\sigma^2 e^x \hat{W} v_{\varepsilon,n}.
\end{aligned}$$

This gives

$$\hat{\mathfrak{T}}[\hat{W}] - \hat{\mathfrak{T}}[0] = -\beta_\varepsilon(\cdot) + \beta_\varepsilon(\cdot\cdot) + \beta'_\varepsilon(\cdot) v_{\varepsilon,n} + \beta'_\varepsilon(\cdot\cdot) v_{\varepsilon,n}.$$

We claim that $\hat{\mathfrak{T}}[\hat{W}] - \hat{\mathfrak{T}}[0] \geq 0$. From the definition of β_ε, we get

$$\beta_\varepsilon(\cdot\cdot) - \beta_\varepsilon(\cdot) = \beta'_\varepsilon(\eta)(-2v_{\varepsilon,n} + 1 + \lambda(\tau) + 1 - \mu(\tau)), \tag{2.15}$$

where η is a real number between $-v_{\varepsilon,n} + 1 + \lambda(\tau)$ and $v_{\varepsilon,n} + 1 - \mu(\tau)$. There are only the following three possibilities:

(i) If $v_{\varepsilon,n} = \frac{1+\lambda(\tau)+1-\mu(\tau)}{2}$, then $-\beta_\varepsilon(\cdot) + \beta_\varepsilon(\cdot\cdot) = 0$. It follows that

$$\hat{\mathfrak{T}}[\hat{W}] - \hat{\mathfrak{T}}[0] \geq 0.$$

(ii) If $v_{\varepsilon,n} > \frac{1+\lambda(\tau)+1-\mu(\tau)}{2}$, then

$$\beta'_\varepsilon(v_{\varepsilon,n} - 1 + \mu(\tau)) \leq \beta'_\varepsilon(\eta) \leq \beta'_\varepsilon(-v_{\varepsilon,n} + 1 + \lambda(\tau))$$

by the monotonicity of β'_ε. Hence,

$$\begin{aligned}
&\hat{\mathfrak{T}}[\hat{W}] - \hat{\mathfrak{T}}[0] \\
&= \beta'_\varepsilon(\eta)(-2v_{\varepsilon,n} + 1 + \lambda(\tau) + 1 - \mu(\tau)) + \gamma\sigma^2 e^x v_{\varepsilon,n}^2 + \beta'_\varepsilon(\cdot) v_{\varepsilon,n} + \beta'_\varepsilon(\cdot\cdot) v_{\varepsilon,n} \\
&\geq \beta'_\varepsilon(\cdot\cdot)(-2v_{\varepsilon,n} + 1 + \lambda(\tau) + 1 - \mu(\tau)) + \gamma\sigma^2 e^x v_{\varepsilon,n}^2 + \beta'_\varepsilon(\cdot) v_{\varepsilon,n} + \beta'_\varepsilon(\cdot\cdot) v_{\varepsilon,n} \\
&= \beta'_\varepsilon(\cdot\cdot)(-v_{\varepsilon,n} + 1 + \lambda(\tau) + 1 - \mu(\tau)) + \gamma\sigma^2 e^x v_{\varepsilon,n}^2 + \beta'_\varepsilon(\cdot) v_{\varepsilon,n} \\
&\geq 0.
\end{aligned}$$

(iii) If $v_{\varepsilon,n} < \frac{1+\lambda(\tau)+1-\mu(\tau)}{2}$, then

$$\beta'_\varepsilon(v_{\varepsilon,n} - 1 + \mu(\tau)) \geq \beta'_\varepsilon(\eta) \geq \beta'_\varepsilon(-v_{\varepsilon,n} + 1 + \lambda(\tau))$$

by the monotonicity of β'_ε. Hence,

$$\begin{aligned}
&\hat{\mathfrak{T}}[\hat{W}] - \hat{\mathfrak{T}}[0] \\
&= \beta'_\varepsilon(\eta)(-2v_{\varepsilon,n} + 1 + \lambda(\tau) + 1 - \mu(\tau)) + \gamma\sigma^2 e^x v_{\varepsilon,n}^2 + \beta'_\varepsilon(\cdot)v_{\varepsilon,n} + \beta'_\varepsilon(\cdot\cdot)v_{\varepsilon,n} \\
&\geq \beta'_\varepsilon(\cdot\cdot)(-2v_{\varepsilon,n} + 1 + \lambda(\tau) + 1 - \mu(\tau)) + \gamma\sigma^2 e^x v_{\varepsilon,n}^2 + \beta'_\varepsilon(\cdot)v_{\varepsilon,n} + \beta'_\varepsilon(\cdot\cdot)v_{\varepsilon,n} \\
&= \beta'_\varepsilon(\cdot\cdot)(-v_{\varepsilon,n} + 1 + \lambda(\tau) + 1 - \mu(\tau)) + \gamma\sigma^2 e^x v_{\varepsilon,n}^2 + \beta'_\varepsilon(\cdot)v_{\varepsilon,n} \\
&\geq 0.
\end{aligned}$$

By (i), (ii) and (iii), the proof of the claim is complete.

Since

$$\begin{cases} \hat{W}(x,0) = 1 - \mu_0, & -n \leq x \leq n, \\ \hat{W}(x,\tau) = v(x,\tau), & x = \pm n, \quad \tau \in [0,T], \end{cases}$$

from the comparison principle, we see that $\hat{W}(x,\tau) \geq 0$ in Ω_T^n, and $\partial_x v_{\varepsilon,n} + v_{\varepsilon,n} \geq 0$ is proved.

Next, we claim that $\partial_\tau v_{\varepsilon,n} \geq -\mu'(\tau)$. Let $w = \partial_\tau v_{\varepsilon,n}$ and $\tilde{w} = w + \mu'(\tau)$. Differentiating (2.11) with respect to τ, we obtain

$$\text{(2.16)} \qquad \begin{cases} \partial_\tau w - \frac{\sigma^2}{2}\partial_{xx}w - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x w - (\alpha - r)w \\ \quad +\gamma\sigma^2 e^x\left[v_{\varepsilon,n}\partial_x w + (\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n})\, w\right] \\ \quad +\beta'_\varepsilon(\cdot)(w + \mu'(\tau)) - \beta'_\varepsilon(\cdot\cdot)(-w + \lambda'(\tau)) = 0, \quad \text{in } \Omega_T^n, \\ \partial_x w(x,\tau) = 0, \quad x = \pm n, \quad 0 \leq \tau \leq T, \\ w(x,0) = (\alpha - r)(1-\mu_0) - \left[\gamma\sigma^2(1-\mu_0)^2 e^x + \beta_\varepsilon(0)\right] \geq 0 \end{cases}$$

by the definition of β_ε. For simplicity of notation, we let $\tilde{\mathfrak{T}}$ stand for the operator:

$$\begin{aligned}
\tilde{\mathfrak{T}}w := \partial_\tau w - \frac{\sigma^2}{2}\partial_{xx}w - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x w - (\alpha - r)w \\
+ \gamma\sigma^2 e^x\left[v_{\varepsilon,n}\partial_x w + (\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n})\, w\right] + \beta'_\varepsilon(\cdot)w + \beta'_\varepsilon(\cdot\cdot)w.
\end{aligned}$$

Then, we have

$$\tilde{\mathfrak{T}}[\tilde{w} - \mu'(\tau)] = -\beta'_\varepsilon(\cdot)\mu'(\tau) + \beta'_\varepsilon(\cdot\cdot)\lambda'(\tau).$$

From the linearity of $\tilde{\mathfrak{T}}$, we get

$$\begin{aligned}
\tilde{\mathfrak{T}}[\tilde{w}] &= \tilde{\mathfrak{T}}[\mu'(\tau)] - \beta'_\varepsilon(\cdot)\mu'(\tau) + \beta'_\varepsilon(\cdot\cdot)\lambda'(\tau). \\
&= \mu''(\tau) - (\alpha - r)\mu'(\tau) + \gamma\sigma^2 e^x(\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n})\mu'(\tau) \\
&\quad + \beta'_\varepsilon(\cdot)\mu'(\tau) + \beta'_\varepsilon(\cdot\cdot)\mu'(\tau) - \beta'_\varepsilon(\cdot)\mu'(\tau) + \beta'_\varepsilon(\cdot\cdot)\lambda'(\tau) \\
&= (\bar{q} - (\alpha - r))\mu'(\tau) + \gamma\sigma^2 e^x(\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n})\mu'(\tau) + \beta'_\varepsilon(\cdot\cdot)\mu'(\tau) + \beta'_\varepsilon(\cdot\cdot)\lambda'(\tau).
\end{aligned}$$

If we assume that $\bar{q} \geq \alpha - r$ or $\bar{q} = 0$, then we see from the inequality $\partial_\tau v_{\varepsilon,n} \geq -v_{\varepsilon,n}$ that

$$\text{(2.17)} \qquad \tilde{\mathfrak{T}}[\tilde{w}] \geq 0 = \tilde{\mathfrak{T}}[0].$$

By (2.16), we have

$$\begin{aligned} &\partial_x \tilde{w}(x,\tau) = 0, \quad x = \pm n, \\ &\tilde{w}(x,0) = (\alpha - r)(1-\mu_0) - \left[\gamma\sigma^2(1-\mu_0)^2 e^x + \beta_\varepsilon(0)\right] + \mu'(0) \geq 0. \end{aligned} \tag{2.18}$$

Combining (2.17) and (2.18), we get

$$\tilde{w} \geq 0 \quad \text{if and only if} \quad \partial_\tau v_{\varepsilon,n} \geq -\mu'(\tau).$$

From $-C_0 \leq -\beta_\varepsilon(v_{\varepsilon,n} - 1 + \mu(\tau)) \leq 0$ and $-C_0 \leq -\beta_\varepsilon(-v_{\varepsilon,n} + 1 + \lambda(\tau)) \leq 0$, we see that

$$\|v_{\varepsilon,n}\|_{W^{2,1}_{p,\mathrm{loc}}(\Omega^n_T)} \leq c, \tag{2.19}$$

where c is independent of ε and n. Using a C^α-estimate, we have

$$\|v_{\varepsilon,n}\|_{C^{\alpha,\alpha/2}(\bar{\Omega}^n_T)} \leq c \tag{2.20}$$

for some constant $c > 0$ which is independent of ε. Then we deduce that

$$v_{\varepsilon,n} \rightharpoonup v_n \quad \text{in } W^{2,1}_{p,\mathrm{loc}}(\Omega^n_T) \text{ weakly}, \tag{2.21}$$

$$v_{\varepsilon,n} \to v_n \quad \text{in } C(\bar{\Omega}^n_T) \tag{2.22}$$

as $\varepsilon \to 0^+$, where v_n is the solution to the problem (2.9). Moreover, $-v_{\varepsilon,n} \leq \partial_x v_{\varepsilon,n} \leq 0$ and $\partial_\tau v_{\varepsilon,n} \geq -\mu'(\tau)$ become the inequalities (2.10) as $\varepsilon \to 0^+$.

Finally, we prove the uniqueness of a solution. Suppose that v_1, v_2 are two solutions to the problem (2.11) and that the set

$$\mathcal{N} := \{(x,\tau) : v_1(x,\tau) < v_2(x,\tau), \quad |x| < n, \ \ 0 < \tau \leq T\}$$

is nonempty. Then if $(x,\tau) \in \mathcal{N}$, we have

$$\begin{cases} v_1(x,\tau) < 1 + \lambda(\tau) \text{ implies that } \partial_\tau v_1 - \mathcal{L}_x v_1 \geq 0, \\ v_2(x.\tau) > 1 - \mu(\tau) \text{ implies that } \partial_\tau v_2 - \mathcal{L}_x v_2 \leq 0. \end{cases}$$

Define $v^* = v_2 - v_1$. Then v^* satisfies

$$\begin{cases} \partial_\tau v^* - \frac{\sigma^2}{2}\partial_{xx} v^* - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x v^* - (\alpha - r)v^* \\ \qquad\qquad + \gamma\sigma^2 e^x \left[v_2 \partial_x v^* + (v_1 + v_2 + \partial_x v_1)\, v^*\right] \leq 0, \quad \text{in } \mathcal{N}, \\ v^*(x,0) = 0, \quad \text{on } \partial_p\mathcal{N} \cap \{|x| < n\}, \\ \partial_x v^*(x,0) = 0, \quad \text{on } \partial_p\mathcal{N} \cap \{|x| = n\}, \end{cases}$$

where $\partial_p\mathcal{N}$ is the parabolic boundary of the domain $\mathcal{N}$. Using the ABP maximum principle (see [7] and [11]), we get $v^* \leq 0$ in $\mathcal{N}$, which contradicts the definition of $\mathcal{N}$. $\square$

Theorem 2.2. *There exists a unique solution* $v \in C(\bar{\Omega}_T) \cap W^{2,1}_{p,\mathrm{loc}}(\Omega^R_T)$ *to problem (2.9) for all* $R > 0$, $1 < p < +\infty$. *Also,*

$$\partial_x v \le 0 \ \text{in}\ \Omega_T; \qquad \partial_\tau v \ge -\mu'(\tau) \ \text{a.e. in}\ \Omega_T. \tag{2.23}$$

Moreover, for any fixed $K \in \mathbb{R}$, $v \in C^{\alpha,\alpha/2}(\overline{(-\infty,K) \times (0,T)})$ *for some* $0 < \alpha < 1$, *with*

$$|v|_{C^{\alpha,\alpha/2}(\overline{(-\infty,K)\times(0,T)})} \le C_K, \tag{2.24}$$

where C_K *is a positive constant depending on* K.

Proof. Since the solution v_n of the problem (2.9) belongs to $W^{2,1}_{p,\mathrm{loc}}(\Omega^n_T)$, we rewrite problem (2.9) as

$$\begin{cases} \partial_\tau v_n - \mathcal{L}_x v_n = f(x,\tau), & \text{in } \Omega^n_T, \\ \partial v_n(x,\tau) = 0, & x = \pm n, \quad 0 \le \tau \le T, \\ v_n(x,0) = 1 - \mu_0, & -n \le x \le n, \end{cases}$$

where

$$\begin{aligned} f(x,\tau) = \chi_{\{v_n = 1+\lambda(\tau)\}}(x,\tau) \cdot \left[\lambda'(\tau) - (\alpha - r)(1+\lambda(\tau)) + \gamma\sigma^2 e^x (1+\lambda(\tau))^2\right] \\ + \chi_{\{v_n = 1-\mu(\tau)\}}(x,\tau) \cdot \left[-\mu'(\tau) - (\alpha - r)(1-\mu(\tau)) + \gamma\sigma^2 e^x (1-\mu(\tau))^2\right]. \end{aligned}$$

Then we see that

$$|f(x,\tau)| \le c(R) \quad \text{for } (x,\tau) \in \Omega^R_T,$$

where the constant $c(R)$ depends on R, but is independent of n. Therefore, for any fixed $R > 0$, we choose $n > R$. Then we have the following $W^{2,1}_p$ uniform estimate in $\overline{\Omega}^R_T$:

$$\begin{aligned} \|v_n\|_{W^{2,1}_p(\Omega^R_T)} &\le C\left(\|v_n\|_{L^\infty(\Omega^R_T)} + (1-\mu_0) + \|f(x,\tau)\|_{L^\infty(\Omega^R_T)}\right) \\ &\le C(R) \end{aligned}$$

for some constant $C(R)$ which is independent of n. Letting $n \to +\infty$, we have a subsequence:

$$v_n \rightharpoonup v_R \quad \text{in } W^{2,1}_p(\Omega^R_T) \text{ weakly} \quad \text{and} \quad v_n \to v_R \quad \text{in } C(\overline{\Omega}^R_T) \quad \text{as } n \to +\infty.$$

Define $v = v_R$ if $x \in [-R, R]$. Then it follows that v is well-defined and v is the solution of problem (2.7).

Now, we prove (2.24). Note that

$$\begin{cases} \partial_\tau v - \mathcal{L}_x v = f(x,\tau), & \text{in} \quad \Omega_T, \\ v(x,0) = 1 - \mu_0, & x \in \mathbb{R}. \end{cases}$$

Since $f(x,\tau)$ is bounded on $(-\infty,K)\times(0,T)$, (2.24) follows from the standard C^α theory of parabolic equation. The proof of the uniqueness is the same as that of Lemma 2.1. □

3. Characterizations of the Free Boundaries

In this section, we mainly consider the problem (1.3). We define

$$BR=\{(z,\tau):V(z,\tau)=1+\lambda(\tau)\} \qquad \text{(buy region)},$$
$$NR=\{(z,\tau):1-\mu(\tau)<V(z,\tau)<1+\lambda(\tau)\} \qquad \text{(no transaction region)},$$
$$SR=\{(z,\tau):V(z,\tau)=1-\mu(\tau)\} \qquad \text{(sell region)}.$$

Note that

$$\begin{aligned}&(\partial_\tau-\mathcal{L}_z)(1+\lambda(\tau))\\ &\quad=\lambda'(\tau)-(\alpha-r)(1+\lambda(\tau))+\gamma\sigma^2 z(1+\lambda(\tau))^2\le 0, \qquad \text{in } BR,\\ &(\partial_\tau-\mathcal{L}_z)(1-\mu(\tau))\\ &\quad=-\mu'(\tau)-(\alpha-r)(1-\mu(\tau))+\gamma\sigma^2 z(1-\mu(\tau))^2\ge 0, \qquad \text{in } SR.\end{aligned}$$

Also, we deduce that

$$BR\subset\left\{(z,\tau):\quad z\le\frac{(\alpha-r)(1+\lambda(\tau))-\lambda'(\tau)}{\gamma\sigma^2(1+\lambda(\tau))^2}\right\}, \tag{3.1}$$

$$SR\subset\left\{(z,\tau):\quad z\ge\frac{(\alpha-r)(1-\mu(\tau))+\mu'(\tau)}{\gamma\sigma^2(1-\mu(\tau))^2}\right\}. \tag{3.2}$$

We remark that

$$\begin{aligned}\frac{(\alpha-r)(1+\lambda(\tau))-\lambda'(\tau)}{\gamma\sigma^2(1+\lambda(\tau))^2}&\le\frac{(\alpha-r)(1+\lambda(\tau))+\mu'(0)}{\gamma\sigma^2(1+\lambda(\tau))^2}\\ &\le\frac{(\alpha-r)(1-\mu_0)+\mu'(0)}{\gamma\sigma^2(1-\mu_0)^2}\\ &\le\frac{(\alpha-r)(1-\mu(\tau))+\mu'(\tau)}{\gamma\sigma^2(1-\mu(\tau))^2}.\end{aligned}$$

On the other hand, from the inequalities (2.23) and (2.24), we see that

$$\partial_\tau V\ge-\mu'(\tau), \qquad \partial_z V=e^{-x}\partial_x V\le 0, \tag{3.3}$$

$$|V(z,\tau)|_{C_\tau^{\alpha/2}[0,T]}\le C_K, \qquad 0<z\le K, \tag{3.4}$$

where C_K is a constant depending on K. From the second inequality in (3.3), we can define the free boundaries:

$$z_b(\tau)=\sup\{z:V(z,\tau)=1+\lambda(\tau)\}, \qquad 0<\tau\le T,$$
$$z_s(\tau)=\inf\{z:V(z,\tau)=1-\mu(\tau)\}, \qquad 0<\tau\le T.$$

From the first inequality in (3.3), we deduce that $z_s(\tau)$ is increasing, but $z_b(\tau)$ is not always increasing.

Lemma 3.1. *There is a constant $M_s > 0$ such that*

$$0 \le z_b(\tau) \le \frac{(\alpha - r)(1+\lambda(\tau)) - \lambda'(\tau)}{\gamma\sigma^2(1+\lambda(\tau))^2}, \tag{3.5}$$

$$\frac{(\alpha - r)(1-\mu(\tau)) + \mu'(\tau)}{\gamma\sigma^2(1-\mu(\tau))^2} \le z_s(\tau) \le M_s, \tag{3.6}$$

where M_s is independent of T.

Proof. Since $V(z,\tau) = 1+\lambda(\tau)$ for all $z<0$, $z_b(\tau) \ge 0$. From (3.1) and (3.2), we see that the second parts of (3.5) and the first parts of (3.6) hold.

Next, we claim that $z_s(\tau) \le M_s$. First, we introduce the stationary problem of (1.3):

$$\begin{cases} -\mathcal{L}_z W(z) = 0, & \text{if} \quad 1-\mu(T) < W(z) < 1+\lambda(T), \quad z \in \mathbb{R}_+, \\ -\mathcal{L}_z W(z) \le 0, & \text{if} \quad W(z) + 1 + \lambda(T), \quad z \in \mathbb{R}_+, \\ -\mathcal{L}_z W(z) \ge 0, & \text{if} \quad W(z) = 1-\mu(T), \quad z \in \mathbb{R}_+. \end{cases} \tag{3.7}$$

By the Fichera Theorem in [9], we consider the problem without the boundary value at $z=0$. Using the similar way in Section 2, we can show the existence and uniqueness of the solution to problem (3.7): For each $R>0$ and $1<p<+\infty$,

$$W \in C^1(\mathbb{R}^+) \cap W_p^2\left(\frac{1}{R}, R\right).$$

Furthermore, we have $W'(z) \le 0$. Then we can define

$$SR^* := \left\{ z \in \mathbb{R}^+ : W(z) = 1-\mu(T) \right\}.$$

Set $W^*(z,\tau) := W(z)$. If $1-\mu(\tau) < W^* < 1+\lambda(\tau)$ for each $(z,\tau) \in \mathbb{R}^+ \times (0,T]$, then $1-\mu(T) < W(z) < 1+\lambda(T)$. From the stationary problem (3.7), we have $\partial_\tau W^* - \mathcal{L}_z W^* = -\mathcal{L}_z W(z) = 0$. As a result, we deduce that

$$\begin{cases} \partial_\tau W^* - \mathcal{L}_z W^* = 0, & \text{if } 1-\mu(\tau) < W^* < 1+\lambda(\tau), \quad (z,\tau) \in \mathbb{R}^+ \times (0,T], \\ \partial_\tau W^* - \mathcal{L}_z W^* \ge 0, & \text{if } W^* = 1-\mu(\tau), \quad (z,\tau) \in \mathbb{R}^+ \times (0,T], \\ \partial_\tau W^* - \mathcal{L}_z W^* \le 0, & \text{if } W^* = 1+\lambda(\tau), \quad (z,\tau) \in \mathbb{R}^+ \times (0,T]. \end{cases}$$

Applying the comparison principle, we get $V(z,\tau) \le W^*(z,\tau) = W(z)$.

Next, we claim that there exists a constant $M_s > 0$ such that

$$[M_s, +\infty) \times \{\tau = T\} \subset SR^* \times \{\tau = T\} \subset SR.$$

If it holds, then $[M_s, +\infty) \times [0,T] \subset SR$ by the first parts of the inequality (3.3). It means that $z_s(\tau) \le M_s$, which completes the proof.

To prove the claim, it suffices to show that $[M_s, +\infty) \subset SR^*$. Suppose the claim were not. Then we could find that if $1-\mu(T) < W(z) < 1+\lambda(T)$,

$$\frac{d}{dz}\left[\frac{\sigma^2}{2}z(zW)' + \left(\alpha - \frac{\sigma^2}{2} - r\right)(zW) - \frac{\gamma\sigma^2}{2}(zW)^2\right] = 0.$$

Then we have

$$z(zW)' + \left(\frac{2(\alpha-r)}{\sigma^2} - 1\right)(zW) - \gamma(zW)^2 = \bar{C}, \tag{3.8}$$

where $\bar{C}$ is an unknown constant. Define

$$\widehat{W} := zW + \frac{1}{2\gamma}\left[1 - \frac{2(\alpha-r)}{\sigma^2}\right].$$

Then $z\widehat{W}' - \gamma\widehat{W}^2 = C$, where $C = \bar{C} - \frac{1}{4\gamma^2}\left[1 - \frac{2(\alpha-r)}{\sigma^2}\right]^2$.

If $C < 0$, $z\widehat{W}' - \gamma\widehat{W}^2 = -C_1^2$, where $C_1^2 = -C$. By solving the ODE, we can see that

$$\widehat{W} = \frac{C_1}{\sqrt{\gamma}}\left(\frac{2}{1 - C_2 z^2\sqrt{\gamma}C_1} - 1\right), \quad W = \frac{1}{z}\left(\widehat{W} + \frac{\alpha-r}{\gamma\sigma^2} - \frac{1}{2\gamma}\right).$$

As $z \to +\infty$, $W \to 0$, which contradicts $1-\mu(\tau) \le W \le 1+\lambda(\tau)$.

If $C > 0$, $z\widehat{W}' - \gamma\widehat{W}^2 = C_1^2$, where $C_1^2 = C$. By solving the ODE, we get

$$\widehat{W} = \frac{C_1}{\sqrt{\gamma}}\tan\left(C_1\sqrt{\gamma}\ln z + C_2\right), \quad W = \frac{1}{z}\left(\widehat{W} + \frac{\alpha-r}{\gamma\sigma^2} - \frac{1}{2\gamma}\right).$$

Then we obtain $\liminf_{z\to+\infty} W = 0$, which contradicts $1-\mu(\tau) \le W \le 1+\lambda(\tau)$.

If $C = 0$, by solving the ODE, then we obtain

$$\widehat{W} = \frac{-1}{C_2 + \gamma\ln z}, \quad W = \frac{1}{z}\left(\widehat{W} + \frac{\alpha-r}{\gamma\sigma^2} - \frac{1}{2\gamma}\right).$$

As $z \to +\infty$, $W \to 0$, which is a contradiction. Therefore, there exists a constant $M_s > 0$ which the above claim holds. □

Lemma 3.2. *There exist $z_0 > 0$ and $\tau_0 > 0$ such that $(0, z_0) \times (0, \tau_0) \subset NR$ and that*

(3.9) *all partial derivatives of $V(z,\tau)$ are bounded on $(0, z_0) \times (0, \tau_0)$.*

Proof. Using $V(z, 0) = 1 - \mu_0$ and (3.4), for any fixed $K > 0$, we observe that there exists $\tau_0 > 0$ such that

$$V(z,\tau) < 1 + \lambda(\tau), \qquad (z,\tau) \in (0,K) \times (0,\tau_0). \tag{3.10}$$

On the other hand, from (3.2), for $0 < z_0 < \frac{(\alpha-r)(1-\mu(\tau))+\mu'(\tau)}{\gamma\sigma^2(1-\mu(\tau))^2}$, we get

$$V(z,\tau) > 1-\mu(\tau), \qquad (z,\tau) \in (0,z_0). \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, & \text{in } (0,z_0)\times(0,\tau_0), \\ V(z_0,\tau) \in C^\infty[0,\tau_0], \\ V(z,0) = 1-\mu_0, & 0<z<z_0. \end{cases} \tag{3.12}$$

Set $x = \ln z$, $x_0 = \ln z_0$, and $v(x,\tau) = V(z,\tau)$. Then we have

$$\begin{cases} \partial_\tau v - \frac{\sigma^2}{2}\partial_{xx}v - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x v - (\alpha-r)v + \gamma\sigma^2 e^x v\left(\partial_x v + v\right) = 0, \\ v(x_0,\tau) \in C^\infty[0,\tau_0], \\ v(x,0) = 1-\mu_0, \quad x \in (-\infty, x_0). \end{cases} \tag{3.13}$$

Using (2.24) and Schauder theory in [9], we see that

$$\|v\|_{C^{2+\alpha,1+\alpha/2}\left(\overline{(-\infty,x_0)\times(0,\tau_0)}\right)} \le C_{x_0},$$

where C_{x_0} depends on x_0. Employing a bootstrap argument, it gives that all partial derivatives of $v(x,\tau)$ are bounded on $(-\infty,x_0)\times(0,\tau_0)$.

We proceed to show that $\partial_z V(z,\tau) = e^{-x}\partial_x v(x,\tau)$ is bounded. Differentiating with respect to x in (3.13), we get

$$\begin{aligned} &\partial_\tau(\partial_x v) - \frac{\sigma^2}{2}\partial_{xx}(\partial_x v) - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x(\partial_x v) - (\alpha-r)(\partial_x v) \\ &\quad + \gamma\sigma^2 e^x\left[(\partial_x v + v)^2 + v(\partial_{xx}v + \partial_x v)\right] = 0. \end{aligned}$$

Set $W = e^{-x}\partial_x v$. Then

$$\begin{cases} \partial_\tau W - \dfrac{\sigma^2}{2}\partial_{xx}W - \left(\alpha - r + \dfrac{3}{2}\sigma^2\right)\partial_x W - \left(2\alpha - 2r + \sigma^2\right)W \\ \qquad = -\gamma\sigma^2\left[(\partial_x v + v)^2 + v(\partial_{xx}v + \partial_x v)\right], \quad (x,\tau) \in (-\infty,x_0)\times(0,\tau_0], \\ W(x_0,\tau) \in C^\infty[0,\tau_0], \\ W(x,0) = 0, \quad x \in (-\infty,x_0). \end{cases}$$

Since the right-hand side of the equation is bounded, $\partial_z V = e^{-x}\partial_x v$ is bounded. In the same manner, we can see that $\partial_{zz}V = \partial_z W = e^{-2x}(\partial_{xx}v - \partial_x v)$ is bounded. Furthermore, all partial derivatives of $V(z,\tau)$ are bounded on $(0,z_0)\times(0,\tau_0)$ by the bootstrap argument. □

Theorem 3.3. *$z_s(\tau) \in C[0,T]\cap C^\infty(0,T]$ and is strictly increasing with $z_s(0) = z^*$, where*

$$z^* = \frac{(\alpha-r)(1-\mu_0)+\mu'(0)}{\gamma\sigma^2(1-\mu_0)^2}. \tag{3.14}$$

Proof. First, we show (3.14). Recalling (3.2), we see that

$$z^* \geq \frac{(\alpha - r)(1-\mu_0) + \mu'(0)}{\gamma\sigma^2(1-\mu_0)^2}.$$

We suppose that $z^* > \frac{(\alpha-r)(1-\mu_0)+\mu'(0)}{\gamma\sigma^2(1-\mu_0)^2}$. Then there exists $z_2 < z^*$ such that

$$\begin{cases} \partial_\tau V - \mathcal{L}_z V = 0, & (z,\tau) \in \left(\frac{(\alpha-r)(1-\mu_0)+\mu'(0)}{\gamma\sigma^2(1-\mu_0)^2}, z_2\right) \times (0,T), \\ V(z,0) = 1-\mu_0, & \frac{(\alpha-r)(1-\mu_0)+\mu'(0)}{\gamma\sigma^2(1-\mu_0)^2} \leq z \leq z_2. \end{cases} \tag{3.15}$$

Therefore, $\partial_\tau V(z,0) = (\alpha-r)(1-\mu_0) - \gamma\sigma^2 z(1-\mu_0)^2 < -\mu'(0)$, which contradicts the first part of the inequalities (3.3). We complete the proof of (3.14).

Next, we claim that $z_s(\tau)$ is strictly increasing in $(0,T]$. Suppose that $z_s(\tau)$ is not strictly increasing in $(0,T]$. Then $z_s(\tau_1) = z_s(\tau_2) = z_0$ for some $z_0 \in \mathbb{R}^+$ and $0 < \tau_1 < \tau_2 \leq T$. And there exists $z_1 < z_0$ satisfying $(z_1,\tau) \in NR$ for all $\tau \in [\tau_1,\tau_2]$. Set $D = (z_1,z_0) \times (\tau_1,\tau_2)$. Then we see that

$$\begin{cases} V(z_0,\tau) = 1-\mu(\tau), & \tau \in [\tau_1,\tau_2], \\ \partial_\tau V(z_0,\tau) = -\mu'(\tau), & \tau \in [\tau_1,\tau_2]. \end{cases}$$

Since $V > 1-\mu(\tau)$ in D, we observe that

$$\partial_\tau V - \mathcal{L}_z V = 0, \quad \text{in } D. \tag{3.16}$$

We show that the strong maximum principle implies that $\partial_\tau V \equiv -\mu'(\tau)$ or $\partial_{z\tau} V(z_0,\tau) < 0$ for any $\tau \in (\tau_1,\tau_2)$. Differentiating (3.16) with respect to τ, we obtain

$$\begin{aligned} \mathcal{T}W := \partial_\tau W - \frac{\sigma^2}{2} z^2 \partial_{zz} W - (\sigma^2 + \alpha - r)\partial_z W - (\alpha - r)W \\ + \gamma\sigma^2 z[2VW + (\partial_z V)W + zV\partial_z W] = 0 \qquad \text{in } D. \end{aligned}$$

Assume that $W = -\mu'(\tau_2)$ is a minimum at $(z',\tau') \in (z_1,z_0) \times (\tau_1,\tau_2)$ and that $W > -\mu'(\tau_2)$ is a maximum at $(z^*,\tau^*) \in (z_1,z_0) \times (\tau_1,\tau_2)$. Since $\mathcal{T}W = 0$ in D, $W \equiv -\mu'(\tau_2)$ in $D \cap \{\tau \leq \tau'\}$ or $\partial_z W(z_0,\tau) < 0$ for all $\tau \in [\tau_1,\tau_2]$ by the strong maximal principle.

Since $\partial_z W(z_0,\tau) = \partial_z(1-\mu(\tau)) = 0$ for all $\tau \in [\tau_1,\tau_2]$, we see that

$$W \equiv -\mu'(\tau_2) \qquad \text{in } D \cap \{\tau \leq \tau'\}.$$

This means that $\partial_\tau V$ obtains its minimum value at interior point of $NR \cap \{\tau \leq \tau'\}$. Then we have $\partial_\tau V \equiv -\mu'(\tau_2)$ in $NR \cap \{\tau \leq \tau'\}$ by the strong maximum principle. If

$\bar{q} = 0$, we already know the fact that $z_s(\tau)$ is strictly increasing. Hence, we assume that $\bar{q} \neq 0$. Then, on the segment $\{z_0\} \times (\tau_1, \tau')$,

$$\partial_\tau V(z,\tau) = -\mu'(\tau_2) = -\mu'(\tau),$$

which is a contradiction. Therefore, $z_s(\tau)$ is strictly increasing.

We claim that $z_s(\tau)$ is continuous. Suppose that $z_s(\tau)$ is not continuous in $(0,T]$. Then there exist $\tau_0 \in (0,T]$, $z_0 \in (0,+\infty)$, and small ε_0, δ_0 such that

$$z_s(\tau_0 - \varepsilon) \leq z_0 \quad \text{and} \quad z_s(\tau_0 + \varepsilon) \geq z_0 + \delta_0$$

for all $\varepsilon \in (0, \varepsilon_0)$. Let $D = (z_0, z_0 + \delta) \times (\tau_0, \tau_0 + \varepsilon)$ so that $D \subset NR$. Since $z_s(\tau)$ is strictly increasing, we observe that $z^* < z_0$, i.e., $(\alpha - r)(1 - \mu(\tau_0)) + \mu'(\tau_0) < \gamma\sigma^2 z_0 (1 - \mu(\tau_0))^2$. In D, we see that

(3.17)
$$\begin{aligned}
&(\partial_\tau - \mathcal{L}_z)[V] - (\partial_\tau - \mathcal{L}_z)[1 - \mu(\tau)] \\
&= \mu'(\tau) + (\alpha - r)(1 - \mu(\tau)) - \gamma\sigma^2 z (1 - \mu(\tau))^2 \\
&= \mu'(\tau) + (\alpha - r)(1 - \mu(\tau)) - \gamma\sigma^2 z_0 (1 - \mu(\tau))^2 + \gamma\sigma^2 (z_0 - z)(1 - \mu(\tau))^2 \\
&\leq \mu'(\tau) + (\alpha - r)(1 - \mu(\tau)) - \gamma\sigma^2 z_0 (1 - \mu(\tau_0))^2 + \gamma\sigma^2 (z_0 - z)(1 - \mu(\tau))^2 \\
&= [\mu'(\tau) + (\alpha - r)(1 - \mu(\tau))] - [\mu'(\tau_0) + (\alpha - r)(1 - \mu(\tau_0))] \\
&\quad + \gamma\sigma^2 (z_0 - z)(1 - \mu(\tau))^2 \\
&= \mu'(\tau) - \mu'(\tau_0) + (\alpha - r)(\mu(\tau_0) - \mu(\tau)) + \gamma\sigma^2 (z_0 - z)(1 - \mu(\tau))^2 \\
&= (\bar{q} - (\alpha - r))(\mu(\tau) - \mu(\tau_0)) + \gamma\sigma^2 (z_0 - z)(1 - \mu(\tau))^2.
\end{aligned}$$

Here, we assume that $\bar{q} \leq \alpha - r$. Now (3.17) becomes

$$(\partial_\tau - \mathcal{L}_z)[V] - (\partial_\tau - \mathcal{L}_z)[1 - \mu(\tau)] \leq \gamma\sigma^2 (z_0 - z)(1 - \mu(\tau))^2. \tag{3.18}$$

As $\tau \to \tau_0$, (3.18) leads to

$$\partial_\tau V(z, \tau_0) < -\mu'(\tau),$$

which is impossible. Therefore, $z_s(\tau)$ is continuous in $(0,T]$. Employing a method developed by Friedman in [5], we can see that $z_s(\tau) \in C^\infty(0,T]$. □

Remark 3.4. So far, we have assumed that

1. $0 \leq q \leq \alpha - r$,
2. $\bar{q} = 0$ or $\bar{q} = \alpha - r$.

We next investigate the limit of the solution and the values of the free boundary $z_b(\tau)$ near $\tau = 0$.

Theorem 3.5. *We have* $\lim_{z \to 0^+} V(z,\tau) = V_0(\tau)$ *with*

1. *If $\alpha - r < q$, then $z_b(\tau) = 0$ for each $\tau \in [0, \tau^*]$, where* $\tau^* = \dfrac{1}{(\alpha - r) - q} \ln \dfrac{1+\lambda_0}{1-\mu_0}$, *and*

$$V_0(\tau) = \begin{cases} (1-\mu_0)e^{(\alpha-r)\tau}, & 0 \le \tau \le \tau^*, \\ 1 + \lambda(\tau), & \tau > \tau^*. \end{cases} \tag{3.19}$$

2. *If $\alpha - r = q$, then $z_b(\tau) = 0$ for each $\tau \in [0, \tau_0]$, where $\tau_0 > 0$ is the number in Lemma 3.2, and*

$$V_0(\tau) = (1-\mu_0)e^{(\alpha-r)\tau}, \qquad 0 \le \tau \le \tau_0. \tag{3.20}$$

Proof. Since all partial derivative of $V(z,\tau)$ are bounded on $(0, z_0) \times (0, \tau_0)$, we see that there exists $V_0(\tau) \in C[0,T]$ such that

$$\lim_{z \to 0^+} V(z,\tau) = V_0(\tau), \qquad \lim_{z \to 0^+} \partial_\tau V(z,\tau) = V_0'(\tau).$$

Applying (3.9) and letting $z \to 0^+$ in (3.12), we deduce that

$$\begin{cases} V_0'(\tau) - (\alpha - r)V_0(\tau) = 0, & 0 < \tau < \tau_0, \\ V_0(0) = 1 - \mu_0. \end{cases}$$

Then we obtain

$$V_0(\tau) = (1-\mu_0)e^{(\alpha-r)\tau}, \quad 0 < \tau < \tau_0.$$

If $\alpha - r < q$ and let $V_0(\tau^*) = 1 + \lambda(\tau^*)$, then we have

$$(1-\mu_0)e^{(\alpha-r)\tau} = (1+\lambda_0)e^{q\tau^*}.$$

In short, it follows that

$$\tau^* = \frac{1}{(\alpha - r) - q} \ln \frac{1+\lambda_0}{1-\mu_0},$$

which deduces (3.19). If $\alpha - r = q$, then we see

$$V_0(\tau) = (1-\mu_0)e^{(\alpha-r)\tau}, 0 \le \tau \le T,$$

which implies (3.20). □

Next, we prove that $z_b(\tau) \in C(0,T]$. In order to do so, we use a transformation $\bar{v}(y,\tau) := v(x,\tau)$, where $y = x + k(\tau)$. Fix $T_1 \in (0,T)$ and consider the problem (2.7) only on domain $\mathbb{R} \times [0, T_1]$. Let $M \subseteq \Omega^n_{T_1}$ be a domain defined by

$$M := (-n, x_1(\tau)) \times (0, T_1),$$

which $x_1(\tau)$ is $\inf\{x : v(x,\tau) = 1\}$. Similarly,

$$M^\varepsilon := (-n, x_1^\varepsilon(\tau)) \times (0, T_1),$$

which $x_1^\varepsilon(\tau)$ is $\inf\{x : v_{\varepsilon,n}(x,\tau) = 1\}$. Set some parts of the boundary of M as follows:

$$\partial_1 M := \left\{(x,\tau) \in \overline{\Omega^n_{T_1}} : V(x,\tau) = 1\right\}, \qquad \partial_2 M := \{-n\} \times (0, T_1).$$

Similarly, set

$$\partial_1 M^\varepsilon := \left\{(x,\tau) \in \overline{\Omega^n_{T_1}} : v_{\varepsilon,n}(x,\tau) = 1\right\}.$$

The task is now to find $\partial_\tau \bar{v}(y,\tau) \geq \lambda'(\tau)$ in M. If it holds, this makes it possible that $\bar{z}_b(\tau)$ is monotonic, where $\bar{z}_b(\tau)$ is corresponding to $z_b(\tau)$.

Lemma 3.6. *In M, $\partial_x v_{\varepsilon,n} - k'(\tau)\partial_\tau v_{\varepsilon,n} \geq \lambda'(\tau)$.*

Proof. Step 1. Let $y = x + k(\tau)$ and $\bar{v}(y,\tau) = v(x,\tau)$. Then we calculate

$$\begin{cases} \partial_x v(x,\tau) = \partial_y \bar{v}(y,\tau), \\ \partial_{xx} v(x,\tau) = \partial_{yy} \bar{v}(y,\tau), \\ \partial_\tau \bar{v}(y,\tau) = \partial_\tau v(x,\tau) - k'(\tau)\partial_x v(x,\tau). \end{cases}$$

We claim that $\partial_\tau v(x,\tau) - k'(\tau)\partial_x v(x,\tau) \geq \lambda'(\tau)$. For simplicity, let

$$w_1 = \partial_\tau v_{\varepsilon,n}, \quad w_2 = \partial_x v_{\varepsilon,n},$$

$$w = k'(\tau)\partial_x v_{\varepsilon,n}, \quad Q = w_1 - w = \partial_\tau v_{\varepsilon,n} - k'(\tau)\partial_x v_{\varepsilon,n},$$

where $k(\tau) = k_1\lambda(\tau) + k_2\mu(\tau)$ such that k_1, k_2 are to be chosen later. Differentiating (2.11) with respect to τ, we obtain

$$\begin{aligned} &\partial_\tau w_1 - \frac{\sigma^2}{2}\partial_{xx} w_1 - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x w_1 - (\alpha - r)w_1 \\ &\quad + \gamma\sigma^2 e^x \left[v_{\varepsilon,n}\partial_x w_1 + (\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n})\, w_1\right] \\ &\quad + \beta'_\varepsilon(\cdot)(w_1 + \mu'(\tau)) - \beta'_\varepsilon(\cdot\cdot)(-w_1 + \lambda'(\tau)) = 0, \quad \text{in } \Omega^n_{T_1}. \end{aligned} \tag{3.21}$$

Differentiating (2.11) with respect to x, we have

$$\begin{aligned} &\partial_\tau w_2 - \frac{\sigma^2}{2}\partial_{xx} w_2 - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x w_2 - (\alpha - r)w_2 \\ &\quad + \gamma\sigma^2 e^x \left[v_{\varepsilon,n}\partial_x w_2 + 2v_{\varepsilon,n} w_2 + v_{\varepsilon,n}\partial_x v_{\varepsilon,n} + w_2 \partial_x v_{\varepsilon,n} + v^2_{\varepsilon,n}\right] \\ &\quad + \beta'_\varepsilon(\cdot) w_2 + \beta'_\varepsilon(\cdot\cdot) w_2 = 0, \quad \text{in } \Omega^n_{T_1}. \end{aligned} \tag{3.22}$$

Multiplying (3.22) by $k'(\tau)$, we get

$$\begin{aligned} &\partial_\tau w - \frac{\sigma^2}{2}\partial_{xx} w - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x w - (\alpha - r)w \\ &+ \gamma\sigma^2 e^x \left[v_{\varepsilon,n}\partial_x w + 2v_{\varepsilon,n} w + k'(\tau) v_{\varepsilon,n}\partial_x v_{\varepsilon,n} + (\partial_x v_{\varepsilon,n}) w + k'(\tau) v^2_{\varepsilon,n}\right] \\ &+ \beta'_\varepsilon(\cdot) w + \beta'_\varepsilon(\cdot\cdot) w - k''(\tau)\partial_x v_{\varepsilon,n} = 0, \quad \text{in } \Omega^n_{T_1}. \end{aligned} \tag{3.23}$$

Subtracting (3.23) from (3.21), we see that

$$\begin{aligned}\mathcal{T}[Q] &:= \partial_\tau Q - \frac{\sigma^2}{2}\partial_{xx}Q - \left(\alpha - r + \frac{\sigma^2}{2}\right)\partial_x Q - (\alpha - r)Q \\ &\quad + \gamma\sigma^2 e^x\left[v_{\varepsilon,n}\partial_x Q + (\partial_x v_{\varepsilon,n})Q + 2v_{\varepsilon,n}Q\right] + \beta_\varepsilon'(\cdot)Q + \beta_\varepsilon'(\cdot\cdot)Q \\ &= \gamma\sigma^2 e^x[k'(\tau)v_{\varepsilon,n}\partial_x v_{\varepsilon,n} + k'(\tau)v_{\varepsilon,n}^2] \\ &\quad - \beta_\varepsilon'(\cdot)\mu'(\tau) + \beta_\varepsilon'(\cdot\cdot)\lambda'(\tau) - k''(\tau)\partial_x v_{\varepsilon,n}, \qquad \text{in } \Omega_{T_1}^n.\end{aligned} \tag{3.24}$$

Substituting $Q = \lambda'(\tau)$ into (3.24), we have

$$\begin{aligned}\mathcal{T}[\lambda'(\tau)] = \lambda''(\tau) - (\alpha - r)\lambda'(\tau) + \gamma\sigma^2 e^x[\lambda'(\tau)\partial_x v_{\varepsilon,n} + 2\lambda'(\tau)v_{\varepsilon,n}] \\ + \beta_\varepsilon'(\cdot)\lambda'(\tau) + \beta_\varepsilon'(\cdot\cdot)\lambda'(\tau).\end{aligned} \tag{3.25}$$

Combining (3.24) with (3.25), we see that

$$\begin{aligned}&\mathcal{T}[Q] - \mathcal{T}[\lambda'(\tau)] \\ &= \gamma\sigma^2 e^x[k'(\tau)v_{\varepsilon,n}\partial_x v_{\varepsilon,n} + k'(\tau)v_{\varepsilon,n}^2 - \lambda'(\tau)\partial_x v_{\varepsilon,n} - 2\lambda'(\tau)v_{\varepsilon,n}] \\ &\quad - k''(\tau)\partial_x v_{\varepsilon,n} - \lambda''(\tau) + (\alpha - r)\lambda'(\tau) - \beta_\varepsilon'(v_{\varepsilon,n} - 1 + \mu(\tau))[\mu'(\tau) + \lambda'(\tau)] \\ &=: I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \gamma\sigma^2 e^x[k'(\tau)v_{\varepsilon,n}\partial_x v_{\varepsilon,n} + k'(\tau)v_{\varepsilon,n}^2 - \lambda'(\tau)\partial_x v_{\varepsilon,n} - 2\lambda'(\tau)v_{\varepsilon,n}], \\ I_2 &= -k''(\tau)\partial_x v_{\varepsilon,n} \geq 0, \\ I_3 &= \lambda''(\tau) + (\alpha - r)\lambda'(\tau) = (q + \alpha - r)\lambda'(\tau) \geq 0, \\ I_4 &= -\beta_\varepsilon'(v_{\varepsilon,n} - 1 + \mu(\tau))[\mu'(\tau) + \lambda'(\tau)].\end{aligned}$$

We claim that

$$\mathcal{T}[Q] - \mathcal{T}[\lambda'(\tau)] \geq 0.$$

Since I_2 and I_3 are nonnegative and $I_4 \geq 0$ in M^ε for sufficiently small $\varepsilon > 0$, we have to show that I_1 is nonnegative. Now, we see the boundary condition in M^ε. Define $\widehat{\mathcal{T}}$ by

$$\begin{aligned}\widehat{\mathcal{T}}[w] := \partial_\tau w - \frac{\sigma^2}{2}\partial_{xx}w - \left(\alpha - r + \frac{\sigma^2}{2}\right) - (\alpha - r)w \\ + \gamma\sigma^2 e^x\left(v\partial_x w + 3vw + w^2 + v^2\right).\end{aligned}$$

Differentiating (2.7) with respect to x, we obtain

$$\widehat{\mathcal{T}}[\partial_x v] = 0, \qquad \text{in } M.$$

Let $A = \{(y,\tau) : x - c \leq y \leq x + c, \quad (x,\tau) \in \partial_1 M\}$ be a subset such that $A \subseteq NR$ for sufficiently small constant c. From the strong maximum principle, there exists $\delta_1 > 0$ such that

$$\partial_x v \leq -\delta_1, \qquad \text{in } A.$$

Then, for sufficiently small $\varepsilon > 0$,

$$\partial_x v_{\varepsilon,n} \leq -\frac{\delta_1}{2}, \qquad \text{in } \partial_1 M^\varepsilon. \tag{3.26}$$

Also, let $B = \{(y,\tau) : x - c \leq y \leq x + c, \quad (x,\tau) \in \partial_2 M \cap NR\}$ be a subset such that $B \subseteq NR$ for sufficiently small constant c. From the strong maximum principle, there exists $\delta_2 > 0$ such that

$$\partial_x v \leq -\delta_2, \qquad \text{in } B.$$

Then, for sufficiently small $\varepsilon > 0$,

$$\partial_x v_{\varepsilon,n} \leq -\frac{\delta_2}{2}, \qquad \text{in } \partial_2 M \cap NR. \tag{3.27}$$

Choose

$$k(\tau) := \frac{1}{\delta}[4\lambda(\tau) + 2\mu(\tau)],$$

where $\delta := \min\{\delta_1, \delta_2, \delta_3\} > 0$. Here, $\delta_3 > 0$ is to be selected later. Therefore, we have

$$\begin{aligned}
\partial_\tau v_{\varepsilon,n} - k'(\tau)\partial_x v_{\varepsilon,n} &\geq -\mu'(\tau) - k'(\tau) \cdot \left(-\frac{\delta}{2}\right) \\
&= -\mu'(\tau) + (2\lambda'(\tau) + \mu'(\tau)) \\
&\geq \lambda'(\tau).
\end{aligned}$$

Since $\partial_\tau v_{\varepsilon,n} - k'(\tau)\partial_x v_{\varepsilon,n} = \lambda'(\tau)$ on the boundary $\partial_2 M \cap BR$, combining (3.26) and (3.27) yields

$$\partial_\tau v_{\varepsilon,n} - k'(\tau)\partial_x v_{\varepsilon,n} \geq 0 \qquad \text{in } \partial_1 M \cup \partial_2 M.$$

On the other hand, we get

$$\tilde{\mathfrak{T}}[\partial_x v + v] := \widehat{\mathfrak{T}}[\partial_x v + v] - 2\gamma\sigma^2 e^x v[\partial_x v + v] \geq 0 \qquad \text{in } M.$$

From the strong minimum principle, there exists $\delta_3 > 0$ such that

$$\partial_x v + v \geq \delta_3 \qquad \text{in } M.$$

Then,

$$\partial_x v_{\varepsilon,n} + v_{\varepsilon,n} \geq \frac{\delta_3}{2} \qquad \text{in } M^\varepsilon.$$

Therefore, in the domain M^ε,

$$\begin{aligned}
I_1 &= \gamma\sigma^2 e^x[k'(\tau)v_{\varepsilon,n}(\partial_x v_{\varepsilon,n} + v_{\varepsilon,n}) - \lambda'(\tau)(\partial_x v_{\varepsilon,n} + 2v_{\varepsilon,n})] \\
&\geq \gamma\sigma^2 e^x \left[k'(\tau)v_{\varepsilon,n}\cdot\left(\frac{\delta_3}{2}\right) - 2\lambda'(\tau)v_{\varepsilon,n}\right] \\
&\geq \frac{1}{2}\gamma\sigma^2 e^x v_{\varepsilon,n}\left[k'(\tau)\delta - 4\lambda'(\tau)\right] \\
&\geq \frac{1}{2}\gamma\sigma^2 e^x v_{\varepsilon,n}[2\mu'(\tau)] \\
&\geq 0.
\end{aligned}$$

Therefore, we complete the proof of claim. Note that

$$\begin{cases}
Q(x,\tau) = w_1 - k'(\tau)w_2 \geq \lambda'(\tau), & \text{if } (x,\tau)\in\partial_1 M\cup\partial_2 M,\\
Q(x,0) = (w_1 - k'(\tau)w_2)(x,0) & \\
\qquad = (\alpha - r)(1-\mu_0) - [\gamma\sigma^2(1-\mu_0)^2 e^x + \beta_\varepsilon(0)] - k\lambda'(\tau)\cdot 0. &
\end{cases}$$

Then

$$Q(x,0) - \lambda'(0) = (\alpha - r)(1-\mu_0) + [C_0 - \gamma\sigma^2(1-\mu_0)^2 e^x - \lambda'(0)] \geq 0.$$

Therefore, we conclude that $Q(x,\tau)\geq\lambda'(\tau)$ in $\Omega^n_{T_1}$. □

We consider the domain $\mathcal{S} := \{(y,\tau) : -\infty \leq y \leq x_1(\tau) + k(\tau)\}$. From Lemma 3.6, we see that $\partial_\tau\bar{v}(y,\tau)\geq\lambda'(\tau)$ in $\mathcal{S}$.

Theorem 3.7. *There holds $z_b(\tau)\in C(0,T]$.*

Proof. Suppose that theorem is false. From the problem (2.7) and Lemma 3.6, we see that there exists $(y_1,y_2)\times(0,\tau_1)\subset\mathcal{S}$ such that

$$\begin{cases}
\partial_\tau\bar{v} - \mathcal{L}_y\bar{v} = 0, & (y,\tau)\in(y_1,y_2)\times(0,\tau_1),\\
\bar{v}(y,\tau_1) = 1 + \lambda(\tau_1), & y\in[y_1,y_2],
\end{cases}$$

where

$$\mathcal{L}_y\bar{v} := \frac{\sigma^2}{2}\partial_{yy}\bar{v} + (\alpha - r + \frac{\sigma^2}{2} - k'(\tau))\partial_y\bar{v} + (\alpha - r)\bar{v} - \gamma\sigma^2 e^{y-k(\tau)}\bar{v}(\partial_y\bar{v} + \bar{v}).$$

Set $\bar{w} := \partial_y\bar{v}$. Then $\bar{w}$ satisfies

$$\begin{cases}
\partial_\tau\bar{w} - \frac{\sigma^2}{2}\partial_{yy}\bar{w} - (\alpha - r + \frac{\sigma^2}{2} - k'(\tau))\partial_y\bar{w} - (\alpha - r)\bar{w} \\
\quad +\gamma\sigma^2 e^{y-k(\tau)}[3\bar{v}\bar{w} + \bar{w}\partial_y\bar{v} + \bar{v}\partial_y\bar{w}] \\
\qquad = -\gamma\sigma^2 e^{y-k(\tau)}\bar{v}^2 \leq 0, \quad (y,\tau)\in(y_1,y_2)\times(0,\tau_1),\\
\bar{w}(y,0) = 0, \quad y\in[y_1,y_2].
\end{cases}$$

It follows that $\bar{w}$ achieves non-negative maximum on $\tau = \tau_1$. By the maximum principle, $\partial_y \bar{v} = \bar{w} \equiv 0$ in $(y_1, y_2) \times (0, \tau_1)$. Then $\bar{w} \equiv 0$ in $NR_y \cap \{\tau \le \tau_1\}$, where

$$NR_y := \{(y,\tau) \in \mathcal{S} : 1 - \mu(\tau) < \bar{v}(y,\tau) < 1 + \lambda(\tau)\} .$$

Therefore, by monotonicity, there exist $\tau^* < \tau_1$ and $y^* < y_1$ such that $y^* = \overline{z_b}(\tau^*)$, where

$$\overline{z_b}(\tau) = \sup\{y : \bar{v}(y,\tau) = 1 + \lambda(\tau)\} .$$

Since $\bar{w} \equiv 0$ in $NR_y \cap \{\tau \le \tau_1\}$, we have $\bar{v}(y,\tau) = 1 + \lambda(\tau^*)$ on the line $\{\tau = \tau^*\}$, which is a contradiction. Hence, we see that $\overline{z_b}(\tau)$ is continuous, which deduces that $z_b(\tau) \in C(0,T]$. □

Theorem 3.8. *There holds $z_b(\tau) \in C^\infty(0,T]$.*

Proof. Since $k(\tau)$ is smooth, the proof of Theorem 3.8 follows directly from the fact that $\bar{z}_b(\tau) \in C^\infty(0,T]$, which is clear from [13]. □

4. Equivalence

The equivalence of the double obstacle problem and the original problem is discussed in this section. According to (2.6), there should be two functions $A(\tau)$ and $B(\tau)$ such that

$$V^*(z,\tau) = \begin{cases} A(\tau) - \gamma(1+\lambda(\tau)), & \text{if } (z,\tau) \in BR, \\ B(\tau) - \gamma(1-\mu(\tau)), & \text{if } (z,\tau) \in SR. \end{cases}$$

Since $\partial_\tau V^* - LV^* = 0$ on $z = z_b(\tau)$, recalling (2.4) we obtain

$$A'(\tau) = \gamma\lambda'(\tau)z_b(\tau) - \gamma(\alpha - r)(1+\lambda(\tau))z_b(\tau) + \frac{\sigma^2}{2}\gamma^2(1+\lambda(\tau))^2 z_b^2(\tau). \tag{4.1}$$

Since $V^*(z,0) = -\gamma(1+\lambda_0)z, A(0) = 0$. Therefore, by the Fundamental Theorem of Calculus, we have

$$A(\tau) = \int_0^\tau \gamma\lambda'(t)z_b(t) - \gamma(\alpha - r)(1+\lambda(t))z_b(t) + \gamma^2\frac{\sigma^2}{2}(1+\lambda(t))^2 z_b(t)^2 \, dt. \tag{4.2}$$

From the integral of (2.6) with respect to z, we get

$$V^*(z,\tau) = A(\tau) - \gamma\int_0^z V(\xi,\tau)\,d\xi. \tag{4.3}$$

Lemma 4.1. *$V^*(z,\tau), \partial_\tau V^*(z,\tau), z\partial_z V^*, z^2\partial_{zz}V^* \in C(\mathbb{R} \times [0,T])$. Moreover,*

$$V^*(z,\tau) = \begin{cases} A(\tau) - \gamma(1+\lambda(\tau))z, & \text{if } z \le z_b(\tau), \\ B(\tau) - \gamma(1-\mu(\tau))z, & \text{if } z \ge z_s(\tau), \end{cases} \tag{4.4}$$

where $A(\tau)$ is given by (4.2) and

$$B(\tau) = A(\tau) - \gamma \int_0^{z_s(\tau)} V(\xi,\tau)d\xi + \gamma(1-\mu(\tau))z_s(\tau). \tag{4.5}$$

Proof. First we prove (4.4). If $z \le z_b(\tau)$, we obtain from (4.3) that

$$V^*(z,\tau) = A(\tau) - \gamma \int_0^z (1+\lambda(\tau))\, d\xi = A(\tau) - \gamma(1+\lambda(\tau))z.$$

If $z \ge z_s(\tau)$, we have

$$\begin{aligned} V^*(z,\tau) &= A(\tau) - \gamma \int_0^{z_s(\tau)} V(\xi,\tau)\, d\xi - \gamma \int_{z_s(\tau)}^z (1-\mu(\tau))\, d\xi \\ &= A(\tau) - \gamma \int_0^{z_s(\tau)} V(\xi,\tau)\, d\xi + \gamma(1-\mu(\tau))z_s(\tau) - \gamma(1-\mu(\tau))z \\ &= B(\tau) - \gamma(1-\mu(\tau))z, \quad (by\ (4.5)). \end{aligned}$$

Now, we show the smoothness of $V^*(z,\tau)$. Since $z_b(\tau) \in C[0,T]$, $A(\tau) \in C^1[0,T]$ by (4.2). Furthermore, $V \in L^\infty(\mathbb{R} \times [0,T])$ and is continuous with respect to τ. As a result, $V^*(z,\tau) \in C(\mathbb{R} \times [0,T])$ by (4.3).

Next, we prove $\partial_\tau V^*(z,\tau) \in C(\mathbb{R} \times [0,T])$. Indeed,

$$\partial_\tau V^*(z,\tau) = A'(\tau) - \int_0^z \partial_\tau V(\xi,\tau)\, d\xi. \tag{4.6}$$

It is clear that $\partial_\tau V^*$ is continuous across $z=0$ by (4.6). On the other hand, (4.6) can be rewritten as

$$\partial_\tau V^*(z,\tau) = A'(\tau) - \gamma \int_{z_b(\tau)}^z \partial_\tau V(\xi,\tau)d\xi - \gamma\lambda'(\tau)z_b(\tau). \tag{4.7}$$

If $z \le z_b(\tau)$, then

$$\begin{aligned} \partial_\tau V^*(z,\tau) &= A'(\tau) - \gamma\lambda'(\tau)z_b(\tau) \\ &= -\gamma(\alpha-r)(1+\lambda(\tau))z_b(\tau) + \gamma^2\frac{\sigma^2}{2}(1+\lambda(\tau))^2 z_b^2(\tau). \end{aligned} \tag{4.8}$$

If $z_b(\tau) \le z \le z_s(\tau)$, then

$$\begin{aligned} &\int_{z_b(\tau)}^z \partial_\tau V(\xi,\tau)d\xi \\ &= \int_{z_b(\tau)}^z \mathcal{L}_z V(\xi,\tau)d\xi \\ &= \int_{z_b(\tau)}^z \frac{\partial}{\partial\xi}\left[\frac{\sigma^2}{2}\xi^2\partial_\xi V(\xi,\tau) + (\alpha-r)\xi V(\xi,\tau) - \gamma\frac{\sigma^2}{2}(\xi V(\xi,\tau))^2\right] d\xi \\ &= \left[\frac{\sigma^2}{2}\xi^2\partial_\xi V(\xi,\tau) + (\alpha-r)\xi V(\xi,\tau) - \gamma\frac{\sigma^2}{2}(\xi V(\xi,\tau))^2\right]_{\xi=z_b(\tau)}^{\xi=z}. \end{aligned} \tag{4.9}$$

Combining (4.7) and (4.9), we get

$$\begin{aligned}
&\partial_\tau V^*(z,\tau)\\
&= A'(\tau) - \gamma\lambda'(\tau)z_b(\tau)\\
&\quad - \gamma\left[\frac{\sigma^2}{2}\xi^2\partial_\xi V(\xi,\tau) + (\alpha-r)\xi V(\xi,\tau) - \gamma\frac{\sigma^2}{2}(\xi V(\xi,\tau))^2\right]_{\xi=z_b(\tau)}^{\xi=z}\\
&= -\gamma\left[\frac{\sigma^2}{2}z^2\partial_z V(z,\tau) + (\alpha-r)zV(z,\tau) - \gamma\frac{\sigma^2}{2}(zV(z,\tau))^2\right].
\end{aligned} \tag{4.10}$$

Combining (4.8) and (4.10) implies that $\partial_\tau V^*(z,\tau)$ is continuous across $z = z_b(\tau)$. Moreover, by (4.10),

$$\lim_{z\to z_s^-(\tau)} \partial_\tau V^*(z,\tau) = -\gamma(\alpha-r)(1-\mu(\tau))z_s(\tau) + \gamma^2\frac{\sigma^2}{2}(1-\mu(\tau))^2 z_s^2(\tau). \tag{4.11}$$

On the other hand, if $z \geq z_s(\tau)$,

$$\begin{aligned}
&\partial_\tau V^*(z,\tau)\\
&= B'(\tau) + \gamma\mu'(\tau)z\\
&= A'(\tau) - \gamma\int_0^{z_s(\tau)} \partial_\tau V(\xi,\tau)d\xi + \gamma(z - z_s(\tau))\mu'(\tau)\\
&= A'(\tau) - \gamma\left[\frac{\sigma^2}{2}\xi^2\partial_\xi V(\xi,\tau) + (\alpha-r)\xi V(\xi,\tau) - \gamma\frac{\sigma^2}{2}(\xi V(\xi,\tau))^2\right]_{\xi=z_b(\tau)}^{\xi=z_s(\tau)}\\
&\quad + \gamma(z - z_s(\tau))\mu'(\tau)\\
&= \gamma(z - z_s(\tau))\mu'(\tau) + \gamma^2\frac{\sigma^2}{2}z_s^2(\tau)(1-\mu(\tau))^2 - \gamma(\alpha-r)z_s(\tau)(1-\mu(\tau)).
\end{aligned}$$

If $z = z_s(\tau)$, then

$$\partial_\tau V^*(z,\tau) = -\gamma(\alpha-r)(1-\mu(\tau))z_s(\tau) + \gamma^2\frac{\sigma^2}{2}(1-\mu(\tau))^2 z_s^2(\tau).$$

Therefore, $\partial_\tau V^*(z,\tau)$ is continuous across $z = z_s(\tau)$. As a result, we notice that $\partial_z[zV(z,\tau)]$ is bounded on $\mathbb{R}\times[0,T]$, and that $z\partial_z V^*(z,\tau) = -\gamma zV(z,\tau)$ is continuous on $\mathbb{R}\times[0,T]$. Furthermore,

$$z^2\partial_{zz}V^*(z,\tau) = -\gamma z^2\partial_z V(z,\tau) = -\gamma z\left\{\partial_z[zV(z,\tau)] - V(z,\tau)\right\}$$

is continuous on $\mathbb{R}\times[0,T]$. This completes the proof of Lemma 4.1. □

Theorem 4.2. *$V^*(z,\tau)$, defined by (4.4), is the solution of the problem (2.5). In detail,*

$$\partial_\tau V^* - LV^* \leq 0, \quad \text{in } \mathbb{R}\times(0,T); \tag{4.12}$$

$$-\gamma(1+\lambda(\tau)) \leq \partial_z V^* \leq -\gamma(1-\mu(\tau)), \quad \text{in } \mathbb{R}\times(0,T); \tag{4.13}$$

$$\partial_\tau V^* - LV^* = 0, \quad \text{if } -\gamma(1+\lambda(\tau)) < \partial_z V^* < -\gamma(1-\mu(\tau)); \tag{4.14}$$

$$V^*(z,0) = \begin{cases} -\gamma(1+\lambda_0)z, & \textit{if} \quad z < 0, \\ -\gamma(1-\mu_0)z, & \textit{if} \quad z \geq 0. \end{cases} \tag{4.15}$$

Proof. Since $\partial_z V^* = -\gamma V$ and $1-\mu(\tau) \leq V \leq 1+\lambda(\tau)$, we obtain (4.13). In detail,

$$\begin{cases} \partial_z V^* = -\gamma(1+\lambda(\tau)), & \text{if } z \leq z_b(\tau), \\ -\gamma(1+\lambda(\tau)) < \partial_z V^* < -\gamma(1-\mu(\tau)), & \text{if } z_b(\tau) < z < z_s(\tau), \\ \partial_z V^* = -\gamma(1-\mu(\tau)), & \text{if } z \geq z_s(\tau). \end{cases}$$

Combining $A(0) = 0$ and the initial value of V with (4.3) yields (4.15). Next, we show (4.14). Since $L_z(-\gamma V) = -\frac{1}{\gamma}\frac{\partial}{\partial z}(LV^*)$, we have

$$\partial_z(\partial_\tau V^* - LV^*) = 0 \text{ if } -\gamma(1+\lambda(\tau)) < \partial_z V^* < -\gamma(1-\mu(\tau)). \tag{4.16}$$

Moreover, we get

$$\partial_\tau V^* - LV^* = 0 \qquad \text{on } z = z_b(\tau). \tag{4.17}$$

Combining (4.16) and (4.17) gives (4.14). Finally, we know that

$$\frac{\partial}{\partial z}\left(\partial_\tau V^* - LV^*\right) = -\gamma\left(\partial_\tau V - \mathcal{L}_z V\right) \begin{cases} \geqslant 0, & \text{if} \quad z \leqslant z_b(\tau); \\ = 0, & \text{if} \quad z_b(\tau) < z < z_s(\tau); \\ \leqslant 0, & \text{if} \quad z \geqslant z_s(\tau), \end{cases}$$

which proves (4.12). □

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