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# $h$-almost Ricci Solitons on Generalized Sasakian-space-forms

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Abstract. The aim of this article is to study the $h$-almost Ricci solitons and $h$-almost gradient Ricci solitons on generalized Sasakian-space-forms. First, we consider $h$-almost Ricci soliton with the potential vector field $V$ as a contact vector field on generalized

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Sasakian-space-form of dimension greater than three. Next, we study $h$-almost gradient Ricci solitons on a three-dimensional quasi-Sasakian generalized Sasakian-space-form. In both the cases, several interesting results are obtained.

## 1. Introduction

Nowadays, the Ricci solitons and their generalizations are enjoying rapid growth by providing new techniques in understanding the geometry and topology of arbitrary Riemannian manifolds. Ricci soliton is a natural generalization of Einstein metric, and is also a self-similar solution to Hamilton's Ricci flow [20, 21]. It plays a specific role in the study of singularities of the Ricci flow. A solution $g(t)$ of the non-linear evolution PDE: $\frac{\partial}{\partial t}g(t) = -2S(g(t))$ is called the Ricci flow, where $S$ is the Ricci tensor associated to the metric $g$. In differential geometry, the Ricci flow is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing out irregularities in the metric. A Riemannian manifold $(M, g)$ is called a *Ricci soliton* if there are a smooth vector field $V$ and a scalar $\lambda \in \mathbb{R}$ such that

$$S + \frac{1}{2}\pounds_V g = \lambda g \tag{1.1}$$

on $M$, where $S$ is the Ricci tensor and $\pounds_V g$ is the Lie derivative of the metric $g$ along $V$. If the potential vector field $V$ vanishes identically, then the Ricci soliton becomes trivial, and in this case manifold is an Einstein one. A Ricci soliton is said to be a gradient Ricci soliton if the potential vector field $V$ can be expressed as a gradient of a smooth function $u$ on $M$, i.e., $V = Du$, where $D$ is the gradient operator of $g$ on $M$. An important application of the Ricci flow is the proof for Thurston's Conjecture given recently by Perelman [32]: A Ricci soliton on any compact Riemannian manifold is always a gradient Ricci soliton. We recommend the papers [8, 11, 12, 13, 17, 18, 28, 40] and the references therein for more details about the study of Ricci solitons, gradient Ricci solitons and their generalizations in the context of contact Riemannian geometry.

The generalized version of Ricci soliton, so called *almost Ricci soliton* was introduced in the paper [33] by treating the soliton constant $\lambda$ as a smooth function. It is noted from [5] that a compact almost Ricci soliton with constant scalar curvature is isometric to Euclidean sphere. Later in [38], Sharma studied Ricci almost solitons in $K$-contact geometry and Ghosh [16] studied Ricci almost solitons and gradient Ricci almost solitons in $(\kappa, \mu)$-contact geometry. Recently, Wang-Gomes-Xia [39] extended the notion of almost Ricci soliton to $h$-almost Ricci soliton. According to [39], a complete connected Riemannian manifold $(M^{2n+1}, g)$ is said to be *$h$-almost Ricci soliton* if there exists a smooth vector field $V$ on $M^{2n+1}$ such that

$$S + \frac{h}{2}\pounds_V g = \lambda g \tag{1.2}$$

where $\lambda$ and $h$ are smooth functions on $M^{2n+1}$. Here, $\lambda$ is called soliton function and

$V$ is called the potential vector field of $h$-almost Ricci soliton. This notion is denoted by $(M^{2n+1}, g, V, h, \lambda)$. An $h$-almost Ricci soliton is called: (i) shrinking, when the soliton constant $\lambda$ is positive; (ii) steady, when $\lambda$ is zero and (iii) expanding, when $\lambda$ is negative. If the potential vector field $V$ can be expressed as a gradient of a smooth function $u$ on $M$, i.e., $V = Du$, where $D$ is the gradient operator of $g$ on $M$, then the $h$-almost Ricci soliton equation becomes

$$S + hHess\ u = \lambda g \tag{1.3}$$

(where, $Hess\ u = \nabla^2 u$ denotes the Hessian of the smooth function $u$) and characterizes what is called *$h$-almost gradient Ricci soliton.* The problem of studying $h$-almost Ricci solitons and $h$-almost gradient Ricci solitons in the context of contact metric geometry was initiated by Ghosh-Patra [19]. In particular, they studied $h$-almost Ricci solitons and $h$-almost gradient Ricci solitons on a $K$-contact manifold and proved that if a compact $K$-contact metric is $h$-almost gradient Ricci soliton then it is isometric to a unit sphere $S^{2n+1}$. More recently, Kar-Majhi [26] studied $(\kappa, \mu)$-almost co-Kähler manifold which admits $h$-almost Ricci soliton and $h$-almost gradient Ricci soliton. Also, $h$-almost Ricci solitons on Sasakian 3-manifolds was studied in the paper [29]. Motivated by the above studies, in this paper we undertake the study of $h$-almost Ricci solitons on almost contact metric manifolds, particularly, on generalized Saskian-space-forms. Before to proceed further, we recall that, Ricci solitons and $\eta$-Ricci solitons on generalized Sasakian-space-forms were studied in the paper [31]. Further, the study of invariant submanifolds of generalized Sasakian-space-forms was recorded in [25].

The paper is organized as follows: Section 2 is concerned with the preliminaries on generalized Sasakian-space-forms. In section 3, $h$-almost Ricci soliton on a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ with the potential vector field as a contact vector field is being considered and prove that in such a case the manifold has a constant scalar curvature and the flow vector field is Killing. Next, we also show that the manifold $M^{2n+1}(f_1, f_2, f_3)$ is locally symmetric and has a constant $\phi$-sectional curvature provided the characteristic vector field $\xi$ is Killing. In the last section, we study three-dimensional quasi-Sasakian generalized Sasakian-space-form $M^3(f_1, f_2, f_3)$ with $f_1 \neq f_3$ admitting $h$-almost gradient Ricci soliton and prove that in this situation the manifold $M^3(f_1, f_2, f_3)$ is of constant curvature $f_1 - f_3$.

## 2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold $M^{2n+1}$ is called an almost contact manifold (see, Blair [7]) equipped with the structure $(\phi, \xi, \eta)$ where $\phi$ is a tensor field of type (1,1), $\xi$ a characteristic or Reeb vector field and $\eta$ is a 1-form satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \ \ \eta \circ \phi = 0 \tag{2.1}$$

for all vector field $X$ on $M^{2n+1}$. In general, a differentiable manifold $M^{2n+1}$ together with the almost contact structure $(\phi, \xi, \eta)$ is said to be an almost contact manifold and it is denoted by $(M^{2n+1}, \phi, \xi, \eta)$. If an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ admits a Riemannian metric $g$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields $X$, $Y$ on $M^{2n+1}$, then the manifold is called an almost contact metric manifold and is denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. Then from (2.2), it can be easily deduced that $g(\phi X, Y) = -g(X, \phi Y)$. The fundamental 2-form $d\eta$ associate with the almost contact metric structure is defined by

$$d\eta(X, Y) = g(X, \phi Y) \tag{2.3}$$

for any vector fields $X$ and $Y$.

An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta)$ is said to be a generalized Sasakian-space-form if the curvature tensor of the manifold satisfies

$$R(X, Y)Z = f_1 R_1 + f_2 R_2 + f_3 R_3 \tag{2.4}$$

for some smooth functions $f_1$, $f_2$ and $f_3$ on $M^{2n+1}$, where $R_1$, $R_2$ and $R_3$ are curvature-like tensors given by

$$\begin{aligned} R_1(X, Y)Z &= g(Y, Z)X - g(X, Z)Y, \\ R_2(X, Y)Z &= g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z, \\ R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \end{aligned}$$

for any vector fields $X, Y, Z$ on $M^{2n+1}$. In such case we will write the manifold as $M^{2n+1}(f_1, f_2, f_3)$. Moreover, Sasakian, cosymplectic or/and Kenmotsu space forms are the typical examples of generalized Sasakian-space-forms. This almost contact counterpart was introduced and studied by Alegre-Blair-Carriazo [1] in 2004. Since then, several papers have appeared concerning different aspects of this topic. At this point, we recommend the papers [2, 3, 4, 10, 14, 22, 23, 24, 27, 34, 35, 36, 37] and the references therein to reader for a wide and detailed overview of the results on generalized Sasakian-space-forms.

In addition to the relation (2.4), for a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ the following relations also hold [1]:

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.5}$$

$$R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}, \tag{2.6}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - \{3f_2 + (2n-1)f_3\}\eta(X)\eta(Y), \tag{2.7}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{2.8}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{2.9}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{2.10}$$

for any vector fields $X, Y$ on $M^{2n+1}(f_1, f_2, f_3)$, where $R$, $S$ and $r$ are the curvature tensor, Ricci tensor and scalar curvature of the space-form, respectively.

Also, for a generalized Sasakian-space-form $M^3(f_1, f_2, f_3)$ of dimension three the Ricci operator $Q$ and the curvature tensor $R$ are given by [28]:

$$QX = (\frac{r}{2} - f_1 + f_3)X - (\frac{r}{2} - 3f_1 + 3f_3)\eta(X)\xi \tag{2.11}$$

and

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} - 2f_1 + f_3)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad -(3f_1 - 3f_3 + \frac{r}{2})\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)X\} \end{aligned} \tag{2.12}$$

respectively, for any vector fields $X, Y, Z$ on $M^3(f_1, f_2, f_3)$.

**Definition 2.1.** ([7]) A vector field $V$ on a contact manifold $M^{2n+1}$ is said to be a contact vector field if it preserves the contact form $\eta$, that is,

$$\pounds_V \eta = \rho\eta \tag{2.13}$$

for some smooth function $\rho$ on $M^{2n+1}$. When $\rho = 0$ on $M^{2n+1}$, the vector field $V$ is called a strict contact vector field.

**Definition 2.2.** ([15]) An infinitesimal automorphism $V$ is a smooth vector field such that Lie derivatives of all structure tensor along $V$ vanishes, that is,

$$\pounds_V g = \pounds_V \xi = \pounds_V \phi = \pounds_V \eta = 0. \tag{2.14}$$

## 3. $h$-almost Ricci Solitons on Generalized Sasakian-space-forms

Let $g$ be an $h$-almost Ricci soliton on a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$. Then we have from (1.2) that

$$S(X,Y) + \frac{h}{2}(\pounds_V g)(X,Y) = \lambda g(X,Y). \tag{3.1}$$

Replacing $\xi$ instead of $X$ and $Y$ in (3.1) we get

$$hg(\pounds_V \xi, \xi) = 2n(f_1 - f_3) - \lambda. \tag{3.2}$$

Plugging $Y$ by $\xi$ in (3.1) and then using (2.8) and (2.13) gives

$$h\pounds_V \xi = (h\rho + 4n(f_1 - f_3) - 2\lambda)\xi. \tag{3.3}$$

Observing (3.2) in (3.3) we have

$$h\rho = -2n(f_1 - f_3) - \lambda. \tag{3.4}$$

Making use of (3.4) in (3.3) we get

$$h\pounds_V\xi = \{2n(f_1 - f_3) - \lambda\}\xi. \tag{3.5}$$

On the other hand, from (2.3) we deduce that

$$(\pounds_V d\eta)(X,Y) = (\pounds_V g)(X,\phi Y) + g(X,(\pounds_V\phi)Y). \tag{3.6}$$

Multiplying both sides of (3.6) by $h$ and then using (3.1) we infer

$$h(\pounds_V d\eta)(X,Y) = -2S(X,\phi Y) + 2\lambda g(X,\phi Y) + hg(X,(\pounds_V\phi)Y). \tag{3.7}$$

Feeding (2.7) in (3.7) we get

$$h(\pounds_V d\eta)(X,Y) = \{-2(2nf_1 + 3f_2 - f_3) + 2\lambda\}g(X,\phi Y) + hg(X,(\pounds_V\phi)Y). \tag{3.8}$$

Let us suppose that the potential vector field $V$ of $M^{2n+1}$ be a contact vector field. Then, with the aid of (2.13) we have

$$(\pounds_V d\eta)(X,Y) = \frac{1}{2}\{d\rho(X)\eta(Y) - d\rho(Y)\eta(X)\} + \rho g(X,\phi Y). \tag{3.9}$$

This together with (3.8) provides

$$\begin{aligned} 2h(\pounds_V\phi)Y &= 4\{2nf_1 + 3f_2 - f_3 - \lambda\}\phi Y + 2\rho h\phi Y \\ &+ h\eta(Y)D\rho - h(Y\rho)\xi. \end{aligned} \tag{3.10}$$

Inserting $\xi$ in place of $Y$ we get

$$2h(\pounds_V\phi)\xi = h\{D\rho - (\xi\rho)\xi\}. \tag{3.11}$$

With the help of $\phi\xi = 0$ and (3.5) we obtain

$$h(\pounds_V\phi)\xi = h\pounds_V\phi\xi - \phi(h\pounds_V\xi) = 0. \tag{3.12}$$

Applying (3.12) in (3.11) we have

$$D\rho = (\xi\rho)\xi. \tag{3.13}$$

Taking the inner product of (3.13) with $X$ gives

$$d\rho(X) = (\xi\rho)\eta(X), \tag{3.14}$$

or equivalently,

$$d\rho = (\xi\rho)\eta. \tag{3.15}$$

Taking exterior derivative of (3.15) we get

$$d^2\rho = d(\xi\rho)\wedge\eta + (\xi\rho)d\eta = 0,$$

which implies

$$d(\xi\rho) \wedge \eta + (\xi\rho)d\eta = 0. \tag{3.16}$$

Taking wedge product of (3.16) with $\eta$ we gave

$$(\xi\rho)\eta \wedge d\eta = 0, \tag{3.17}$$

from which it follows that $\xi\rho = 0$. Since $\eta \wedge (d\eta)^n \neq 0$, and by (3.15) one can obtain $d\rho = 0$ and hence $\rho$ is constant.

Further, with the help of (2.13) and noting that $\pounds_V$ and $d$ commutes, we have

$$\pounds_V d\eta = d\pounds_V \eta = (d\rho) \wedge \eta + \rho(d\eta). \tag{3.18}$$

As a volume form, $\omega$ is closed and by thus the Cartan's formula provides

$$\pounds_V \omega = (divV)\omega \tag{3.19}$$

Next, taking the Lie differentiation to volume form $\omega = \eta \wedge (d\eta)^n$ and then using (3.18) and (3.19) we obtain

$$divV = (n+1)\rho. \tag{3.20}$$

Integrating (3.20) over $M^{2n+1}$ and then applying Divergence theorem, we infer

$$\rho = 0, \tag{3.21}$$

(and so $divV = 0$). Thus, we obtain from (3.4) that

$$\lambda = 2n(f_1 - f_3). \tag{3.22}$$

**Theorem 3.1.** *Let $M^{2n+1}(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian-space-form with the potential vector field $V$ as a contact vector field. If $g$ is h-almost Ricci soliton on $M^{2n+1}(f_1, f_2, f_3)$, then the soliton is shrinking, steady or expanding accordingly as $f_1 - f_3$ is positive, zero or negative.*

The trace of (3.1) and with the fact that $\sum_{i=1}^{3}(\pounds_V g)(e_i, e_i) = 2divV$ and (3.20) we deduce

$$r = (2n+1)\lambda. \tag{3.23}$$

Now it is easy to check from (2.10), (3.22) and (3.23) that

$$3f_2 + (2n-1)f_3 = 0. \tag{3.24}$$

By virtue of (3.24) and (2.7) we have

$$S(X,Y) = 2n(f_1 - f_3)g(X,Y). \tag{3.25}$$

That is, $M^{2n+1}(f_1, f_2, f_3)$ is Einstein with Einstein constant $2n(f_1 - f_3)$. Contracting (3.25) we obtain

$$r = 2n(2n+1)(f_1 - f_3).$$

Therefore, the scalar curvature $r$ is constant.

Next, with the help of (3.22) and (3.25), from (3.1) we get $\pounds_V g = 0$, which implies that $V$ is Killing. Since $\rho$ is constant, it follows from (3.10) that

$$h(\pounds_V \phi)Y = \{2nf_1 + 3f_2 - f_3 - \lambda\}\phi Y. \tag{3.26}$$

We employ (3.22) and (3.24) in the above equation to achieve $\pounds_V \phi = 0$ as $h$ is positive. Further, by taking account of (3.22) in (3.5), we have $\pounds_V \xi = 0$. Finally, we substitute (3.21) in (2.13) to deduce $\pounds_V \eta = 0$. Thus, Lie derivatives of all structure tensor along $V$ vanishes and from (2.14), the flow vector field $V$ is an infinitesimal automorphism of the almost contact metric structure of $M^{2n+1}(f_1, f_2, f_3)$. Hence, we summarize the above in the form of a theorem which is as follows:

**Theorem 3.2.** *Let $M^{2n+1}(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form with the potential vector field $V$ as a contact vector field. If $g$ is $h$-almost Ricci soliton on $M^{2n+1}(f_1, f_2, f_3)$, then the scalar curvature of $M^{2n+1}(f_1, f_2, f_3)$ is constant and the flow vector field $V$ is Killing. Moreover, $V$ is an infinitesimal automorphism of the almost contact metric structure of $M^{2n+1}(f_1, f_2, f_3)$.*

**Remark 3.3.** In [14], De and Sarkar studied projective curvature tensor on a generalized Sasakian-space-forms and proved that a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form is projectively flat if and only if $3f_2 + (2n-1)f_3 = 0$. So by virtue of (3.24) it is evident that a $(2n+1)$-dimensional $(n > 1)$ generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ admits an $h$-almost Ricci soliton is projectively flat.

Now, at this junction, we recall the following theorem due to Kim [27]:

**Theorem 3.4.** *Let $M^{2n+1}$ be a $(2n+1)$-dimensional generalized Sasakian-space-form. Then we have following:*
*(i) If $n > 1$, then $M^{2n+1}$ is conformally flat if and only if $f_2 = 0$.*
*(ii) If $M^{2n+1}$ is conformally flat and $\xi$ is a Killing vector field, then $M^{2n+1}$ is locally symmetric and has constant $\phi$-sectional curvature.*

Also, it is known that projectively flat and conformally flat conditions for a generalized Sasakian-space-form of dimension greater than three are equivalent. By taking account of this fact along with previous discussion, we are able to conclude the following:

**Theorem 3.5.** *Let $M^{2n+1}(f_1, f_2, f_3)$ be a generalized Sasakian-space-form of dimension greater than 3. If $(g, V)$ is an $h$-almost Ricci soliton on $M^{2n+1}(f_1, f_2, f_3)$ with the potential vector field $V$ as a contact vector field, then $M^{2n+1}(f_1, f_2, f_3)$ is conformally flat. In addition, if the characteristic vector field $\xi$ of $M^{2n+1}$ is a*

*Killing vector field, then $M^{2n+1}(f_1, f_2, f_3)$ is locally symmetric and has constant $\phi$-sectional curvature.*

**4.$h$-almost Gradient Ricci Solitons on Three-dimensional Quasi-Sasakian Generalized Sasakian-space-forms**

In [28], authors have studied the notion of quasi-Sasakian generalized Sasakian-space-forms. This notion is an analogous version of the trans-Sasakian generalized Sasakian-space-forms studied in [2]. An almost contact metric manifold $M^3$ is a three-dimensional quasi-Sasakian manifold if and only if [30]

$$\nabla_X \xi = -\beta\phi X \tag{4.1}$$

for any vector field $X$ on $M^3$ and for a certain function $\beta$, such that $\xi\beta = 0$. Here, $\nabla$ denotes the operator of the covariant differentiation with respect to the Levi-Civita connection of $M^3$. If $\beta$ = constant, then the manifold reduces to a $\beta-$Sasakian manifold and if in particular $\beta = 1$, the manifold becomes a Sasakian manifold. As a consequence of (4.1), we have

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\}. \tag{4.2}$$

From (4.2) it follows that

$$R(X,\xi)\xi = \beta^2 X \quad \textit{and} \quad R(X,\phi X)\xi = d\beta(\phi X)\phi X + d\beta(X)X, \tag{4.3}$$

for any vector field $X$ on $M^3$, orthogonal to $\xi$. Also, from (2.4) we obtain

$$R(X,\xi)\xi = (f_1 - f_3)X \quad \textit{and} \quad R(X,\phi X)\xi = 0. \tag{4.4}$$

Therefore (4.3) and (4.4) give us $\beta^2 = f_1 - f_3$ and $\beta$ is constant. Also, in a three-dimensional quasi-Sasakian generalized Sasakian-space-form $\beta$ is non-zero, provided $f_1 \neq f_3$.

In this section, before entering into the main part we prove the following:

**Lemma 4.1.** On a three-dimensional quasi-Sasakian generalized Sasakian-space-form $M^3(f_1, f_2, f_3)$, we have

$$(\nabla_X Q)\xi - (\nabla_\xi Q)X = \beta\{\phi QX - 2(f_1 - f_3)\phi X\} \tag{4.5}$$

for any vector field $X$ on $M^3(f_1, f_2, f_3)$.

*Proof.* For a three-dimensional quasi-generalized generalized Sasakian-space-form $M^3(f_1, f_2, f_3)$ we have

$$Q\xi = 2(f_1 - f_3)\xi. \tag{4.6}$$

Taking covariant differentiation of (4.6) along an arbitrary vector field $X$ on $M^3(f_1, f_2, f_3)$ and using (4.1), we get

$$(\nabla_X Q)\xi = \beta\{Q\phi X - 2(f_1 - f_3)\phi X\}. \tag{4.7}$$

Since $\xi$ is Killing on a three-dimensional quasi-Sasakian generalized Sasakian-space-form $M^3(f_1, f_2, f_3)$, we have $(\pounds_\xi Q) = 0$ on $M^3(f_1, f_2, f_3)$. This follows that $\pounds_\xi(QX) = Q(\pounds_\xi X)$. Now, taking into account (4.1) it follows that

$$(\nabla_\xi Q)X = \beta\{Q\phi X - \phi QX\} \tag{4.8}$$

for any vector field $X$ on $M^3(f_1, f_2, f_3)$. Subtraction of (4.8) from (4.7) gives (4.5). This completes the proof.

Next, suppose that in a three-dimensional quasi-Sasakian generalized Sasakian-space-form $M^3(f_1, f_2, f_3)$, the metric $g$ admits $h$-almost gradient Ricci soliton. Then the soliton equation defined by (1.3) with the potential function $u$ can be exhibited as

$$\beta\nabla_X Du = -QX + \lambda X \tag{4.9}$$

for any vector field $X$ on $M^3$; where $D$ is the gradient operator of $g$ on $M^3(f_1, f_2, f_3)$. By straightforward computations, using the well known expression of the curvature tensor:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

and the repeated use of equation (4.9) gives

$$\begin{aligned} hR(X, Y)Du &= \frac{1}{h}(Xh)\{QY - \lambda Y\} - \frac{1}{h}(Yh)\{QX - \lambda X\} \\ &- \{(\nabla_X Q)Y - (\nabla_Y Q)X\} - \{(X\lambda)Y - (Y\lambda)X\}. \end{aligned} \tag{4.10}$$

Replacing $\xi$ instead of $X$ in (4.10) and making use of (2.12) and (4.5) we get

$$\begin{aligned} hR(\xi, Y)Du &= \frac{(\lambda - 2(f_1 - f_3))}{h}(Yh)\xi + \frac{1}{h}(\xi h)(QY - \lambda Y) \\ &- \beta\{2(f_1 - f_3)\phi Y - \phi QY\} + (\xi\lambda)Y - (Y\lambda)\xi. \end{aligned}$$

for any vector field $Y$ on $M^3(f_1, f_2, f_3)$. Scalar product of the last equation with an arbitrary vector field $X$ and using (2.6), we obtain

$$\begin{aligned} &h(f_1 - f_3)\{g(Y, Du)\eta(X) - (\xi u)g(X, Y)\} \\ = &\frac{(\lambda - 2(f_1 - f_3))}{h}(Yh)\eta(Y) + \frac{1}{h}(\xi h)\{g(QX, Y) - \lambda g(X, Y)\} \\ - &\beta\{2(f_1 - f_3)g(\phi Y, X) - g(\phi QY, X)\} + (\xi\lambda)g(X, Y) - (Y\lambda)\eta(Y) \end{aligned} \tag{4.11}$$

for any vector fields $X$ and $Y$ on $M^3(f_1, f_2, f_3)$. Next, substituting $X$ by $\phi X$ and $Y$ by $\phi Y$ in (4.11) and then using (2.1) provides

$$\begin{aligned} &\{(\xi\lambda) - \frac{\lambda}{h}(\xi h) + h(f_1 - f_3)(\xi u)\}\{g(X, Y) - \eta(X)\eta(Y)\} \\ + &\frac{1}{h}(\xi h)g(Q\phi X, \phi Y) - \beta\{2(f_1 - f_3)g(X, \phi Y) - g(Q\phi Y, X)\} = 0 \end{aligned} \tag{4.12}$$

for all vector fields $X, Y$ on $M^3(f_1, f_2, f_3)$. Adding the preceding equation with (4.11) yields

$$\begin{aligned} & \{2h(f_1-f_3)(\xi u)+2(\xi\lambda)-\frac{2\lambda}{h}(\xi h)\}g(X,Y)+4\beta(f_1-f_3)g(\phi X,Y) \\ + \quad & \{\frac{\lambda}{h}(\xi h)-(\xi\lambda)-h(\xi u)\}\eta(X)\eta(Y)-\beta g(Q\phi X+\phi QX,Y) \\ + \quad & \{\frac{(\lambda-2(f_1-f_3))}{h}(Yh)-h(f_1-f_3)(Yu)-(Y\lambda)\}\eta(X) \\ + \quad & \frac{1}{h}(\xi h)\{g(QX,Y)+g(Q\phi X,\phi Y)\}=0. \end{aligned} \tag{4.13}$$

Anti-symmetrizing the foregoing equation provides

$$\begin{aligned} & \{\frac{(\lambda-2(f_1-f_3))}{h}(Yh)-h(f_1-f_3)(Yu)-(Y\lambda)\}\eta(X) \\ - \quad & \{\frac{(\lambda-2(f_1-f_3))}{h}(Xh)-h(f_1-f_3)(Xu)-(X\lambda)\}\eta(Y) \\ + \quad & 8\beta(f_1-f_3)g(\phi X,Y)-2\beta g(Q\phi X+\phi QX,Y)=0 \end{aligned} \tag{4.14}$$

for all vector fields $X, Y$ on $M^3(f_1, f_2, f_3)$. Moreover, substituting $X$ by $\phi X$ and $Y$ by $\phi Y$ in the last equation and using (2.9) and (2.1) gives

$$\beta\{g((Q\phi+\phi Q)X,Y)-4(f_1-f_3)g(\phi X,Y)\}=0$$

for all vector fields $X, Y$ on $M^3(f_1, f_2, f_3)$. It follows from last equation that either $\beta = 0$ or

$$(Q\phi+\phi Q)X=4(f_1-f_3)\phi X. \tag{4.15}$$

for any vector field $X$ on $M^3(f_1, f_2, f_3)$. Let us assume that $f_1 \neq f_3$. Then we know that $\beta$ is non zero. Hence, the equation (4.15) stands. Let $\{e, \phi e, \xi\}$ be an orthonormal $\phi$-basis of $M^3(f_1, f_2, f_3)$ such that $Qe = \sigma e$. Thus, we have $\phi Qe = \sigma\phi e$. Substituting $e$ for $X$ in (4.15) and using the foregoing equation, we obtain $Q\phi e = (4(f_- f_3) - \sigma)\phi e$. Using $\phi$-basis and (2.9) the scalar curvature is given by

$$\begin{aligned} r &= g(Q\xi,\xi)+g(Qe,e)+g(Q\phi e,\phi e) \\ &= 2(f_1-f_3)+\sigma+4(f_1-f_3)-\sigma \\ &= 6(f_1-f_3). \end{aligned}$$

For the scalar curvature $r = 6(f_1 - f_3)$, (2.12) gives us

$$R(X,Y)Z=(f_1-f_3)\{g(Y,Z)X-g(X,Z)Y\} \tag{4.16}$$

for any vector fields $X, Y, Z$ on $M^3(f_1, f_2, f_3)$. Hence from (4.16) it follows that $M^3(f_1, f_2, f_3)$ is of constant curvature $f_1 - f_3$. Thus, we state the following:

**Theorem 4.2.** *Let $M^3(f_1, f_2, f_3)$ be a three-dimensional quasi-Sasakian generalized Sasakian-space-form with $f_1 \neq f_3$. If $g$ is an $h$-almost gradient Ricci soliton, then $M^3(f_1, f_2, f_3)$ is of constant curvature $f_1 - f_3$.*

It is noted that the Weyl tensor vanishes on any three dimensional Riemannian manifold. Therefore we may consider Cotton tensor which is another conformal invariant of a three-dimensional Riemannian manifold. The Cotton tensor $\mathfrak{C}(X, Y)$ of type (1,1) is defined by: (see [6, 9, 41])

$$\mathfrak{C}(X,Y) = (\nabla_X Q)(Y) - (\nabla_Y Q)(X) - \frac{1}{4}\{dr(X)(Y) - dr(Y)(X)\}$$

for any vector fields $X$ and $Y$ on $M^3$. A three-dimensional Riemannian manifold is said to be conformally flat if the Cotton tensor $\mathfrak{C}$ vanishes.

Since the manifold under consideration is of constant curvature, that is, the scalar curvature $r$ is constant, therefore the Cotton tensor vanishes. From the above discussions, we conclude the following:

**Corollory 4.3.** Let $M^3(f_1, f_2, f_3)$ be a 3-dimensional quasi-Sasakian generalized Sasakian-space-form with $f_1 \neq f_3$. If $g$ is an $h$-almost gradient Ricci soliton, then the Cotton tensor $\mathfrak{C}$ vanishes on $M^3(f_1, f_2, f_3)$.

# References


[1] P. Alegre, D. E. Blair and A. Carriazo, *Generalized Sasakian-space-forms*, Israel J. Math., **141**(2004), 157–183.

[2] P. Alegre and A. Carriazo, *Structures on generalized Sasakian-space-forms*, Differential Geom. Appl., **26(6)**(2008), 656–666.

[3] P. Alegre and A. Carriazo, *Submanifolds generalized Sasakian-space-forms*, Taiwanese J. Math., **13(3)**(2009), 923–941.

[4] P. Alegre and A. Carriazo, *Generalized Sasakian-space-forms and conformal change of metric*, Results Math., **59**(2011), 485–493.

[5] A. Barros and Jr. Ribeiro, *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc., **140(3)**(2012), 1033–1040.

[6] A. L. Besse, *Einstein manifolds*, Classics in mathematics, Springer-Verlag, berlin(1987).

[7] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., Springer-Verlag(1976).

[8] C. Calin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malays. Math. Soc., **33(3)**(2010), 361–368.

[9] S. K. Chaubey, U. C. De and Y. J. Suh, *Kenmotsu manifolds satisfying the Fischer-Marsden equation*, J. Korean Math. Soc. **58(3)** (2021), 597–607.

[10] S. K. Chaubey and Y. J. Suh, *Ricci-Bourguignon solitons and Fischer-Marsden conjecture on generalized Sasakian-space-forms with $\beta$-Kenmotsu structure*, J. Korean Math. Soc. (2023), https://doi.org/10.4134/JKMS.j220057.

[11] S. K. Chaubey and G.-E. Vlcu, *Gradient Ricci solitons and Fischer-Marsden equation on cosymplectic manifolds*, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. RACSAM **116(4)** (2022), Paper No. 186, 14 pp.

[12] J. T. Cho, *Ricci solitons in almost contact geometry*, Proceedings of the 17th International Workshop on Differential Geometry [Vol. 17], 8595, Natl. Inst. Math. Sci. (NIMS), Taejon(2013).

[13] J. T. Cho and R. Sharma, *Contact geometry and Ricci solitons*, Int. J. Geom. Methods Mod. Phys., **7(6)**(2010), 951–960.

[14] U. C. De and A. Sarkar, *On the projective curvature tensor of generalized Sasakian-space-forms*, Quaest. Math., **33(2)**(2010), 245–252.

[15] K. Erken, *Yamabe solitons on three-dimensional normal almost paracontact metric manifolds*, Period. Math. Hungar., **80(2)**(2020), 172–184.

[16] A. Ghosh, *Certain contact metrics as Ricci almost solitons*, Results Math., **65 (1-2)**(2014), 81–94.

[17] A. Ghosh and R. Sharma, *Sasakian metric as a Ricci soliton and related results*, J. Geom. Phys., **75**(2013), 1–6.

[18] A. Ghosh and R. Sharma, *K-contact metrics as Ricci solitons*, Beitr. Algebra Geom., **53(1)**(2012), 25–30.

[19] A. Ghosh and D. S. Patra, *The k-almost Ricci solitons and contact geometry*, J. Korean Math. Soc., **55(1)**(2018), 161–174.

[20] R. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Int. Press, Cambridge(1995).

[21] R. Hamilton, *The Ricci flow on surfaces*, Contemp. Math., **71**(1988), 237–262.

[22] S. K. Hui and A. Sarkar, *On the $W_2$-curvature tensor of generalized Sasakian-space-forms*, Math. Pannon., **23(1)**(2012), 113–124.

[23] S. K. Hui and D. G. Prakasha, *On the C-Bochner curvature tensor of generalized Sasakian-space-forms*, Proc. Nat. Acad. Sci. India Sect. A, **85(3)**(2015), 401–405.

[24] S. K. Hui, D. G. Prakasha and V. Chavan, *On generalized $\phi$-recurrent generalized Sasakian-space-forms*, Thai J. Math., **15(2)**(2017), 323–332.

[25] S. K. Hui, S. Uddin, A. H. Alkhaldi and P. Mandal, *Invariant submanifolds of generalized Sasakian-space-forms*, Int. J. Geom. Methods Mod. Phys., **15(9)**(2018), 21pp.

[26] D. Kar and P. Majhi, *$\beta$-almost Ricci solitons on almost co-Kähler manifolds,* Korean J. Math., **27(3)**(2019), 691–705.

[27] U. K. Kim, *Conformally flat generalized Sasakian-space-forms and locally symmetric generalized Sasakian-space-forms*, Note Mat., **26(1)**(2006), 55–67.

[28] P. Majhi and U. C. De, *On three dimensional generalized Sasakian-space-forms*, J. Geom., **108(3)**(2017), 1039–1053.

[29] P. Majhi and D. Kar, *$\beta$-almost Ricci solitons on Sasakian 3-manifolds,* Cubo, **21(3)**(2019), 63–74.

[30] Z. Olszak, *Normal almost contact metric manifolds of dimension three*, Ann. Polon. Math., **47**(1986), 41–50.

[31] S. Pahan, T Dutta and A. Bhattacharyya, *Ricci solitons and $\eta$-Ricci solitons on generalized Sasakian-space-forms*, Filomat, **31(13)**(2017), 4051–4062.

[32] G. Perelman, *Ricci flow with surgery on three-manifolds*, preprint, arXiv:math.DG/0303109.

[33] S. Pigola, M. Rigoli, M. Rimoldi and A. Setti, *Ricci almost solitons*, Ann. Sc. Norm. Super. Pisa Cl. Sci., **10(4)**(2011), 757–799.

[34] D. G. Prakasha, *On generalized Sasakian-space-forms with Weyl-conformal curvature tensor*, Lobachevskii J. Math., **33(3)**(2012), 223–228.

[35] D. G. Prakasha and H. G. Nagaraja, *On quasi-conformally flat and quasi-conformally semisymmetric generalized Sasakian-space-forms*, Cubo, **15(3)**(2013), 59–70.

[36] D. G. Prakasha and V. Chavan, *E-Bochner curvature tensor on generalized Sasakian-space-forms*, C. R. Math. Acad. Sci. Paris, **354(8)**(2016), 835–841.

[37] A. Sarkar, M. Sen and A. Akbar, *Generalized Sasakian-space-forms with conharmonic curvature tensor*, Palest. J. Math., **4(1)**(2015), 84–90.

[38] R. Sharma, *Almost Ricci solitons and K-contact geometry*, Monatsh. Math., **175**(2014), 621–628.

[39] Q. Wang, J. N. Gomes and C. Xia, *On the h-almost Ricci soliton*, J. Geom. Phys., **114**(2017), 216–222.

[40] Y. Wang, U. C. De and X. Liu, *Gradient Ricci solitons on almost Kenmotsu manifolds,* Publ. Inst. Math., **98**(2015), 227–235.

[41] Y. Wang, *Cotton tensors on almost coKahler 3-manifolds*, Ann. Polon. Math., **120(2)**(2017), 135–148.