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Generalized G-Metric Spaces

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Abstract. In this paper, we propose the notion of a distance between n points, called a g-metric, which is a further generalized G-metric. Indeed, it is shown that the g-metric with dimension 2 is the ordinary metric and the g-metric with dimension 3 is equivalent to the G-metric.

1. Introduction

A metric is a measurement how far apart each pair elements of a given set are. Without a doubt, a metric is one of the most important notions in mathematics and many other scientific fields. For instance, a metric is used to quantify a dissimilarity (or equivalently similarity) between two objects in some sense. The definition of a metric was proposed by M. Fréchet [4] in 1906.

Definition 1.1. [4] Let Ω be a nonempty set. A function $d : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ is called a *metric* or *distance function* on Ω if it satisfies the following conditions:

(1) (identity) $d(x, y) = 0$ if and only if $x = y$,
(2) (non-negativity) $d(x, y) > 0$ if $x \neq y$,
(3) (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in \Omega$,
(4) (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \Omega$.

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The pair (Ω, d) is called a *metric space*.

In 1963, Gahler [5] generalized an ordinary metric space, called a 2-metric space. It, however, was shown in [6] that not every 2-metric is continuous and there is no strong connection between fixed point theorems in an ordinary metric space and in a 2-metric space, which means that a 2-metric space is not a natural generalization of an ordinary metric space. For this reason, Dhage [2] introduced a newly generalized metric space, called D-metric space, and related fixed point theorems. However, Mustafa and Sims [8] pointed out that similar problems occur in the setting of Dhage, and they [9] proposed an appropriate notion of a generalized metric space. See [1] and references therein for more details.

Definition 1.2. [9] Let Ω be a nonempty set. A function $G : \Omega \times \Omega \times \Omega \longrightarrow \mathbb{R}_+$ is called a *G-metric* on Ω if it satisfies the following conditions:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $G(x, x, y) > 0$ for all $x, y \in \Omega$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in \Omega$ with $y \neq z$,
(G4) $G(x, y, z) = G(x, z, y) = \cdots$ (symmetry in all three variables x, y, z),
(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in \Omega$.

The pair (Ω, G) is called a *G-metric space*. A G-metric space (Ω, G) is said to be *symmetric* if

(G6) $G(x, y, y) = G(x, x, y)$ for all $x, y \in \Omega$.

More generalized measurement methods are required to be considered in order to analyze more complex data sets such as grouped multivariate data. In this paper, we propose a generalized notion of a metric between n points, called a g-metric. It coincides with the ordinary distance between two points and with the G-metric between three points. Furthermore, we establish fundamental topological notions and properties on the g-metric space including the convergence of sequences and continuity of mappings.

2. Structure of A g-Metric Space

Let $\mathbb{N}$ (resp. $\mathbb{R}$) be the set of all nonnegative integers (resp. all real numbers). We denote as $\mathbb{R}_+$ the set of all nonnegative real numbers. For a finite set A, we denote the number of distinct elements of A by $n(A)$.

We now propose a new definition of a generalized metric for n number of points instead of two or three points in a given set. For a set Ω, we denote $\Omega^n := \prod_{i=1}^{n} \Omega$.

Definition 2.1. Let Ω be a nonempty set. A function $g : \Omega^n \longrightarrow \mathbb{R}_+$ is called a *generalized metric* or simply *g-metric with dimension* n ($n \geq 2$) on Ω if it satisfies the following conditions:

($g1$) (positive definiteness) $g(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n$,
($g2$) (permutation invariancy) $g(x_1, \ldots, x_n) = g(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for any permutation σ on $\{1, \ldots, n\}$,

($g3$) (monotonicity) $g(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \Omega^n$ with $\{x_i : i = 1, \ldots, n\} \subsetneq \{y_i : i = 1, \ldots, n\}$,

($g4$) (triangle inequality) for all $x_1, \ldots, x_s, y_1, \ldots, y_t, w \in \Omega$ with $s + t = n$

$$g(x_1, \ldots, x_s, y_1, \ldots, y_t) \leq g(x_1, \ldots, x_s, w, \ldots, w) + g(y_1, \ldots, y_t, w, \ldots, w).$$

The pair (Ω, g) is called a *g-metric space*.

Definition 2.2. A g-metric on Ω is called *multiplicity-independent* if the following holds

$$g(x_1, \ldots, x_n) = g(y_1, \ldots, y_n)$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \Omega^n$ with $\{x_i : i = 1, \ldots, n\} = \{y_i : i = 1, \ldots, n\}$.

Note that for a given multiplicity-independent g-metric with dimension 3, it holds that $g(x, y, y) = g(x, x, y)$. For a given multiplicity-independent g-metric with dimension 4, it holds that $g(x, y, y, y) = g(x, x, y, y) = g(x, x, x, y)$ and $g(x, x, y, z) = g(x, y, y, z) = g(x, y, z, z)$.

Remark 2.3. If we allow equality under the condition of monotonicity in Definition 2.1, i.e., "$g(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)$ for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \Omega^n$ with $\{x_i : i = 1, \ldots, n\} \subseteq \{y_i : i = 1, \ldots, n\}$", then every g-metric becomes multiplicity-independent.

Let us explain why the condition ($g4$) can be considered as a generalization of the triangle inequality. Recall that the triangle inequality condition for a distance function d is $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z.

The point w is required to measure approximately the distance between x and y with the distances between x and w and between w and y. Note that one cannot measure the distance between x and y by the distances $d(x, w_1)$ and $d(y, w_2)$ with $w_1 \neq w_2$. Consider $d(x, y)$ as a dissimilarity between x and y. Clearly, if $x = y$, then the dissimilarity is 0, vice versa. Also, the dissimilarity between x and y is same as the dissimilarity between y and x. If x (resp. y) and z (resp. z) are sufficiently similar, then by the triangle inequality x and y must be sufficiently similar.

In the similar way, one can generalize the definition of triangle inequality for the g-metric. Specifically, one can see from the definition of triangle inequality for the g-metric that if both $g(x_1, \ldots, x_s, w, \ldots, w)$ and $g(y_1, \ldots, y_t, w, \ldots, w)$ are sufficiently small, then $g(x_1, \ldots, x_s, y_1, \ldots, y_t)$ must be sufficiently small. That is, the higher similarities two data sets $\{x_1, \ldots, x_s, w\}$ and $\{y_1, \ldots, y_t, w\}$ have, the higher similarity data set $\{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ does. Note that w is a necessary point to combine information about similarity for each data set.

The following theorem shows us that g-metrics generalize the notions of ordinary metric and G-metric.

Theorem 2.4. *Let Ω be a given nonempty set. The following are true.*

(1) *d is a g-metric with dimension 2 on Ω if and only if d is a metric on Ω.*

(2) *d is a (resp. multiplicity-independent) g-metric with dimension 3 on Ω if and only if d is a (resp. symmetric) G-metric on Ω.*

Remark that since a g-metric with dimension 3 on a nonempty set Ω is a G-metric, any g-metrics with dimension 3 satisfy all properties of the G-metric as shown in [9].

A new g-metric can be constructed from given g-metrics. The proof is left to the reader.

Lemma 2.5. *Let (Ω, g) and $(\Omega, \tilde{g})$ be g-metric spaces. Then the following functions, denoted by d, are g-metrics on Ω.*

(1) $d(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) + \tilde{g}(x_1, x_2, \ldots, x_n)$.

(2) $d(x_1, x_2, \ldots, x_n) = \psi(g(x_1, x_2, \ldots, x_n))$ *where ψ is a function on* $[0, \infty)$ *satisfies*
   (i) *ψ is increasing on* $[0, \infty)$*;*
   (ii) $\psi(0) = 0$*;*
   (iii) $\psi(x + y) \le \psi(x) + \psi(y)$ *for all* $x, y \in [0, \infty)$.

Example 2.6. The following functions, denoted by ψ, satisfy the conditions in Lemma 2.5 (2). Thus, each $\psi \circ g$ is a g-metric for any g-metric g.

(1) $(\psi \circ g)(x_1, \ldots, x_n) = kg(x_1, \ldots, x_n)$ where $\psi(x) = kx$ with a fixed $k > 0$.

(2) $(\psi \circ g)(x_1, \ldots, x_n) = \dfrac{g(x_1, \ldots, x_n)}{1 + g(x_1, \ldots, x_n)}$ where $\psi(x) = \dfrac{x}{1 + x}$.

(3) $(\psi \circ g)(x_1, \ldots, x_n) = \sqrt{g(x_1, \ldots, x_n)}$ where $\psi(x) = \sqrt{x}$. Furthermore, it is true for $\psi(x) = x^{1/p}$ with a fixed $p \ge 1$.

(4) $(\psi \circ g)(x_1, \ldots, x_n) = \log(g(x_1, \ldots, x_n) + 1)$ where $\psi(x) = \log(x + 1)$.

(5) $(\psi \circ g)(x_1, \ldots, x_n) = \min\{k, g(x_1, \ldots, x_n)\}$ where $\psi(x) = \min\{k, x\}$ with a fixed $k > 0$.

Lemma 2.7. *Let g be a g-metric with dimension n on a nonempty set Ω. The following are true:*

(1) $g(\underbrace{x, \ldots, x}_{s\ times}, y, \ldots, y) \le g(\underbrace{x, \ldots, x}_{s\ times}, w, \ldots, w) + g(\underbrace{w, \ldots, w}_{s\ times}, y, \ldots, y)$,

(2) $g(x, y, \ldots, y) \le g(x, w, \ldots, w) + g(w, y, \ldots, y)$,

(3) $g(\underbrace{x, \ldots, x}_{s\ times}, w, \ldots, w) \le sg(x, w, \ldots, w)$ *and*

$g(\underbrace{x, \ldots, x}_{s\ times}, w, \ldots, w) \le (n - s)g(w, x, \ldots, x)$,

(4) $g(x_1, x_2, \ldots, x_n) \le \displaystyle\sum_{i=1}^{n} g(x_i, w, \ldots, w)$,

(5) $\big|g(y, x_2, \ldots, x_n) - g(w, x_2, \ldots, x_n)\big| \le \max\{g(y, w, \ldots, w), g(w, y, \ldots, y)\}$,

(6) $\big|g(\underbrace{x, \ldots, x}_{s\ times}, w, \ldots, w) - g(\underbrace{x, \ldots, x}_{\tilde{s}\ times}, w, \ldots, w)\big| \le \big|s - \tilde{s}\big| g(x, w, \ldots, w)$.

(7) $g(x, w, \ldots, w) \le (1 + (s - 1)(n - s))g(\underbrace{x, \ldots, x}_{s\ times}, w, \ldots, w)$,

Proof. (1) and (2) follow from the condition $(g4)$. Note that for a multiplicity-independent g-metric g, it is true that $g(y, w, \ldots, w) = g(w, y, \ldots, y)$.

(3) By the condition $(g4)$, it follows that

$$\begin{aligned} g(\underbrace{x,\dots,x}_{s\text{ times}},w,\dots,w) &\le g(\underbrace{x,\dots,x}_{s-1\text{ times}},w,w)+g(x,w,\dots,w)\\ &\le g(\underbrace{x,\dots,x}_{s-2\text{ times}},w,w,w)+g(x,w,\dots,w)+g(x,w,\dots,w)\\ &\ \ \vdots\\ &\le sg(x,w,\dots,w). \end{aligned}$$

(4) By the condition $(g2)$ and $(g4)$, it follows that

$$\begin{aligned} g(x_1,x_2,\dots,x_n) &\le g(x_1,w,\dots,w)+g(x_2,x_3,\dots,x_n,w)\\ &\le g(x_1,w,\dots,w)+g(x_2,w,\dots,w)+g(x_3,\dots,x_n,w,w)\\ &\ \ \vdots\\ &\le \sum_{i=1}^{n} g(x_i,w,\dots,w). \end{aligned}$$

(5) By the condition $(g4)$, we get the inequality

$$g(y,x_2,\dots,x_n)\le g(w,x_2,\dots,x_n)+g(y,w,\dots,w).$$

So

$$g(y,x_2,\dots,x_n)-g(w,x_2,\dots,x_n)\le g(y,w,\dots,w).$$

Similarly, we have

$$g(w,x_2,\dots,x_n)-g(y,x_2,\dots,x_n)\le g(w,y,\dots,y).$$

(6) By (3), it is trivial.

(7) By Lemma 2.7 (3), we have

$$\begin{aligned} g(x,w,\dots,w) &\le g(x,x,w,\dots,w)+g(w,x,\dots,x)\\ &\le g(x,x,x,w,\dots,w)+g(w,x,\dots,x)+g(w,x,\dots,x)\\ &\ \ \vdots\\ &\le g(\underbrace{x,\dots,x}_{s\text{ times}},w,\dots,w)+(s-1)g(w,x,\dots,x)\\ &\le g(\underbrace{x,\dots,x}_{s\text{ times}},w,\dots,w)+(s-1)(n-s)g(\underbrace{x,\dots,x}_{s\text{ times}},w,\dots,w)\\ &= (1+(s-1)(n-s))g(\underbrace{x,\dots,x}_{s\text{ times}},w,\dots,w). \end{aligned}$$

□

For a given g-metric, we can construct a distance function.

Proposition 2.8. *For any g-metric space (Ω, g), the following are distance functions.*

(1) $d(x,y) = g(\underbrace{x,\ldots,x}_{s\ times}, y,\ldots,y) + g(\underbrace{y,\ldots,y}_{s\ times} x,\ldots,x)$,

(2) $d(x,y) = g(x,y,\ldots,y) + g(x,x,y,\ldots,y) + \cdots + g(x,x,\ldots,x,y)$,

(3) $d(x,y) = \max\{g(x_1,x_2,\ldots,x_n) : x_i \in \{x,y\},\ 1 \le i \le n\}$.

We give several interesting examples of g-metric on a variety of settings in the following.

Example 2.9. (1) (Discrete g-metric) For a nonempty set Ω, define $d : \Omega^n \to \mathbb{R}_+$ by

$$d(x_1,\ldots,x_n) = \begin{cases} 0 & \text{if } x_1 = \cdots = x_n, \\ 1 & \text{otherwise} \end{cases}$$

for all $x_1,\ldots,x_n \in \Omega$. Then d is a g-metric on Ω.

(2) (Diameter g-metric) Define $d : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ by

$$d(x_1,\ldots,x_n) = \max_{1\le i\le n} x_i - \min_{1\le j\le n} x_j$$

for all $x_1,\ldots,x_n \in \mathbb{R}_+$. Then d is a g-metric on $\mathbb{R}_+$.

(3) (Average g-metric) For a given metric space (Ω,δ), define $d : \Omega^n \longrightarrow \mathbb{R}_+$ by

$$d(x_1,\ldots,x_n) = \frac{1}{n^2}\sum_{i,j=1}^{n} \delta(x_i,x_j)$$

for all $x_1,\ldots,x_n \in \Omega$. Then d is a g-metric on Ω.

(4) (Max g-metric) For a given metric space (Ω,δ), define $d : \Omega^n \longrightarrow \mathbb{R}_+$ by

$$d(x_1,\ldots,x_n) = \max_{1\le i,j\le n} \delta(x_i,x_j)$$

for all $x_1,\ldots,x_n \in \Omega$. Then d is a g-metric on Ω.

(5) (Shortest path g-metric) For a given metric space (Ω,δ), define $d : \Omega^n \longrightarrow \mathbb{R}_+$ by

$$d(x_1,\ldots,x_n) = \min_{\pi\in\mathcal{S}} \sum_{i=1}^{n-1} \delta(x_{\pi(i)}, x_{\pi(i+1)})$$

for all $x_1,\ldots,x_n \in \Omega$.

Here, $\mathcal{S}$ denotes the set of all permutations on $\{1,\ldots,n\}$. So $d(x_1,\ldots,x_n)$ is the length of the shortest path connecting $x_1,\ldots,x_n$. Finding the shortest path is very important problem in operations research and theoretical computer science, which is also known as the traveling salesman problem[10, 12].

(6) (Smallest ball g-metric) Let Ω be a nonempty subset of $\mathbb{R}^n$, i.e., Ω can be considered as an n-dimensional data set. Define $d : \Omega^n \longrightarrow \mathbb{R}_+$ by $d(x_1,\ldots,x_n)$ is the diameter of the smallest closed ball, B, such that $\{x_1,\ldots,x_k\} \subseteq B$. This is called the smallest enclosing circle problem, which was introduced by Sylvester[11]. For more information, see [3, 7]. It

is an open problem that d is a g-metric for any $n \geq 4$.

Remark 2.10.

(1) For a nonempty normed space $(\Omega, \|\cdot\|)$, let us define a map $d : \Omega^n \longrightarrow \mathbb{R}_+$ by

$$d(x_1, \ldots, x_n) = \max_{1 \leq i \leq n} \|x_i\| - \min_{1 \leq j \leq n} \|x_j\|$$

for all $x_1, \ldots, x_n \in \Omega$. Then it is not a g-metric on Ω. In fact, it holds ($g2$), ($g3$), and ($g4$), but does not hold ($g1$) in general. Indeed, there possibly exist $x_1, x_2, \ldots, x_n \in \Omega$ such that $\|x_1\| = \|x_2\| = \cdots = \|x_n\|$ although $x_i \neq x_j$ for some $i \neq j$.

(2) In Example 2.9 (3), on a given metric space (Ω, δ)

$$d(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} \delta(x_i, x_j)$$

is a g-metric by Example 2.6 (1). Then this g-metric and the max g-metric in Example 2.9 (4) can be considered as

$$d(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} \delta(x_i, x_j) = ||M||_1,$$
$$d(x_1, \ldots, x_n) = \max_{1 \leq i,j \leq n} \delta(x_i, x_j) = ||M||_\infty,$$

where $M = [m_{ij}]_{1 \leq i,j \leq n}$ is the $n \times n$ matrix whose entries are $m_{ij} = \delta(x_i, x_j)$. Here, $|| \cdot ||_1$ and $|| \cdot ||_\infty$ are ℓ_1 and ℓ_∞ matrix norms, respectively. So it is a natural question whether or not $||M||_p$ for $1 < p < \infty$ is a g-metric on the metric space (Ω, δ).

3. Topology on A g-Metric Space

For a given metric space (Ω, d), we denote the ball centered at x with radius r by $B_d(x, r)$. We define a ball on a g-metric space.

Definition 3.1. Let (Ω, g) be a g-metric space. For $x \in \Omega$ and $r > 0$, the ball centered at x with radius r is

$$B_g(x, r) = \{y \in \Omega : g(x, y, \ldots, y) < r\}.$$

Proposition 3.2. *Let (Ω, g) be a g-metric space. Then the following hold.*

(1) *If $g(x_1, x_2, \ldots, x_n) < r$ and $n(\{x_1, x_2, \ldots, x_n\}) \geq 3$, then $x_i \in B_g(x_1, r)$ for all $i = 1, \ldots, n$.*

(2) *If g is multiplicity-independent and $g(x_1, x_2, \ldots, x_n) < r$, then $x_i \in B_g(x_1, r)$ for all $i = 1, \ldots, n$.*

(3) *Let $y \in B_g(x_1, r_1) \cap B_g(x_2, r_2)$. Then there exists $\delta > 0$ such that $B_g(y, \delta) \subseteq B_g(x_1, r_1) \cap B_g(x_2, r_2)$.*

Proof. Suppose that $g(x_1, x_2, \ldots, x_n) < r$. Set $X = \{x_1, x_2, \ldots, x_n\}$.

(1) Since $n(X) \geq 3$, clearly $\{x_1, x_i, x_i, \ldots, x_i\} \subsetneq X$ for each $i \in \mathbb{N}$. By monotonicity of the g-metric, we have $g(x_1, x_i, \ldots, x_i) \leq g(x_1, x_2, \ldots, x_n) < r$. So $x_i \in B_g(x_1, r)$ for all $i \in \mathbb{N}$.

(2) It suffices to show that it holds for $n(X) = 2$. Since a g-metric is multiplicity-independent, $g(x_1, x_i, \ldots, x_i) \leq g(x_1, x_2, \ldots, x_n) < r$.

(3) Since $y \in B_g(x_1, r_1) \cap B_g(x_2, r_2)$, it holds that $g(x_i, y, \ldots, y) < r_i$ for $i = 1, 2$. We take $\delta = \min\{r_i - g(x_i, y, \ldots, y) : i = 1, 2\}$. Then for every $z \in B_g(y, \delta)$, by Lemma 2.7 (2) we have $g(x_i, z, \ldots, z) \leq g(x_i, y, \ldots, y) + g(y, z, \ldots, z) < g(x_i, y, \ldots, y) + \delta < r_i$ for each $i = 1, 2$. Therefore, $B_g(y, \delta) \subseteq B_g(x_1, r_1) \cap B_g(x_2, r_2)$.

□

Due to the preceding proposition, the collection of all balls, $\mathcal{B} = \{B_g(x, r) : x \in \Omega, r > 0\}$ forms a basis for a topology on Ω. We call the topology generated by $\mathcal{B}$ the *g-metric topology* on Ω.

Theorem 3.3. *Let (Ω, g) be a g-metric space and let $d(x, y) = g(x, y, \ldots, y) + g(y, x, \ldots, x)$. Then*

$$B_g\Big(x_1, \frac{r}{n}\Big) \subseteq B_d(x_1, r) \subseteq B_g(x_1, r).$$

Proof. Recall that $y \in B_g(x_1, r) \iff g(x_1, y, \ldots, y) < r$.

(i) Let $x \in B_g\Big(x_1, \frac{r}{n}\Big)$. Then $g(x_1, x, \ldots, x) < \frac{r}{n}$. It follows that

$$\begin{aligned} d(x_1, x) &= g(x_1, x, \ldots, x) + g(x, x_1, \ldots, x_1) \\ &\leq g(x_1, x, \ldots, x) + (n-1)g(x_1, x, \ldots, x) \\ &\leq ng(x_1, x, \ldots, x) < r. \end{aligned}$$

So, $x \in B_d(x_1, r)$.

(ii) Let $x \in B_d(x_1, r)$. Then $d(x_1, x) = g(x_1, x, \ldots, x) + g(x, x_1, \ldots, x_1) < r$. Since $g(x_1, x, \ldots, x) \leq (n-1)g(x, x_1, \ldots, x_1)$, it follows that

$$\frac{n}{n-1} g(x_1, x, \ldots, x) \leq g(x_1, x, \ldots, x) + g(x, x_1, \ldots, x_1) < r.$$

Thus, $g(x_1, x, \ldots, x) < r$, i.e., $x \in B_g(x_1, r)$ as desired. □

Remark 3.4. Every g-metric space is topologically equivalent to a metric space arising from the metric d defined in Theorem 3.3. This makes it possible to transport many concepts and results from metric spaces into the g-metric setting.

Definition 3.5. Let (Ω, g) be a g-metric space. Let $x \in \Omega$ be a point and $\{x_k\} \subseteq \Omega$ be a sequence.

(1) $\{x_k\}$ *converges* to x, denoted by $\{x_k\} \xrightarrow{g} x$, if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$i_1, \ldots, i_{n-1} \geq N \Longrightarrow g(x, x_{i_1}, \ldots, x_{i_{n-1}}) < \varepsilon.$$

For such a case, $\{x_k\}$ is said to be *convergent* in Ω and x is called the *limit* of $\{x_k\}$.

(2) $\{x_k\}$ is said to be *Cauchy* if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$i_1, \ldots, i_n \geq N \Longrightarrow g(x_{i_1}, \ldots, x_{i_n}) < \varepsilon.$$

(3) (Ω, g) is *complete* if every Cauchy sequence in (Ω, g) is convergent in (Ω, g).

Proposition 3.6. *The following are true.*

(1) *The limit of a convergent sequence in a g-metric space is unique.*
(2) *Every convergent sequence in a g-metric space is a Cauchy sequence.*

Proof. (1) Let (Ω, g) be a g-metric space and let $\{x_k\} \subseteq \Omega$ be a convergent sequence. Suppose that $x, y \in \Omega$ are the limits of $\{x_k\}$. By Definition 3.5 (1), there exists $N_1, N_2 \in \mathbb{N}$ such that

$$g(x, x_{i_1}, \ldots, x_{i_{n-1}}) < \frac{\varepsilon}{n} \quad \text{for all } i_1, \ldots, i_n \geq N_1,$$
$$g(y, x_{i_1}, \ldots, x_{i_{n-1}}) < \frac{\varepsilon}{n} \quad \text{for all } i_1, \ldots, i_n \geq N_2.$$

Set $N = \max\{N_1, N_2\}$. If $m \geq N$, then by the condition $(g4)$ and Lemma 2.7 (3), it follows that

$$\begin{aligned} g(x, y, y, \ldots, y) &\leq g(x, x_m, x_m, \ldots, x_m) + g(x_m, y, y, \ldots, y) \\ &\leq g(x, x_m, x_m, \ldots, x_m) + (n-1)g(y, x_m, x_m, \ldots, x_m) \\ &< \frac{\varepsilon}{n} + \frac{(n-1)\varepsilon}{n} = \varepsilon. \end{aligned}$$

Since ε is arbitrary, $g(x, y, y, \ldots, y) = 0$. Thus, $x = y$ by the condition $(g1)$.

(2) Let (Ω, g) be a g-metric space and let $\{x_k\} \subseteq \Omega$ be a convergent sequence with the limit x. By Definition 3.5 (1), there exists $N \in \mathbb{N}$ such that

$$g(x, x_{i_1}, \ldots, x_{i_{n-1}}) < \frac{\varepsilon}{n} \quad \text{for all } i_1, \ldots, i_{n-1} \geq N.$$

By Lemma 2.7 (4) and the monotonicity condition for the g-metric, it follows that

$$g(x_{i_1}, \ldots, x_{i_n}) \leq \sum_{k=1}^{n} g(x_{i_k}, x, x, \ldots, x) < \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon.$$

Thus, $\{x_k\}$ is a Cauchy sequence in (Ω, g).

□

Lemma 3.7. *Let (Ω, g) be a g-metric space. Let $\{x_k\} \subseteq \Omega$ be a sequence and $x \in \Omega$. The following are equivalent.*

(1) $\{x_k\} \xrightarrow{g} x$.
(2) *For a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_k \in B_g(x, \varepsilon)$ for all $k \geq N$.*

(3) $\lim_{k_1,\dots,k_s\to\infty} g(\underbrace{x_{k_1},\dots,x_{k_s}}_{s\ times},x,\dots,x)=0$ *for a fixed* $1\le s\le n-1$. *That is, for all* $\varepsilon>0$, *there exists* $N\in\mathbb{N}$ *such that* $k_1,\dots,k_s\ge N$ *implies* $g(x_{k_1},\dots,x_{k_s},x,\dots,x)<\varepsilon$.

Proof. ((1) $\Longleftrightarrow$ (2)) It is clear by the definition of convergence.

((2) $\Longrightarrow$ (3)) Assume that for a given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $k\ge N$ implies $x_k\in B_g\Big(x,\dfrac{\varepsilon}{s}\Big)$, i.e., $g(x,x_k,\dots,x_k)<\dfrac{\varepsilon}{s}$. If $k_1,\dots,k_s\ge N$, then by Lemma 2.7 (4), we have that $g(x_{k_1},\dots,x_{k_s},x,\dots,x)\le\displaystyle\sum_{j=1}^{s} g(x,x_{k_j},\dots,x_{k_j})<\varepsilon$.

((3) $\Longrightarrow$ (2)) Let $\varepsilon>0$. Assume that there exists $N\in\mathbb{N}$ such that

$$k_1,\dots,k_s\ge N \Longrightarrow g(k_1,\dots,k_s,x,\dots,x)<\frac{\varepsilon}{(1+(s-1)(n-s))}.$$

If $k\ge N$, then by Lemma 2.7 (7) it follows that

$$g(x,x_k,\dots,x_k)\le(1+(s-1)(n-s))g(\underbrace{x_k,\dots,x_k}_{s\ \text{times}},x,\dots,x)<\varepsilon.$$

□

Lemma 3.8. *Let* (Ω,g) *be a* g*-metric space. Let* $\{x_k\}\subseteq\Omega$ *be a sequence. The following are equivalent.*

(1) $\{x_k\}$ *is Cauchy.*
(2) $g(x_k,x_{k+1},x_{k+1},\dots,x_{k+1})\longrightarrow 0$ *as* $k\longrightarrow\infty$.
(3) $\lim_{k,\ell\to\infty} g(\underbrace{x_k,\dots,x_k}_{s\ times},x_\ell,\dots,x_\ell)=0$ *for a fixed* $1\le s\le n-1$.

Proof. ((1) $\Longrightarrow$ (2)) It is trivial by Definition 3.5 (2).

((2) $\Longrightarrow$ (3)) Without loss of generality, we can assume $k<\ell$. Let $\varepsilon>0$ be given. Then for each $m=0,\dots,\ell-k-1$ there exists $N_m\in\mathbb{N}$ such that $g(x_{k+m},x_{k+m+1},\dots,x_{k+m+1})<\dfrac{\varepsilon}{n(\ell-k)}$. Let $N=\max\{N_0,\dots,N_{\ell-k-1}\}$. Then by Lemma 2.7 (3),(4), and the conditions ($g4$), we have that

$$\begin{aligned}
g(\underbrace{x_k,\dots,x_k}_{s\ \text{times}},x_\ell,\dots,x_\ell) &\le sg(x_k,x_\ell,\dots,x_\ell)\\
&\le s\big(g(x_k,x_{k+1},\dots,x_{k+1})+g(x_{k+1},x_\ell,\dots,x_\ell)\big)\\
&\ \ \vdots\\
&\le s\sum_{i=k}^{\ell-1} g(x_i,x_{i+1},\dots,x_{i+1})<\varepsilon,
\end{aligned}$$

for all $k\ge N$. If $k,\ell\ge N$, then $g(\underbrace{x_k,\dots,x_k}_{s\ \text{times}}x_\ell,\dots,x_\ell)<\varepsilon$.

((3) $\Longrightarrow$ (1)) Let $\varepsilon > 0$ be given. Assume that there exists $N \in \mathbb{N}$ such that

$$k, \ell \geq N \implies g(\underbrace{x_k, \ldots, x_k}_{s \text{ times}}, x_\ell, \ldots, x_\ell) < \frac{\varepsilon}{n(1+(s+1)(n-s))}.$$

If $i_0, i_1, \ldots, i_n \geq N$, then by Lemma 2.7 (4),(7) it follows that

$$\begin{aligned} g(x_{i_0}, x_{i_1}, \ldots, x_{i_n}) &\leq \sum_{k=0}^{n} g(x_{i_k}, x_{i_0}, \ldots, x_{i_0}) \\ &\leq \sum_{k=0}^{n} (1+(s+1)(n-s)) g(\underbrace{x_{i_k}, \ldots, x_{i_k}}_{s \text{ times}}, x_{i_0}, \ldots, x_{i_0}) < \varepsilon. \end{aligned}$$

□

Definition 3.9. Let (Ω, g) be a g-metric space, and let $\varepsilon > 0$ be given.

(1) A set $A \subseteq \Omega$ is called an *ε-net* of (Ω, g) if for each $x \in \Omega$, there exists $a \in A$ such that $x \in B_g(a, \varepsilon)$. If the set A is finite then A is called a *finite ε-net* of (Ω, g).

(2) A g-metric space (Ω, g) is called *totally bounded* if for every $\varepsilon > 0$ there exists a finite ε-net.

(3) A g-metric space (Ω, g) is called *compact* if it is complete and totally bounded.

Definition 3.10. Let (Ω_1, g_1) and (Ω_2, g_2) be g-metric spaces.

(1) A mapping $T : \Omega_1 \longrightarrow \Omega_2$ is said to be *continuous at a point* $x \in \Omega_1$ provided that for each open ball $B_{g_2}(T(x), \varepsilon)$, there exists an open ball $B_{g_1}(x, \delta)$ such that $T(B_{g_1}(x, \delta)) \subseteq B_{g_2}(T(x), \varepsilon)$.

(2) $T : \Omega_1 \longrightarrow \Omega_2$ is said to be *continuous* if it is continuous at every point of Ω_1.

(3) $T : \Omega_1 \longrightarrow \Omega_2$ is called a *homeomorphism* if T is bijective, and T and T^{-1} are continuous. In this case, the spaces Ω_1 and Ω_2 are said to be *homeomorphic*.

(4) A property P of g-metric spaces is called a *topological invariant* if P satisfies the condition:
If a space Ω_1 has the property P and if Ω_1 and Ω_2 are homeomorphic, then Ω_2 also has the property P.

Proposition 3.11. *Let (Ω_1, g_1) and (Ω_2, g_2) be g-metric spaces, and let $T : \Omega_1 \longrightarrow \Omega_2$ be a mapping. Then the following are equivalent.*

(1) *T is continuous.*

(2) *For each point $x \in \Omega_1$ and for each sequence $\{x_k\}$ in Ω_1 converging to x, $\{T(x_k)\}$ converges to $T(x)$.*

Proof. ((1) $\Longrightarrow$ (2)) Let $x \in \Omega_1$, and let $\{x_k\}$ be a sequence in Ω_1 converging to x. Since $T : \Omega_1 \longrightarrow \Omega_2$ is continuous, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $T(B_{g_1}(x, \delta)) \subseteq B_{g_2}(T(x), \varepsilon(n-1)^{-2})$. Since $\{x_k\} \xrightarrow{g} x$, there is $N \in \mathbb{N}$ such that

$g(x, x_{i_1}, \ldots, x_{i_{n-1}}) < \delta$ for all $i_1, \ldots, i_{n-1} \geq N$. Thus $g(x, x_{i_k}, \ldots, x_{i_k}) < \delta$ for each $k = 1, \ldots, n-1$. Then the continuity of T gives rise to the inequality

$$g(T(x), T(x_{i_k}), \ldots, T(x_{i_k})) < \frac{\varepsilon}{(n-1)^2}$$

for each $k \in \mathbb{N}$. By Lemma 2.7 (3) and (4) we have

$$\begin{aligned} g(T(x), T(x_{i_1}), \ldots, T(x_{i_{n-1}})) &\leq \sum_{k=1}^{n-1} g(T(x_{i_k}), T(x), \ldots, T(x)) \\ &\leq \sum_{k=1}^{n-1} (n-1) g(T(x), T(x_{i_k}), \ldots, T(x_{i_k})) < \varepsilon. \end{aligned}$$

Therefore, $\{T(x_k)\}$ converges to $T(x)$.

((2) $\Longrightarrow$ (1)) Suppose that T is not continuous, i.e. there exists $x \in \Omega_1$ such that T is not continuous at x. Then there exists $\varepsilon > 0$ such that for each $\delta > 0$ there is $y \in \Omega_1$ with $g(x, y, \ldots, y) < \delta$ but $g(T(x), T(y), \ldots, T(y)) \geq \varepsilon$. Then for each $k \in \mathbb{N}$ we can take $x_k \in \Omega_1$ such that $g(x, x_k, \ldots, x_k) < \frac{1}{k}$ but $g(T(x), T(x_k), \ldots, T(x_k)) \geq \varepsilon$. Hence, $\{x_k\}$ converges to x but $\{T(x_k)\}$ does not converges to $T(x)$, which contradicts to (2). □

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References

[1] R. P. Agarwal, E. Karapinar, D. O'Regan, and A. F. R.-L. de Hierro. *Fixed Point Theory in Metric Type Spaces*. Springer International Publishing (2015).

[2] B. C. Dhage. *Generalised metric spaces and mappings with fixed point*. Bull. Calcutta Math. Soc., **84(4)**(1992), 329–336.

[3] L. D. Drager, J. M. Lee, and C. F. Martin. *On the geometry of the smallest circle enclosing a finite set of points*. J. Franklin Inst., **344(7)**(2007), 929–940.

[4] M. Fréchet. *Sur quelques points du calcul fonctionnel*. Rendiconti del Circolo Matematico di Palermo, **22(1)**(1906), 1–74.

[5] S. Gahler. *2-metrische Raume und ihre topologische Strukture*. Math. Nachr, **26**(1963), 115–148.

[6] K. S. Ha, Y. J. Cho, and A. White. *Strictly convex and strictly 2-convex 2-normed spaces*. Math. Japonica, **33(3)**(1988), 375–384.

[7] B. Mordukhovich, N. M. Nam, and C. Villalobos. *The smallest enclosing ball problem and the smallest intersecting ball problem: existence and uniqueness of solutions*. Optim. Lett., **7(5)**(2013), 839–853.

[8] Z. Mustafa and B. Sims. *Some remarks concerning D-metric spaces*. International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama (2004), 189–198.

[9] Z. Mustafa and B. Sims. *A new approach to generalized metric spaces*. J. Nonlinear Convex Anal., **7(2)**(2006), 289–297.

[10] M. Padberg and G. Rinaldi. *A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems*. SIAM Review, **33(1)**(1991), 60–100.

[11] J. J. Sylvester. *A question in the geometry of situation*. Q. J. Pure Appl. Math., **79(1)**.

[12] S. Verblunsky. *On the shortest path through a number of points*. Proc. Am. Math. Soc., **2(6)**(1951), 904–913.