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ITERATIVE ALGORITHM FOR RANDOM GENERALIZED NONLINEAR MIXED VARIATIONAL INCLUSIONS WITH RANDOM FUZZY MAPPINGS

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Abstract. In this paper, we consider a class of random generalized nonlinear mixed variational inclusions with random fuzzy mappings and random relaxed cocoercive mappings in real Hilbert spaces. We suggest and analyze an iterative algorithm for finding the approximate solution of this class of inclusions. Further, we discuss the convergence analysis of the iterative algorithm under some appropriate conditions. Our results can be viewed as a refinement and improvement of some known results in the literature.

1. Introduction

In 1994, Hassouni and Moudafi [10] used the resolvent operator technique for maximal monotone mapping to study a class of mixed type variational inequalities with single-valued mappings which was called variational inclusions

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and developed a perturbed algorithm for finding the approximate solutions of the mixed type variational inequalities. Since then, many researchers have obtained some important extensions and generalizations of the results given in [10] in various different directions. For details, we refer to (see [1-8, 10-12, 14-19]).

In 1965, Zadeh [20] gave the notion of fuzzy sets as an extension of crisp sets, the usual two-valued sets in ordinary set theory, by enlarging the truth value set to the real unit interval $[0, 1]$. Ordinary fuzzy sets are characterized by, and mostly identified with, mapping called 'membership function' into $[0, 1]$. The basic operations and properties of fuzzy sets or fuzzy relations are defined by equations or inequalities between the membership functions. Heilpern [11] initiated the study of fuzzy mappings and established a fuzzy analogue of the Nadler's fixed point theorem [16] for multivalued mappings. Random variational inequality theory is an important part of random functional analysis. The fuzzy variational inequality (inclusions) problems have a close relation with fuzzy optimization problems. These topics have attracted many scholars and experts due to the extensive applications of random problems in modeling, optimization, engineering sciences and decision making problems(see [1-3, 5-8, 14-20]).

In 1989, Chang and Zhu [5] initiated the study of a class of variational inequalities with fuzzy mappings. The concept of random fuzzy mapping was first introduced by Huang [14]. In recent past, various classes of random variational inequalities involving fuzzy mappings have been introduced and studied by Cho and Lan [6], Ding [7], Huang [13], Noor [17] and Park and Jeong [18], etc.

Recently, Huang [14] developed an iterative scheme for a class of random variational inclusions with random fuzzy mappings and discussed its convergence analysis in real Hilbert space. Very recently, Ahmad and Bazan [1], Ahmad and Farajzadeh [2], Alshehri *et al.* [3], Ding and Park [8], Lan *et al.* [15] and Park and Jeong [19] introduced and studied some generalized classes of random variational inclusions with random fuzzy mappings in the setting of Hilbert and Banach spaces.

Motivated by the recent research work going in this field, in this paper, we consider a class of random generalized nonlinear mixed variational inclusions with random fuzzy mappings and random relaxed cocoercive mappings in real Hilbert space. Further, we suggest and analyze an iterative algorithm for finding the approximate solution of this class of inclusions. Furthermore, we discuss the convergence analysis of iterative algorithm under some appropriate conditions. Our results can be viewed as a refinement and improvement of some known results given in [1-3, 5-8, 13-15, 17-19].

2. Preliminaries

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let (Ω,Σ) be a measurable space, where Ω is a set in H and Σ is σ-algebra of subsets of Ω. Let $B(H)$ be the class of Borel σ-fields in H, $CB(H)$ be the collection of all nonempty, bounded and closed subsets of H and 2^H be the power set of H. The Hausdorff metric $\tilde{H}(\cdot,\cdot)$ on $CB(H)$ is defined by

$$\tilde{H}(A,B) \;=\; \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\},\ A,B\in CB(H). \tag{2.1}$$

First, we recall and define the following concepts and known results.

Definition 2.1. ([3]) A mapping $x:\Omega\to H$ is said to be *measurable*, if for any $B\in B(H)$, $\{t\in\Omega : x(t)\in B\}\in\Sigma$.

Definition 2.2. ([3]) A mapping $f:\Omega\times H\to H$ is said to be *random*, if for any $x\in H$, $f(t,x)=x(t)$ is measurable. A random mapping f is said to be *continuous* (resp. *linear, bounded*), if for any $t\in\Omega$, the mapping $f(t,\cdot):H\to H$ is continuous (resp. *linear, bounded*).

Remark 2.3. ([3]) It is well known that a measurable mapping is necessarily a random mapping. Similarly, we can define a random mapping $a:\Omega\times H\times H\to H$. We will write $f_t(x)=f(t,x(t))$ and $a_t(x,y)=a(t,x(t),y(t))$, for all $t\in\Omega$ and $x(t),y(t)\in H$.

Definition 2.4. ([3]) A multivalued mapping $G:\Omega\to 2^H$ is said to be *measurable*, if for any $B\in\ B(H)$, $G^{-1}(B)=\{t\in\Omega : G(t)\cap B\neq\emptyset\}\in\Sigma$.

Definition 2.5. ([3]) A mapping $u:\Omega\to H$ is said to be *measurable selection* of a multivalued measurable mapping $G:\Omega\to 2^H$ if u is measurable and for any $t\in\Omega$, $u(t)\in G(t)$.

Definition 2.6. ([3]) A multivalued mapping $G:\Omega\times H\to 2^H$ is said to be *random*, if for any $x\in H$, $G(\cdot,x)$ is measurable. A random multivalued mapping $G:\Omega\times H\to CB(H)$ is said to be $\tilde{H}$-*continuous*, if for any $t\in\Omega$, $G(t,\cdot)$ is continuous in the Hausdorff metric.

Definition 2.7. Let $F(H)$ be the family of all fuzzy sets on H. A mapping $F:H\to F(H)$ is called a *fuzzy mapping* on H.

Definition 2.8. ([3]) If F is a fuzzy mapping on H, then $F(x)$ (denoted by F_x, in the sequel) is fuzzy set on H and $F_x(y)$ is the membership function of y in F_x.

Definition 2.9. ([3]) Let $M \in F(H)$, $\alpha \in [0,1]$. Then the set $(M)_\alpha = \{x \in H : M(x) \geq \alpha\}$ is called a *α-cut* set of fuzzy set M.

Definition 2.10. ([3]) A fuzzy mapping $F : \Omega \to F(H)$ is called *measurable*, if for any $\alpha \in (0,1]$, $(F(\cdot))_\alpha : \Omega \to 2^H$ is a measurable multivalued mapping.

Definition 2.11. ([3]) A fuzzy mapping $F : \Omega \times H \to F(H)$ is said to be a *random fuzzy mapping*, if for any $x \in H$, $F(\cdot, x) : \Omega \to F(H)$ is a measurable fuzzy mapping.

Remark 2.12. We note that the random fuzzy mappings include multivalued mappings, random multivalued mappings and fuzzy mappings as special cases.

Definition 2.13. ([2]) A random mapping $P : \Omega \times H \to H$ is said to be

(i) *monotone*, if

$$\langle P(t, x_1(t)) - P(t, x_2(t)),\, x_1(t) - x_2(t)\rangle \geq 0,$$

for all $x_1(t), x_2(t) \in H,\ t \in \Omega$;

(ii) *r-strongly monotone*, if there exists a measurable function $r : \Omega \to (0, \infty)$ such that

$$\langle P(t, x_1(t)) - P(t, x_2(t)),\, x_1(t) - x_2(t)\rangle \ \geq \ r(t)\|x_1(t) - x_2(t)\|^2,$$

for all $x_1(t),\ x_2(t) \in H,\ t \in \Omega$;

(iii) *m-relaxed monotone*, if there exists a measurable function $m : \Omega \to (0, \infty)$ such that

$$\langle P(t, x_1(t)) - P(t, x_2(t)),\, x_1(t) - x_2(t)\rangle \ \geq \ -m(t)\|x_1(t) - x_2(t)\|^2,$$

for all $x_1(t),\ x_2(t) \in H,\ t \in \Omega$;

(iv) *(λ, μ)-relaxed cocoercive*, if there exist measurable functions $\lambda, \mu : \Omega \to (0, \infty)$ such that

$$\begin{aligned}&\langle P(t, x_1(t)) - P(t, x_2(t)),\, x_1(t) - x_2(t)\rangle \\ &\geq \lambda(t)\|P(t, x_1(t)) - P(t, x_2(t))\|^2 + \mu(t)\|x_1(t) - x_2(t)\|^2,\end{aligned}$$

for all $x_1(t),\ x_2(t) \in H,\ t \in \Omega$;

(v) *l_P-Lipschitz continuous*, if there exists a measurable function $l_P : \Omega \to (0, \infty)$ such that

$$\|P(t, x_1(t)) - P(t, x_2(t))\| \ \leq \ l_P(t)\|x_1(t) - x_2(t)\|,$$

for all $x_1(t),\ x_2(t) \in H,\ t \in \Omega$.

Definition 2.14. ([2]) A random multivalued mapping $W : \Omega \times H \to 2^H$ is said to be

(i) *monotone*, if

$$\langle u(t)-v(t),\, x_1(t)-x_2(t)\rangle \;\geq\; 0,$$

for all $u \in W(t,x_1(t))$, $v \in W(t,x_2(t))$, $x_1(t)$, $x_2(t) \in H,\ t \in \Omega$;

(ii) r-*strongly monotone*, if there exists a measurable function $r : \Omega \to (0,\infty)$ such that

$$\langle u(t)-v(t),\, x_1(t)-x_2(t)\rangle \;\geq\; r(t)\|x_1(t)-x_2(t)\|^2,$$

for all $u \in W(t,x_1(t))$, $v \in W(t,x_2(t))$, $x_1(t)$, $x_2(t) \in H,\ t \in \Omega$;

(iii) m-*relaxed monotone*, if there exists a measurable function $m : \Omega \to (0,\infty)$ such that

$$\langle u(t)-v(t),\, x_1(t)-x_2(t)\rangle \;\geq\; -m(t)\|x_1(t)-x_2(t)\|^2,$$

for all $u \in W(t,x_1(t))$, $v \in W(t,x_2(t))$, $x_1(t)$, $x_2(t) \in H,\ t \in \Omega$.

Definition 2.15. Let $A, C, R : \Omega \times H \to CB(H)$ be random multivalued mappings. A random mapping $N : \Omega \times H \times H \times H \to H$ is said to be $(l_{(N,2)}(t), l_{(N,3)}(t), l_{(N,4)}(t))$-*mixed Lipschitz continuous*, if there exist measurable functions $l_{(N,2)}, l_{(N,3)}, l_{(N,4)} : \Omega \to (0,\infty)$ such that

$$\begin{aligned}&\|N(t,x_1(t),y_1(t),z_1(t)) - N(t,x_2(t),y_2(t),z_2(t))\|\\ &\leq l_{(N,2)}(t)\|x_1(t)-x_2(t)\| + l_{(N,3)}(t)\|y_1(t)-y_2(t)\| + l_{(N,4)}(t)\|z_1(t)-z_2(t)\|,\end{aligned}$$

for all $x_i(t), y_i(t), z_i(t) \in H,\ t \in \Omega,\ i = 1,2$.

Definition 2.16. ([2]) Let $P : \Omega \times H \to H$ be a single-valued mapping. Then a random multivalued mapping $W : \Omega \times H \to 2^H$ is said to be P-*monotone* if:

(i) W is m-relaxed monotone,

(ii) $\big[P(t,x(t)) + \rho(t)W(t,x(t))\big](H) = H$, for all $x(t) \in H$, $t \in \Omega$ and $\rho(t) > 0$.

Definition 2.17. Let $P : \Omega \times H \to H$ be r-strongly monotone and $W : \Omega \times H \to 2^H$ be P-monotone. Then P-resolvent operator $J_{W_t}^{\rho(t),P_t} : \Omega \times H \to H$ associated with P and W is defined by

$$J_{W_t}^{\rho(t),P_t}(x) \;=\; (P_t + \rho(t)W_t)^{-1}(x),$$

where $P_t(x) = P(t,x(t))$ and $W_t(x) = W(t,x(t))$, for all $x(t) \in H$, $t \in \Omega$ and $\rho(t) > 0$.

Lemma 2.18. ([2, 15]) Let $P : \Omega \times H \to H$ be r-strongly monotone and $W : \Omega \times H \to 2^H$ be P-monotone. Then P-resolvent operator $J_{W_t}^{\rho(t),P_t} : \Omega \times H \to H$ is $1/(r(t)-\rho(t)m(t))$-Lipschitz continuous for $\rho(t) \in (0, r(t)/m(t))$.

Definition 2.19. ([2, 15]) A random multivalued mapping $A : \Omega \times H \to CB(H)$ is said to *$\tilde{H}$-Lipschitz continuous*, if there exists a measurable function $l_{\tilde{H}_A} : \Omega \to (0, \infty)$ such that

$$\tilde{H}(A(t, x_1(t)),\ A(t, x_2(t)))\ \leq\ l_{\tilde{H}_A}(t)\|x_1(t) - x_2(t)\|,$$

for all $x_1(t), x_2(t) \in H$, $t \in \Omega$.

3. Formulation of problem

Let $A, C, R, S, T : \Omega \times H \to F(H)$ be random fuzzy mappings satisfying the following condition (**C**):

(C): There exist mappings $a, b, c, d, e : H \to [0, 1]$ such that

$$(A_{t,x})_{a(x)} \in CB(H),\ (C_{t,x})_{b(x)} \in CB(H),\ (R_{t,x})_{c(x)} \in CB(H),$$
$$(S_{t,x})_{d(x)} \in CB(H),\ (T_{t,x})_{e(x)} \in CB(H), \text{ for all l } (t, x) \in \Omega \times H.$$

By using the random fuzzy mappings A, C, R, S and T, we can define respectively the multivalued mappings $\tilde{A}, \tilde{C}, \tilde{R}, \tilde{S}, \tilde{T} : \Omega \times H \to CB(H)$ by $\tilde{A}(t, x) = (A_{t,x})_{a(x)}$, $\tilde{C}(t, x) = (C_{t,x})_{b(x)}$, $\tilde{R}(t, x) = (R_{t,x})_{c(x)}$, $\tilde{S}(t, x) = (S_{t,x})_{d(x)}$, $\tilde{T}(t, x) = (T_{t,x})_{e(x)}$, for each $(t, x) \in \Omega \times H$. It means that

$$\begin{aligned}
\tilde{A}(t, x) &= (A_{t,x})_{a(x)} = \{z \in H, (A_{t,x})(z) \geq a(x)\} \in CB(H),\\
\tilde{C}(t, x) &= (C_{t,x})_{b(x)} = \{z \in H, (C_{t,x})(z) \geq b(x)\} \in CB(H),\\
\tilde{R}(t, x) &= (R_{t,x})_{c(x)} = \{z \in H, (R_{t,x})(z) \geq c(x)\} \in CB(H),\\
\tilde{S}(t, x) &= (S_{t,x})_{d(x)} = \{z \in H, (S_{t,x})(z) \geq d(x)\} \in CB(H),\\
\tilde{T}(t, x) &= (T_{t,x})_{e(x)} = \{z \in H, (T_{t,x})(z) \geq e(x)\} \in CB(H).
\end{aligned}$$

In the sequel, $\tilde{A}, \tilde{C}, \tilde{R}, \tilde{S}$ and $\tilde{T}$ are called the random multivalued mappings induced by the random fuzzy mappings A, C, R, S and T, respectively.

Given mappings $a, b, c, d, e : H \to [0, 1]$, random fuzzy mappings $A, C, R, S, T : \Omega \times H \to F(H)$, random mappings $f, g : \Omega \times H \to H$, $N : \Omega \times H \times H \times H \to H$ and $W : \Omega \times H \to 2^H$ with $\mathrm{Im(g)} \cap \mathrm{dom}(W(t, \cdot)) \neq \emptyset$, for $t \in \Omega$. We consider the following random generalized nonlinear mixed variational inclusion problem involving random fuzzy mappings (RGNMVIP):

Find measurable mappings $x, u, v, w, p, q : \Omega \to H$ with for all $t \in \Omega$, $x(t) \in H$, $A_{t,x(t)}(u(t)) \geq a(x(t))$, $C_{t,x(t)}(v(t)) \geq b(x(t))$, $R_{t,x(t)}(w(t)) \geq c(x(t))$, $S_{t,x(t)}(p(t)) \geq d(x(t))$, $T_{t,x(t)}(q(t)) \geq e(x(t))$ and $g(t, q(t)) \cap \mathrm{dom}(W(t, \cdot)) \neq \emptyset$, such that, for $t \in \Omega$

$$0 \in N(t, u(t), v(t), w(t)) - \big\{f(t, p(t)) - g(t, q(t))\big\} + W\big(t, g(t, q(t))\big). \qquad (3.1)$$

The set of measurable mappings (x, u, v, w, p, q) is called a random solution of RGNMVIP (3.1).

For suitable choices of the mappings involved in the problem (3.1) and the space H, RGNMVIP (3.1) reduces to various known classes of random variational inclusions (inequalities) involving random fuzzy mappings (see [1-3, 5-8, 14-19]).

4. Random iterative algorithm

First, we recall the following useful lemmas.

Lemma 4.1. ([4]) *Let $A : \Omega \times H \to CB(H)$ be a $\tilde{H}$-continuous random multivalued mapping. Then for any measurable mapping $u : \Omega \to H$, the multivalued mapping $A(\cdot, u(\cdot)) : \Omega \to CB(H)$ is measurable.*

Lemma 4.2. ([4]) *Let $A, C : \Omega \times H \to CB(H)$ be two measurable multivalued mappings, $\epsilon > 0$ be a constant and $u : \Omega \to H$ be a measurable selection of A. Then there exists a measurable selection $v : \Omega \to H$ of C such that for all $t \in \Omega$,*

$$\|u(t) - v(t)\| \ \leq \ (1 + \epsilon)\ \tilde{H}(A(t)\,, C(t)).$$

Lemma 4.3. *The set of measurable mappings $x, u, v, w, p, q : \Omega \to H$ is a random solution of RGNMVIP (3.1) if and only if for all $t \in \Omega$, $x(t) \in H$, $u(t) \in \tilde{A}(t, x(t))$, $v(t) \in \tilde{C}(t, x(t))$, $w(t) \in \tilde{R}(t, x(t))$, $p(t) \in \tilde{S}(t, x(t))$, $q(t) \in \tilde{T}(t, x(t))$ and*

$$\begin{aligned} g(t, q(t)) \ = \ J_{W_t}^{\rho(t), P_t}\big[&P_t(g(t, q(t))) - \rho(t)\{N(t, u(t), v(t), w(t)) \\ &- (f(t, p(t)) - g(t, q(t)))\}\big], \end{aligned}$$

where $\rho : \Omega \to (0, \infty)$ is a measurable function.

Proof. The proof directly follows from the definition of $J_{W_t}^{\rho(t), P_t}$. □

Using Lemma 4.3, we develop an iterative algorithm for finding the approximate random solution of RGNMVIP (3.1) as follows.

Algorithm 4.4. Let $A, C, R, S, T : \Omega \times H \to F(H)$ be random fuzzy mappings satisfying the condition (**C**). Let $\tilde{A}, \tilde{C}, \tilde{R}, \tilde{S}, \tilde{T} : \Omega \times H \to CB(H)$ be $\tilde{H}$-continuous random multivalued mappings induced by A, C, R, S, T, respectively. Let $f, g, P : \Omega \times H \to H$ and $N : \Omega \times H \times H \times H \to H$ be the single-valued random mappings and $W : \Omega \times H \to 2^H$ be a multivalued random mapping such that for each $t \in \Omega, W(t, \cdot) : H \to 2^H$ is P-monotone with $\mathrm{Im(g)} \cap \mathrm{dom}\,(W(t, \cdot)) \neq \emptyset$. For any given measurable mapping $x_0 : \Omega \to H$, the multivalued mappings $\tilde{A}(\cdot,\ x_0(\cdot)), \tilde{C}(\cdot, x_0(\cdot)), \tilde{R}(\cdot, x_0(\cdot)), \tilde{S}$

$(\cdot, x_0(\cdot))$, $\tilde{T}(\cdot,\ x_0(\cdot)) : \Omega \to CB(H)$ are measurable by Lemma 4.1. Hence by Himmelberg [12], there exist measurable selections $u_0 : \Omega \to H$ of $\tilde{A}(\cdot, x_0(\cdot))$, $v_0 : \Omega \to H$ of $\tilde{C}(\cdot, x_0(\cdot))$, $w_0 : \Omega \to H$ of $\tilde{R}(\cdot, x_0(\cdot))$, $p_0 : \Omega \to H$ of $\tilde{S}(\cdot, x_0(\cdot))$ and $q_0 : \Omega \to H$ of $\tilde{T}(\cdot, x_0(\cdot))$. Let

$$\begin{aligned} x_1(t) \ = \ & x_0(t) - g(t, q_0(t)) \\ & + J_{W_t}^{\rho(t),P_t}\big[P_t(g(t,q_0(t))) - \rho(t)\{N(t, u_0(t), v_0(t), w_0(t)) \\ & - (f(t, p_0(t)) - g(t, q_0(t)))\}\big], \end{aligned}$$

where $\rho(t)$ is same as in Lemma 4.3. Then, it is easy to observe that $x_1 : \Omega \to H$ is measurable. By Lemma 4.2, there exist measurable selections $u_1 : \Omega \to H$ of $\tilde{A}(\cdot, x_1(\cdot))$, $v_1 : \Omega \to H$ of $\tilde{C}(\cdot, x_1(\cdot))$, $w_1 : \Omega \to H$ of $\tilde{R}(\cdot, x_1(\cdot))$, $p_1 : \Omega \to H$ of $\tilde{S}(\cdot, x_1(\cdot))$ and $q_1 : \Omega \to H$ of $\tilde{T}(\cdot, x_1(\cdot))$ such that for all $t \in \Omega$,

$$\begin{aligned} \|u_0(t) - u_1(t)\| \le \ & (1 + (1+0)^{-1})\, \tilde{H}\,(\tilde{A}(t, x_0(t)),\ \tilde{A}(t, x_1(t))), \\ \|v_0(t) - v_1(t)\| \le \ & (1 + (1+0)^{-1})\, \tilde{H}\,(\tilde{C}(t, x_0(t)),\ \tilde{C}(t, x_1(t))), \\ \|w_0(t) - w_1(t)\| \le \ & (1 + (1+0)^{-1})\, \tilde{H}\,(\tilde{R}(t, x_0(t)),\ \tilde{R}(t, x_1(t))), \\ \|p_0(t) - p_1(t)\| \le \ & (1 + (1+0)^{-1})\, \tilde{H}\,(\tilde{S}(t, x_0(t)),\ \tilde{S}(t, x_1(t))), \\ \|q_0(t) - q_1(t)\| \le \ & (1 + (1+0)^{-1})\, \tilde{H}\,(\tilde{T}(t, x_0(t)),\ \tilde{T}(t, x_1(t))). \end{aligned}$$

Let

$$\begin{aligned} x_2(t) = \ & x_1(t) - g(t, q_1(t)) \\ & + J_{W_t}^{\rho(t),P_t}\big[P_t(g(t,q_1(t))) - \rho(t)\{N(t, u_1(t), v_1(t), w_1(t)) \\ & - (f(t, p_1(t)) - g(t, q_1(t)))\}\big], \end{aligned}$$

then $x_2 : \Omega \to H$ is measurable. Continuing the above process inductively, we can define the following random iterative sequences $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$, $\{w_n(t)\}$, $\{p_n(t)\}$ and $\{q_n(t)\}$ for solving problem (3.1) as follows:

$$\begin{aligned} x_{n+1}(t) = \ & x_n(t) - g(t, q_n(t)) \\ & + J_{W_t}^{\rho(t),P_t}\big[P_t(g(t,q_n(t))) \\ & - \rho(t)\{N(t, u_n(t), v_n(t), w_n(t)) - (f(t, p_n(t)) - g(t, q_n(t)))\}\big], \quad (4.1) \\ u_n(t) \in \ & \tilde{A}(t, x_n(t)),\ v_n(t) \in \tilde{C}(t, x_n(t)),\ w_n(t) \in \tilde{R}(t, x_n(t)), \\ p_n(t) \in \ & \tilde{S}(t, x_n(t)),\ q_n(t) \in \tilde{T}(t, x_n(t)), \end{aligned}$$

such that

$$\begin{aligned}
\|u_n(t)-u_{n+1}(t)\| &\le (1+(1+n)^{-1})\,\tilde{H}\,(\tilde{A}(t,x_n(t)),\ \tilde{A}(t,x_{n+1}(t))),\\
\|v_n(t)-v_{n+1}(t)\| &\le (1+(1+n)^{-1})\,\tilde{H}\,(\tilde{C}(t,x_n(t)),\ \tilde{C}(t,x_{n+1}(t))),\\
\|w_n(t)-w_{n+1}(t)\| &\le (1+(1+n)^{-1})\,\tilde{H}\,(\tilde{R}(t,x_n(t)),\ \tilde{R}(t,x_{n+1}(t))),\\
\|p_n(t)-p_{n+1}(t)\| &\le (1+(1+n)^{-1})\,\tilde{H}\,(\tilde{S}(t,x_n(t)),\ \tilde{S}(t,x_{n+1}(t)))\\
\|q_n(t)-q_{n+1}(t)\| &\le (1+(1+n)^{-1})\,\tilde{H}\,(\tilde{T}(t,x_n(t)),\ \tilde{T}(t,x_{n+1}(t))),
\end{aligned}$$

for any $t \in \Omega$, $n = 0, 1, 2, \ldots$, and $\rho : \Omega \to (0, \infty)$ is a measurable function.

5. Convergence of Algorithm 4.4 for RGNMVIP (3.1)

Theorem 5.1. *Let H be a real Hilbert space. Let $f, g : \Omega \times H \to H$ be Lipschitz continuous random mappings with constants $l_f(t)$ and $l_g(t)$, respectively and g be (λ, μ)-relaxed cocoercive. Let the random mapping $N : \Omega \times H \times H \times H \to H$ be $(l_{(N,2)}(t), l_{(N,3)}(t), l_{(N,4)}(t))$-mixed Lipschitz continuous and $W : \Omega \times H \to 2^H$ be a random multivalued mapping such that for each $t \in \Omega$, $W(t, \cdot) : H \to 2^H$ is P-monotone mapping. Let $P : \Omega \times H \to H$ be r-strongly monotone and Lipschitz continuous with constant $l_P(t)$. Let $A, C, R, S, T : \Omega \times H \to F(H)$ be random fuzzy mappings satisfying the condition* (**C**) *and the random multivalued mappings $\tilde{A}, \tilde{C}, \tilde{R}, \tilde{S}, \tilde{T} : \Omega \times H \to CB(H)$ be $\tilde{H}$-Lipschitz continuous with measurable functions $l_{\tilde{H}_{\tilde{A}}}(t)$, $l_{\tilde{H}_{\tilde{C}}}(t)$, $l_{\tilde{H}_{\tilde{R}}}(t)$, $l_{\tilde{H}_{\tilde{S}}}(t)$, $l_{\tilde{H}_{\tilde{T}}}(t)$, respectively. Suppose that the following condition holds, for all $t \in \Omega$,*

$$\begin{aligned}
\theta(t) = &\sqrt{1-2\mu(t)+(l_g(t)l_{\tilde{H}_{\tilde{T}}}(t))^2[1+2\lambda(t)]}\\
&+\frac{1}{L(t)}\Big[l_g(t)l_P(t)l_{\tilde{H}_{\tilde{T}}}(t)+\rho(t)\big(L_N(t)+l_f(t)l_{\tilde{H}_{\tilde{S}}}(t)+l_g(t)l_{\tilde{H}_{\tilde{T}}}(t)\big)\Big]\\
&<1, \qquad (5.1)
\end{aligned}$$

where

$$L(t) = r(t) - \rho(t)m(t)$$

and

$$L_N(t) = l_{(N,2)}(t)l_{\tilde{H}_{\tilde{A}}}(t) + l_{(N,3)}(t)l_{\tilde{H}_{\tilde{C}}}(t) + l_{(N,4)}(t)l_{\tilde{H}_{\tilde{R}}}(t).$$

Then there exist measurable mappings $x, u, v, w, p, q : \Omega \to H$ such that (3.1) *holds. Moreover, $x_n(t) \to x(t)$, $u_n(t) \to u(t)$, $v_n(t) \to v(t)$, $w_n(t) \to w(t)$, $p_n(t) \to p(t)$ and $q_n(t) \to q(t)$, where $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$, $\{w_n(t)\}$, $\{p_n(t)\}$ and $\{q_n(t)\}$ are the random sequences generated by iterative Algorithm* 4.4.

Proof. It follows from (4.1) and Lemma 2.18 that

$$
\begin{aligned}
\|x_{n+1}(t) - x_n(t)\| = \Big\| & x_n(t) - g(t, q_n(t)) + J_{W_t}^{\rho(t),P_t}[P_t(g(t, q_n(t))) \\
& - \rho(t)\{N(t, u_n(t), v_n(t), w_n(t)) - (f(t, p_n(t)) - g(t, q_n(t)))\}] \\
& - \big[x_{n-1}(t) - g(t, q_{n-1}(t)) + J_{W_t}^{\rho(t),P_t}[P_t(g(t, q_{n-1}(t))) \\
& - \rho(t)\{N(t, u_{n-1}(t), v_{n-1}(t), w_{n-1}(t)) \\
& - (f(t, p_{n-1}(t)) - g(t, q_{n-1}(t)))\}]\big]\Big\| \\
\leq\ & \|x_n(t) - x_{n-1}(t) - (g(t, q_n(t)) - g(t, q_{n-1}(t)))\| \\
& + \frac{1}{r(t) - \rho(t)m(t)} \|P_t(g(t, q_n(t))) - P_t(g(t, q_{n-1}(t)))\| \\
& + \frac{\rho(t)}{r(t) - \rho(t)m(t)} \|N(t, u_n(t), v_n(t), w_n(t)) \\
& - N(t, u_{n-1}(t), v_{n-1}(t), w_{n-1}(t))\| \\
& + \frac{\rho(t)}{r(t) - \rho(t)m(t)} \Big(\|f(t, p_n(t)) - f(t, p_{n-1}(t))\| \\
& + \|g(t, q_n(t)) - g(t, q_{n-1}(t))\|\Big).
\end{aligned} \tag{5.2}
$$

Using the $\tilde{H}$-Lipschitz continuity of $\tilde{A}, \tilde{C}, \tilde{R}$ and $(l_{(N,2)}(t), l_{(N,3)}(t), l_{(N,4)}(t))$-mixed Lipschitz continuity of N, we have

$$
\begin{aligned}
& \|N(t, u_n(t), v_n(t), w_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t), w_{n-1}(t))\| \\
& \leq\ l_{(N,2)}(t)\|u_n(t) - u_{n-1}(t)\| + l_{(N,3)}(t)\|v_n(t) - v_{n-1}(t)\| \\
& \quad + l_{(N,4)}(t)\|w_n(t) - w_{n-1}(t)\| \\
& \leq\ (1 + (1 + n)^{-1})\ \big(l_{(N,2)}(t)\ \tilde{H}(\tilde{A}(t, x_n(t)), \tilde{A}(t, x_{n-1}(t))) \\
& \quad + l_{(N,3)}(t)\ \tilde{H}(\tilde{C}(t, x_n(t)), \tilde{C}(t, x_{n-1}(t))) \\
& \quad + l_{(N,4)}(t)\tilde{H}(\tilde{R}(t, x_n(t)), \tilde{R}(t, x_{n-1}(t)))\big) \\
& \quad \leq (1 + (1 + n)^{-1})\ \big(l_{(N,2)}(t)l_{\tilde{H}_{\tilde{A}}}(t) + l_{(N,3)}(t)l_{\tilde{H}_{\tilde{C}}}(t) \\
& \quad + l_{(N,4)}(t)l_{\tilde{H}_{\tilde{R}}}(t)\big)\|x_n(t) - x_{n-1}(t)\|.
\end{aligned} \tag{5.3}
$$

Using the Lipschitz continuity of f and g, we have

$$
\|f(t, p_n(t)) - f(t, p_{n-1}(t))\| \leq l_f(t)l_{\tilde{H}_{\tilde{S}}}(t)(1 + (1+n)^{-1})\ \|x_n(t) - x_{n-1}(t)\|, \tag{5.4}
$$

$$
\|g(t, q_n(t)) - g(t, q_{n-1}(t))\| \leq l_g(t)l_{\tilde{H}_{\tilde{T}}}(t)(1 + (1+n)^{-1})\ \|x_n(t) - x_{n-1}(t)\|. \tag{5.5}
$$

Using (λ, μ)-relaxed cocoercivity of g, Lipschitz continuity of g and (5.5), we have

$$\begin{aligned}
&\|x_n(t) - x_{n-1}(t) - (g(t, q_n(t)) - g(t, q_{n-1}(t)))\|^2 \\
&\le \|x_n(t) - x_{n-1}(t)\|^2 - 2\langle g(t, q_n(t)) - g(t, q_{n-1}(t)), x_n(t) - x_{n-1}(t)\rangle \\
&\quad + \|g(t, q_n(t)) - g(t, q_{n-1}(t))\|^2 \\
&\le \|x_n(t) - x_{n-1}(t)\|^2 + 2\lambda(t)\|g(t, q_n(t)) \\
&\quad - g(t, q_{n-1}(t))\|^2 - 2\mu(t)\|x_n(t) - x_{n-1}(t)\|^2 \\
&\quad + \|g(t, q_n(t)) - g(t, q_{n-1}(t))\|^2 \\
&\le \|x_n(t) - x_{n-1}(t)\|^2 - 2\mu(t)\|x_n(t) - x_{n-1}(t)\|^2 \\
&\quad + (l_g(t) l_{\tilde{H}_{\tilde{T}}}(t)(1 + (1+n)^{-1}))^2[1 + 2\lambda(t)]\|x_n(t) - x_{n-1}(t)\|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|x_n(t) - x_{n-1}(t) - (g(t, q_n(t)) - g(t, q_{n-1}(t)))\| \\
&\le \sqrt{1 - 2\mu(t) + (l_g(t) l_{\tilde{H}_{\tilde{T}}}(t)(1 + (1+n)^{-1}))^2[1 + 2\lambda(t)]}\ \|x_n(t) - x_{n-1}(t)\|. \quad (5.6)
\end{aligned}$$

Using the Lipschitz continuity of P and combining (5.2)-(5.6), we have

$$\begin{aligned}
&\|x_{n+1}(t) - x_n(t)\| \\
&\le \sqrt{1 - 2\mu(t) + (l_g(t) l_{\tilde{H}_{\tilde{T}}}(t) L(n))^2[1 + 2\lambda(t)]}\ \|x_n(t) - x_{n-1}(t)\| \\
&\quad + \frac{l_g(t) l_P(t) l_{\tilde{H}_{\tilde{T}}}(t) L(n)}{L(t)}\ \|x_n(t) - x_{n-1}(t)\| \\
&\quad + \frac{\rho(t)(l_{(N,2)}(t) l_{\tilde{H}_{\tilde{A}}}(t) + l_{(N,3)}(t) l_{\tilde{H}_{\tilde{C}}}(t) + l_{(N,4)}(t) l_{\tilde{H}_{\tilde{R}}}(t)) L(n)}{L(t)} \|x_n(t) - x_{n-1}(t)\| \\
&\quad + \frac{\rho(t)(l_f(t) l_{\tilde{H}_{\tilde{S}}}(t) + l_g(t) l_{\tilde{H}_{\tilde{T}}}(t)) L(n)}{L(t)}\ \|x_n(t) - x_{n-1}(t)\|,\ \forall\, t \in \Omega, \quad (5.7)
\end{aligned}$$

where $L(n) = (1 + (1+n)^{-1})$ and $L(t) = r(t) - \rho(t) m(t)$. Thus, we have

$$\|x_{n+1}(t) - x_n(t)\| \ \le\ \theta_n(t)\, \|x_n(t) - x_{n-1}(t)\|,\ \forall\, t \in \Omega, \quad (5.8)$$

where

$$\begin{aligned}
\theta_n(t) = &\sqrt{1 - 2\mu(t) + (l_g(t) l_{\tilde{H}_{\tilde{T}}}(t) L(n))^2[1 + 2\lambda(t)]} \\
&+ \frac{L(n)}{L(t)} \Big[l_g(t) l_P(t) l_{\tilde{H}_{\tilde{T}}}(t) + \rho(t)\big(l_{(N,2)}(t) l_{\tilde{H}_{\tilde{A}}}(t) + l_{(N,3)}(t) l_{\tilde{H}_{\tilde{C}}}(t) \\
&+ l_{(N,4)}(t) l_{\tilde{H}_{\tilde{R}}}(t) + l_f(t) l_{\tilde{H}_{\tilde{S}}}(t) + l_g(t) l_{\tilde{H}_{\tilde{T}}}(t)\big)\Big],
\end{aligned}$$

where $L(n) = (1 + (1 + n)^{-1})$ and $L(t) = r(t) - \rho(t)m(t)$. Letting $n \to \infty$, we have $\theta_n(t) \to \theta(t)$ for all $t \in \Omega$, where

$$\theta(t) = \sqrt{1 - 2\mu(t) + (l_g(t)l_{\tilde{H}_{\tilde{T}}}(t))^2[1 + 2\lambda(t)]} + \frac{1}{L(t)}\Big[l_g(t)l_P(t)l_{\tilde{H}_{\tilde{T}}}(t)+\rho(t)\big(L_N(t)+l_f(t)l_{\tilde{H}_{\tilde{S}}}(t)+l_g(t)l_{\tilde{H}_{\tilde{T}}}(t)\big)\Big], \forall\, t \in \Omega \tag{5.9}$$

where,

$$L(t) = r(t) - \rho(t)m(t),$$

and

$$L_N(t) = l_{(N,2)}(t)l_{\tilde{H}_{\tilde{A}}}(t) + l_{(N,3)}(t)l_{\tilde{H}_{\tilde{C}}}(t) + l_{(N,4)}(t)l_{\tilde{H}_{\tilde{R}}}(t).$$

By condition (5.1), $\theta(t) \in (0, 1)$ for all $t \in \Omega$. Hence for any $t \in \Omega$, $\theta_n(t) < 1$ for n sufficiently large. Therefore (5.8) implies that $\{x_n(t)\}$ is a Cauchy sequence in H. Since H is complete, there exists a measurable mapping $x : \Omega \to H$ such that $x_n(t) \to x(t)$ for all $t \in \Omega$. Further, it follows from $\tilde{H}$-Lipschitz continuity of $\tilde{A}$ and iterative Algorithm 4.4, we have

$$\|u_{n+1}(t) - u_n(t)\| \;\leq\; (1 + (1 + n)^{-1})l_{\tilde{H}_{\tilde{A}}}(t)\|x_{n+1}(t) - x_n(t)\|,$$

which implies that $\{u_n(t)\}$ is a Cauchy sequence in H.

Similarly, we can prove that $\{v_n(t)\}$, $\{w_n(t)\}$, $\{p_n(t)\}$, $\{q_n(t)\}$ are Cauchy sequences in H. Hence, there exist measurable mappings $v, w, p, q : \Omega \to H$ such that $v_n(t) \to v(t)$, $w_n(t) \to w(t)$, $p_n(t) \to p(t)$ $q_n(t) \to q(t)$ as $n \to \infty$ for all $t \in \Omega$.

Furthermore, for any $t \in \Omega$, we have

$$\begin{aligned} d(u(t),\, \tilde{A}(t, x(t))) \;&\leq\; \|u(t)-u_n(t)\|+d(u_n(t)\,, \tilde{A}(t, x(t))) \\ &\leq\; \|u(t) - u_n(t)\| + \tilde{H}(\tilde{A}(t, x_n(t))\,, \tilde{A}(t, x(t))) \\ &\leq\; \|u(t) - u_n(t)\| + l_{\tilde{H}_{\tilde{A}}}(t)\|x_n(t) - x(t)\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $u(t) \in \tilde{A}(t, x(t))$ for all $t \in \Omega$.

Similarly, we can show that $v(t) \in \tilde{C}(t, x(t))$, $w(t) \in \tilde{R}(t, x(t))$, $p(t) \in \tilde{S}(t, x(t))$, $q(t) \in \tilde{T}(t, x(t))$ for all $t \in \Omega$. This completes the proof. □

Remark 5.2. For all $t \in \Omega$, and measurable functions $\rho, r, m : \Omega \to (0, \infty)$, it is clear that $r(t) > \rho(t)m(t)$, $2\mu(t) < 1 + (l_g(t)l_{\tilde{H}_{\tilde{T}}}(t))^2[1 + 2\lambda(t)]$ and $\rho(t) \in (0, r(t)/m(t))$. Further, $\theta(t) \in (0\,, 1)$ and condition (5.1) of Theorem 5.1 holds for some appropriate values of constants.

Remark 5.3. Since the RGNMVIP (3.1) includes many known classes of random variational inclusions (inequalities) involving random fuzzy mappings as special cases, so the technique utilized in this paper can be used to extend and advance the theorems given by many researchers (see [1-3, 5-8, 14-19]).

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