Nonlinear Functional Analysis and Applications
Vol. 27, No. 4 (2022), pp. 839-853
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2022.27.04.11
http://nfaa.kyungnam.ac.kr/journal-nfaa
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ERGODIC SHADOWING, $\underline{d}$-SHADOWING AND EVENTUAL SHADOWING IN TOPOLOGICAL SPACES

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Abstract. We define the notions of ergodic shadowing property, $\underline{d}$-shadowing property and eventual shadowing property in terms of the topology of the phase space. Secondly we define these notions in terms of the compatible uniformity of the phase space. When the phase space is a compact Hausdorff space, we establish the equivalence of the corresponding definitions of the topological approach and the uniformity approach. In case the phase space is a compact metric space, the notions of ergodic shadowing property, $\underline{d}$-shadowing property and eventual shadowing property defined in terms of topology and uniformity are equivalent to their respective standard definitions.

1. Introduction

Generally by a dynamical system we mean a pair (X, f), where X is a compact metric space called phase space with metric ρ, and f a continuous self map. Recently, the standard notions of a dynamical system have been extended to the case when X is a uniform space. Almost all the standard notions of dynamical systems are defined in terms of the metric except for some cases (Topological transitivity etc.) where the definitions are purely

[0]Received March 7, 2022. Revised April 3, 2022. Accepted April 7, 2022.
[0]2020 Mathematics Subject Classification: 54H99, 37B02, 54E15.
[0]Keywords: Hausdorff ergodic shadowing, Hausdorff $\underline{d}$−shadowing, Hausdorff eventual shadowing.
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topological. It is interesting to know that sensitivity defined based on metric is related to transitivity and dense periodic points defined based on topology (see [5], [6], [13], [19]).

The fact that distance between two points x, y is less than ϵ, that is, $\rho(x,y) < \epsilon$ for metric space can be analogously expressed in terms of suitable entourage U, that is, $(x,y) \in U$ for uniform space. One can see such approach in ([9], [10], [15]). In general, the uniformity of a uniformizable space is not unique just like the metric of a non compact metric space is not unique. Therefore the question of inconsistency of the definitions of various notions arises, that is, a dynamical system may be expansive with respect to one compatible uniformity $\mathcal{V}$ whereas the same dynamical system may not be so with respect to another compatible uniformity $\mathcal{U}$. The inconsistency ceases when one takes a compact uniform space in which the uniformity is unique.

In more generalized way, many researchers have extended various notions of a dynamical system to Hausdorff topological spaces using the concept of finite open cover. Indeed they are defining the various notions of dynamical systems in terms of the topology of the phase space. It is quite interesting to know how the metric oriented definitions of the notions of dynamical systems are analogously defined in terms of finite open covers of Hausdorff topological spaces. One can see such generalization in ([8], [14]). When we study dynamical systems on the class of Hausdorff topological spaces, indeed, we are extending the standard dynamical notions of compact metric spaces to a bigger class of Hausdorff topological spaces.

In this paper, by a Hausdorff dynamical system we mean a pair (X, f), where f is continuous self-map, and X is a Hausdorff space. When the phase space X is a Hausdorff uniform space we refer to (X, f) as a uniform dynamical system whereas we simply say that (X, f) is a dynamical system when X is a metric space. We prefix Hausdorff in the nomenclature of each notion of Hausdorff dynamical system whereas we prefix uniform in the case of uniform dynamical system. When the definition of a notion is purely based on the topology of the space, we use the same nomenclature in all cases. For example, topological transitivity is defined purely based on the topology of the space, therefore the definition *"A system (X, f) is topological transitive if for each pair U, V of nonempty open sets there exists a positive integer n such that $f^n(U) \cap V \neq \phi$ or $(U) \cap f^{-n}(V) \neq \phi$"* is used for Hausdorff dynamical systems as well as for uniform dynamical systems.

In [14], Good and Macias have defined Hausdorff sensitivity and Hausdorff shadowing and further they have proved that they are equivalent to their respective standard notions when the phase space is a compact metric space. In [20], the author defines Hausdorff equicontinuity, Hausdorff equicontinuous point, and Hausdorff sensitivity point in terms of finite open covers. In case the phase space is a compact metric space, he proves that Hausdorff equicontinuity, uniform equicontinuity, and standard equicontinuity are equivalent. In [11], the authors define positive expansivity, chain transitivity, rigidity, and specification property in terms finite open covers. In [3], the authors extended the study of dynamical systems on the completion of totally bounded uniform spaces. In [2], we extend notions such as expansiveness, pseudo orbit tracing property, chain transitivity, periodic shadowing property to uniform dynamical system (X, f), where X is a compact uniform space.

The notion of ergodic shadowing was introduced by Fakhari and Ghane in [12]. They established the fact that ergodic shadowing is stronger than the shadowing property in the sense that any mapping with the ergodic shadowing property has the shadowing property. In [9] Silberstein and Coornaert give sufficient conditions for sensitivity of continuous group actions on uniform spaces. In [10] the authors introduced the notion of weakly ergodic shadowing and proved the equivalence of ergodic shadowing, weakly ergodic shadowing, shadowing, $\underline{d}$-shadowing, and $\overline{d}$-shadowing. In [18], the author defined Eventual shadowing property on a closed f-invariant subset $\Lambda \subset X$. He proves that a map f has eventual shadowing property if and only if f has the eventual shadowing property on Λ.

Motivated by the works on the generalizations of dynamical systems, we want to extend the notions of ergodic shadowing, $\underline{\text{d}}$-shadowing, eventual shadowing to the systems where the phase space is a Hausdorff space or a uniform space. In section 2, we give brief accounts of uniform spaces, and some standard notions of dynamical systems which are used in the paper. We try to simplify the notations as much possible as we can.

In section 3, we give definitions and results. We give various notions of dynamical systems when the phase space is a Hausdorff space or a uniform space. Let (X, f) be a dynamical system where X is a compact Hausdorff space and $\mathcal{U}$ be the unique uniformity on X which induces the topology for X. For $U \in \mathcal{U}$, let $\varepsilon_U = \{x \in X : \exists D \in U \text{ such that for all } n \geq 0, (f^n(y), f^n(z)) \in U\}$. Then we prove that ε_U is inversely invariant, open and $\varepsilon = \bigcap\limits_{U \in \mathcal{U}} \varepsilon_U$.

It is a well known that a transitive system (X, f) is either sensitive or almost equicontinuous ([1]). We extend this result in uniform dynamical systems. Since every compact metric space is both a Baire space and a second countable space, the expression $\varepsilon = \bigcap\limits_{U \in \mathcal{U}} \varepsilon_U$ can be written as countable intersection of dense open sets and thereby proving that the system is almost equicontinuous.

In case the phase space is a compact Hausdorff space we prove the result as follows: A transitive system on a compact Hausdorff space is either uniformly sensitive or ε_U is dense for each $U \in \mathcal{U}$. We define the notions of Hausdorff ergodic shadowing property and uniform ergodic shadowing property. In case the phase space is a compact Hausdorff space, we prove that the two notions are equivalent and further they are equivalent to ergodic shadowing property when the phase space is a compact metric space. We define Hausdorff $\underline{d}$-shadowing property and uniform $\underline{d}$-shadowing property. We prove that if (X, f) has Hausdorff ergodic shadowing property then for any natural number k, (X, f^k) has Hausdorff ergodic shadowing property. Lastly we extend the notion of eventual shadowing to Hausdorff dynamical systems and uniform dynamical systems.

2. Preliminaries

In this section, we give a brief account of a uniform space and its associated terminologies, and as well as some of standard notions of a dynamical system.

Weil [21] introduced the notion of uniformity. Let X be a non-empty set and U, V subsets of $X \times X$. Define $U \circ V$ by $U \circ V = \{(x, y) \in X \times X : \exists z \in X$, such that $(x, z) \in U$, and $(z, y) \in V\}$ and U^{-1} by $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$. U^{-1} is called the inverse of U. If $U = U^{-1}$, we say that U is symmetric. Also, $U \cap U^{-1}$ is symmetric. A non-empty collection $\mathcal{U}$ of subsets $U \subset X \times X$ is said to have uniform structure if the following are satisfied:

(a) Each member U of $\mathcal{U}$ contains the diagonal $\triangle$, where $\triangle = \{(x, x) : x \in X\}$,
(b) $U \in \mathcal{U} \Rightarrow V \circ V \subset U$ for some $V \in \mathcal{U}$,
(c) $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$,
(d) $U \in \mathcal{U}$ and $V \in \mathcal{U} \Rightarrow U \cap V \in \mathcal{U}$,
(e) $U \in \mathcal{U}$ and $U \subset V \subset X \times X \Rightarrow V \in \mathcal{U}$.

$\mathcal{U}$ is called a uniformity for X and the pair $(X, \mathcal{U})$ is called a uniform space. Members of $\mathcal{U}$ are known as entourages. For $x \in X$ and $U \in \mathcal{U}$, the set $U[x] = \{y \in X : (x, y) \in U\}$ is called the U-neighborhood of x. If $D \subseteq X \times X$, and E a symmetric entourage, then

$$E \circ D \circ E = \bigcup_{(x,y) \in D} E[x] \times E[y].$$

For a finite sequence of points $\{y_i\}_{i=1}^{k}$, we have

$$\bigcup_{i=1}^{k} E[y_i] \times E[y_i] \subseteq E \circ \triangle \circ E = E \circ E.$$

For an entourage $V \in \mathcal{U}$, the family $\mathcal{C}(V) = \{V[x] : x \in X\}$ is a cover for X. A cover $\mathcal{B}$ is said to refine another cover $\mathcal{A}$ if for each $B \in \mathcal{B}$, there exists

$A \in \mathcal{A}$ such that $B \subset A$. A cover $\mathcal{A}$ of a uniform space $(X, \mathcal{U})$ is said to be a uniform cover if there is an entourage $V \in \mathcal{U}$ such that $\mathcal{C}(V)$ refines $\mathcal{A}$.

Let (X, f) be a uniform dynamical system. A point $x \in X$ is uniformly sensitive if there is a symmetric entourage $E \in \mathcal{U}$ such that for any neighborhood V of x, there are $y \in V$ and $n \geq 1$ with $(f^n(x), f^n(y)) \notin E$. The system (X, f) is uniformly sensitive if there is a symmetric entourage $E \in \mathcal{U}$ such that for any nonempty open subset O of X, there are $x, y \in O$ and $n \geq 1$ with $(f^n(x), f^n(y)) \notin E$.

The notion of shadowing was first introduced in 1970 by Anosov [4] and Bowen [7]. For any $A \subset Z^+$, define the upper density and lower density of A by

$$\overline{d}(A) = \lim_{n\to\infty} \sup \frac{1}{n} |A \cap \{0, 1, ..., n-1\}|$$

and

$$\underline{d}(A) = \lim_{n\to\infty} \inf \frac{1}{n} |A \cap \{0, 1, ..., n-1\}|$$

respectively, where $|.|$ denotes the cardinality of set. If there exists a number $d(A)$ such that $\overline{d}(A) = \underline{d}(A) = d(A)$, then we say that the set A has density $d(A)$.

Let (X, f) be a dynamical system, and $\delta > 0$. A sequence $\xi = \{\xi_j\}_{j\in\mathbb{Z}^+}$ in (X, f) is said to be $\delta-$pseudo orbit if $\rho(f(\xi_j), \xi_{j+1}) < \delta$ for all $j \in \mathbb{Z}^+$. We say that the $\delta-$pseudo orbit ξ is $\epsilon-$shadowed by a point y if $\rho(f^j(y), \xi_j) < \epsilon$ for all $j \in \mathbb{Z}^+$, where ϵ is a positive number.

For a sequence ξ, and for $\delta > 0$, denote by the sets

$$N(\xi, \delta) := \{j \in \mathbb{Z}^+ : \rho(f(\xi_j), \xi_{j+1}) < \delta\}$$

and

$$N^c(\xi, \delta) := \{j \in \mathbb{Z}^+ : \rho(f(\xi_j), \xi_{j+1}) \not< \delta\}$$

respectively. If ξ is a $\delta-$pseudo orbit, then $N(\xi, \delta) = \mathbb{Z}^+$.

For a given sequence ξ, a point y, and $\epsilon > 0$, denote by the sets

$$N(y, \xi, \epsilon) := \{j \in \mathbb{Z}^+ : \rho(f^j(y), \xi_j) < \epsilon\}$$

and

$$N^c(y, \xi, \epsilon) := \{j \in \mathbb{Z}^+ : \rho(f^j(y), \xi_j) \not< \epsilon\}$$

respectively. If ξ is $\epsilon-$shadowed by y, then $N(y, \xi, \epsilon) = \mathbb{Z}^+$.

In [12] Fakhari and Ghane introduced the notion of $\delta-$ergodic pseudo orbit and ergodic shadowing property. A sequence ξ is said to be δ-ergodic pseudo orbit if $d(N^c(\xi, \delta)) = 0$. We say that the $\delta-$ergodic pseudo orbit ξ is $\epsilon-$ergodic shadowed by some point y if $d(N^c(y, \xi, \epsilon)) = 0$. A dynamical system (X, f) is said to have ergodic shadowing property if for each ϵ there exists δ such that every $\delta-$ergodic pseudo orbit is $\epsilon-$ergodic shadowed by some point. In [10] Das and Das define the notion of $\underline{d}$-shadowing. A dynamical system (X, f) is said to have $\underline{d}$-shadowing if for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-ergodic pseudo orbit ξ is ϵ-shadowed by some point y in such a way that $\underline{d}(N(y, \xi, \epsilon)) > 0$. A dynamical system (X, f) is said to have eventual shadowing property if for every $\epsilon > 0$ there exists $\delta > 0$ such that for each δ-pseudo orbit ξ there exist $N > 0$, $y \in X$ such that $\rho(f^i(y), \xi_i) < \epsilon$ for all $i \geq N$.

We want to extend the notations $N(\xi, \delta), N^c(\xi, \delta), N(y, \xi, \epsilon), N^c(y, \xi, \epsilon)$ to uniform dynamical systems and Hausdorff dynamical systems. We put suffixes U and H in the corresponding notations of uniform dynamical systems and Hausdorff dynamical systems respectively. Let (X, f) be a uniform dynamical system, and D an entourage. A sequence $\xi = \{\xi_j\}_{j\in\mathbb{Z}^+}$ in (X, f) is said to be $D-$pseudo orbit if $(f(\xi_j), \xi_{j+1}) \in D$ for all $j \in \mathbb{Z}^+$. We say that the $D-$pseudo orbit ξ is $E-$shadowed by a point y if $(f^j(y), \xi_j) \in E$ for all $j \in \mathbb{Z}^+$, where E is an entourage. For a sequence ξ, and for an entourage D, denote by the sets

$$N_U(\xi, D) := \{j \in \mathbb{Z}^+ : (f(\xi_j), \xi_{j+1}) \in D\}$$

and

$$N_U^c(\xi, D) := \{j \in \mathbb{Z}^+ : (f(\xi_j), \xi_{j+1}) \notin D\}$$

respectively. If ξ is a $D-$pseudo orbit, then $N_U(\xi, D) = \mathbb{Z}^+$. For a given sequence ξ, a point y, and an entourage E, denote by the sets

$$N_U(y, \xi, E) := \{j \in \mathbb{Z}^+ : (f^j(y), \xi_j) \in E\}$$

and

$$N_U^c(y, \xi, E) := \{j \in \mathbb{Z}^+ : (f^j(y), \xi_j) \notin E\}$$

respectively. If ξ is $E-$shadowed by y, then $N_U(y, \xi, E) = \mathbb{Z}^+$. A sequence ξ is said to be uniform ergodic D-pseudo orbit if $d(N_U^c(\xi, D)) = 0$. We say that the uniform ergodic $D-$pseudo orbit ξ is uniform ergodic $E-$shadowed by some point y if $d(N_U^c(y, \xi, E)) = 0$.

Let (X, f) be a Hausdorff dynamical system, and $\mathcal{A}, \mathcal{C}$ be finite open covers. A sequence $\xi = \{\xi_j\}_{j\in\mathbb{Z}^+}$ in (X, f) is said to be $\mathcal{A}-$pseudo orbit if for each $j \in \mathbb{Z}^+$, there exists $A \in \mathcal{A}$ such that $\{f(\xi_j), \xi_{j+1}\} \subseteq A$. We say that the $\mathcal{A}-$pseudo orbit ξ is $\mathcal{C}-$shadowed by a point y if for each $j \in \mathbb{Z}^+$, there exists $C \in \mathcal{C}$ such that $\{f^j(y), \xi_j\} \subseteq C$. For a sequence ξ, and a finite open cover $\mathcal{A}$, denote by the sets

$$N_H(\xi, \mathcal{A}) := \{j \in \mathbb{Z}^+ : \{f(\xi_j), \xi_{j+1}\} \subseteq A, \text{ for some } A \in \mathcal{A}\}$$

and

$$N_H^c(\xi, \mathcal{A}) := \{j \in \mathbb{Z}^+ : \{f(\xi_j), \xi_{j+1}\} \nsubseteq A, \text{ for any } A \in \mathcal{A}\}$$

respectively. If ξ is a $\mathcal{A}-$pseudo orbit, then $N_H(\xi,\mathcal{A}) = \mathbb{Z}^+$. For a given sequence ξ, a point y, and a finite open cover $\mathcal{C}$, denote by the sets

$$N_H(y,\xi,\mathcal{C}) := \{j \in \mathbb{Z}^+ : \{f^j(y),\xi_j\} \subseteq C, \ \text{ for some } C \in \mathcal{C}\}$$

and

$$N_H^c(y,\xi,\mathcal{C}) := \{j \in \mathbb{Z}^+ : \{f^j(y),\xi_j\} \nsubseteq C, \ \text{ for any } C \in \mathcal{C}\}$$

respectively. If ξ is $\mathcal{C}-$shadowed by y, then $N_H(y,\xi,\mathcal{C}) = \mathbb{Z}^+$.

3. Main results

Definition 3.1. Let (X, f) be a uniform dynamical system, where X is a Hausdorff Uniform space. A point $x \in X$ is said to be uniformly equicontinuous if for each entourage $U \in \mathcal{U}$, there is $D \in \mathcal{U}$ such that $(f^n(x), f^n(y)) \in U$ whenever $y \in D[x]$ and $n \in \mathbb{Z}^+$. The set of uniformly equicontinuous points is denoted by ε. We say that (X, f) is almost uniformly equicontinuous if the set of equicontinuous points is dense. (X, f) is uniformly equicontinuous if every point of X is a uniformly equicontinuous point.

Put $\varepsilon_U = \{x \in X : \exists D \in \mathcal{U}$, such that for all $y, z \in D[x]$, for all $n \geq 0, (f^n(y), f^n(z)) \in U\}$. We call ε_U the $U-$equicontinuous set of (X, f). In the following theorem we prove that ε_U is inversely invariant and open. It is an extension of the Proposition 2.30 ([17]) to uniform dynamical systems.

Theorem 3.2. *Let (X, f) be a Hausdorff dynamical system where X is a compact Hausdorff space. Let $\mathcal{U}$ be the unique Uniformity on X which induces the topology of X. Then, ε_U is inversely invariant, open and* $\varepsilon = \bigcap\limits_{U\in\mathcal{U}} \varepsilon_U$.

Proof. First, we show that ε_U is inversely invariant. Let $x \in f^{-1}(\varepsilon_U)$. We can find a symmetric entourage $D \in \mathcal{U}$ such that $D \circ D \subset U$, and $(f^n(y), f^n(z)) \in U$ for all $n \geq 0$ whenever $y, z \in D[f(x)]$. By uniform continuity, we can find a symmetric entourage E with $E \subset D$ such that $(f(z), f(y)) \in D$ whenever $(z, y) \in E$. For $y, z \in E[x]$, we have $f(y), f(z) \in D[f(x)]$. It follows that $(f^{n+1}(y), f^{n+1}(z)) \in U$ for all $n \geq 0$ whenever $y, z \in E[x]$.

Now, it is clear that $x \in \varepsilon_U$. Therefore ε_U is inversely invariant. Next, we show that ε_U is open.

Let $x \in \varepsilon_U$ be an arbitrary point. As in above, we can find a symmetric entourage D with $D \circ D \subset U$ such that $(f^n(y), f^n(z)) \in U$ for all $n \geq 0$, whenever $y, z \in D[x]$. Let $D' \in \mathcal{U}$ be a symmetric entourage such that $D' \circ D' \subset D$. Now, $D'[x] \subset \varepsilon_U$. Indeed, if $w \in D'[x]$ and if $y, z \in D'[w]$, it can easily be shown that $y, z \in D[x]$, and therefore $(f^n(y), f^n(z)) \in U$ for all $n \geq 0$. It follows that $w \in \varepsilon_U$. Therefore, $D'[x] \subset \varepsilon_U$. Hence ε_U is open.

If $x \in \varepsilon_U$ for all $U \in \mathcal{U}$, then $x \in \varepsilon$. Conversely, let $x \in \varepsilon$, and $U \in \mathcal{U}$ be arbitrary. Take $U' \in \mathcal{U}$ such that $U' \circ U' \subset U$. There exists $D \in \mathcal{U}$ such that for all $y \in D[x]$, and for all n, we have $(f^n(x), f^n(y)) \in U'$. For $y, z \in D[x]$, we have $(f^n(y), f^n(z)) \in U' \circ U' \subset U$. Therefore, $\varepsilon = \bigcap_{U \in \mathcal{U}} \varepsilon_U$. □

In [1], it is proved that a transitive system is either sensitive or almost equicontinuous. Every compact Hausdorff space is a Baire space, therefore countable intersections of open dense sets is dense. In case of a compact metric space, owing to second countability, one can express $\varepsilon = \bigcap_{U \in \mathcal{U}} \varepsilon_U$ as countable intersections of dense open sets and thereby showing that ε is dense.

In the following we give similar result for a compact Hausdorff space.

Theorem 3.3. *A transitive system on a compact Hausdorff space is either uniformly sensitive or ε_U is dense for each $U \in \mathcal{U}$.*

Proof. Let (X, f) be a transitive system, where X is a compact Hausdorff space, and $\mathcal{U}$ the unique uniformity which induce the topology of X. Let $U \in \mathcal{U}$ be an entourage. Then ε_U is inversely invariant and open. Therefore $f^{-n}(\varepsilon_U) \subset \varepsilon_U$ for all $n \geq 1$. Assume that ε_U is nonempty and non-dense. Then, $O = X - \overline{\varepsilon_U}$ is open and nonempty. So, $\phi \neq O \cap f^{-n}(\varepsilon_U) \subseteq O \cap \varepsilon_U = \phi$, this is a contradiction. Therefore, ε_U is either empty or dense. If $\varepsilon_U = \phi$ for some $U \in \mathcal{U}$, then the system is sensitive with sensitivity entourage U', where U' is a symmetric entourage such that $U' \circ U' \subset U$. Indeed for any $x \in X$ and for any open neighborhood $V(x)$ of x, there exist $y, z \in V(x)$ and $n \geq 0$ such that $(f^n(y), f^n(z)) \notin U$. It follows that either $(f^n(y), f^n(x)) \notin U'$ or $(f^n(z), f^n(x)) \notin U'$. This completes the proof. □

We want to extend the notion of ergodic shadowing property to Hausdorff dynamical system and uniform dynamical system. We prove that Hausdorff ergodic shadowing property is equivalent to uniform ergodic shadowing property when the phase space is a compact Hausdorff space. Further, we prove that they are equivalent to ergodic shadowing property when the phase space is a compact metric space.

Definition 3.4. Let (X, f) be a Hausdorff dynamical system, and $\mathcal{A}, \mathcal{C}$ finite open covers of X. A sequence ξ is said to be Hausdorff ergodic $\mathcal{A}-$pseudo orbit if $d(N_H^c(\xi, \mathcal{A})) = 0$ and it is said to be Hausdorff ergodic $\mathcal{C}-$shadowed by a point y if $d(N_H^c(y, \xi, \mathcal{C})) = 0$.

Definition 3.5. A Hausdorff dynamical system (X, f) is said to have Hausdorff ergodic shadowing property if for any finite open cover $\mathcal{C}$, there exists a finite open cover $\mathcal{A}$ such that every ergodic $\mathcal{A}$-pseudo orbit ξ is Hausdorff ergodic $\mathcal{C}$-shadowed by some point y, that is,$d(N_H^c(y,\xi,\mathcal{C}))=0$.

Definition 3.6. Let (X,f) be a uniform dynamical system. (X,f) is said to have uniform ergodic shadowing property if for each entourage $E\in\mathcal{U}$, there exists an entourage $D\in\mathcal{U}$ such that every uniform ergodic D-pseudo orbit ξ is uniform ergodic E-shadowed by some point y, that is, $d(N_U^c(y,\xi,E))=0$.

Theorem 3.7. *Let (X,f) be a Hausdorff dynamical system, where X is a compact Hausdorff space. Then the following claims are equivalent:*

(1) *(X,f) has Hausdorff ergodic shadowing property.*
(2) *(X,f) has uniform ergodic shadowing property.*

If X is metric, then (1) *and* (2) *are equivalent to:*

(3) *(X,f) has ergodic shadowing property.*

Proof. (1) $\Rightarrow$ (2): Assume that (X,f) is Hausdorff ergodic shadowing. Let $\mathcal{U}$ be the unique uniformity on X that induces its topology. Let $E\in\mathcal{U}$ be a symmetric entourage, and $E^{'}$ a symmetric entourage such that $E^{'}\circ E^{'}\subset E$. Then $\{E^{'}[z]:z\in X\}$ is an open cover of X. By compactness, there are $z_1,z_2,...,z_m$ in X such that $\mathcal{C}=\{E'[z_i]:i=1,2,...,m\}$ is a finite subcover. By (1), there exists a finite open cover $\mathcal{A}=\{A_1,A_2,...,A_k\}$ such that every ergodic $\mathcal{A}$-pseudo orbit is ergodic $\mathcal{C}$-shadowed by some point. Now, $D=\bigcup_{i=1}^{k}A_i\times A_i$ is a symmetric entourage. It is obvious that every uniform ergodic $D-$pseudo orbit is a Hausdorff ergodic $\mathcal{A}-$pseudo orbit and vice-versa. Let ξ be an ergodic $\mathcal{A}-$pseudo orbit. Then there exists $y\in X$ such that

$$d(N_H^c(y,\xi,\mathcal{C}))=0. \tag{3.1}$$

Equation (3.1) is equivalent to $d\{j\in\mathbb{Z}^+:(f^j(y),\xi_j)\notin E^{'}\circ E^{'}\subset E\}=0$. It follows that $d(N_U^c(y,\xi,E))=0$.

(2) $\Rightarrow$ (1): Assume that (X,f) has uniform ergodic shadowing property. Let $\mathcal{C}=\{C_1,C_2,...,C_k\}$ be a finite open cover of X. Put $E=\bigcup\limits_{i=1}^{k}C_i\times C_i$, then E is a symmetric entourage. By hypothesis, there exists an entourage $D\in\mathcal{U}$ such that every uniform ergodic D-pseudo orbit is uniform ergodic E-shadowed by some point. Let $D^{'}$ be a symmetric entourage such that $D^{'}\circ D^{'}\subset D$. Since X is compact, there exists a set of finite points $z_1,z_2,...,z_m$ such that $X=\bigcup\limits_{i=1}^{m}D^{'}[z_i]$. Put $\mathcal{A}=\{D^{'}[z_1],D^{'}[z_2],...,D^{'}[z_m]\}$. Then it is obvious that every Hausdorff ergodic $\mathcal{A}$-pseudo orbit is a uniform ergodic D-pseudo orbit. Let ξ be a Hausdorff ergodic $\mathcal{A}-$pseudo orbit. By (2), there exists a point $y\in X$ such that

$$d(N_U^c(y,\xi,E))=0. \tag{3.2}$$

Equation (3.2) is equivalent to $d\{j \in \mathbb{Z}^+ : (f^j(y), \xi_j) \notin C_i \times C_i$, for any $C_i \in \mathcal{C}\} = 0$. It follows that $d(N_H^c(y, \xi, \mathcal{C})) = 0$.

For the metric space X:

(1) $\Rightarrow$ (3): Let $\epsilon > 0$ be arbitrary. Denote by $B(z, \epsilon)$, the open ball of radius ϵ with center at z. Then $\{B(z, \epsilon/2) : z \in X\}$ is an open cover of X. Since X is compact, there exists a finite set $\{z_1, z_2, ..., z_m\}$ of points in X such that $\mathcal{C} = \{B(z_i, \epsilon/2) : 1 \leq i \leq m\}$ is a finite open cover of X. By (1), there exists a finite open cover $\mathcal{A}$ such that every ergodic $\mathcal{A}$-pseudo orbit is ergodic $\mathcal{C}$-shadowed by some point. Let δ be a Lebesgue number of $\mathcal{A}$. Let ξ be an ergodic δ-pseudo orbit, that is, $d\{j \in \mathbb{Z}^+ : \rho(f(\xi_j), \xi_{j+1}) \not< \delta\} = 0$. It implies that $d\{j \in \mathbb{Z}^+ : \{f(\xi_j), \xi_{j+1}\} \nsubseteq A$, for any $A \in \mathcal{A}\} = 0$. By (1), there exists $y \in X$ such that $d\{j \in \mathbb{Z}^+ : \{f^j(y), \xi_j\} \nsubseteq B(z_i, \epsilon/2)$, for any $i \in \{1, 2, ..., m\}\} = 0$. It is equivalent to $d(N^c(y, \xi, \epsilon)) = 0$. Hence (3) holds.

(3) $\Rightarrow$ (1): Assume that (X, f) has ergodic shadowing property. Let $\mathcal{C}$ be a finite open cover of X, and $\epsilon > 0$ a Lebesgue number for $\mathcal{C}$. Since (X, f) has ergodic shadowing property, for $\epsilon > 0$ there exists $\delta > 0$ such that every ergodic δ-pseudo orbit of f is ergodic ϵ-shadowed by some point in X. Since $\{B(z, \delta/2) : z \in X\}$ is an open cover of X, there are finite points $z_1, z_2, ..., z_m \in X$ such that $\mathcal{A} = \{B(z_i, \delta/2) : 1 \leq i \leq m\}$ is a finite open cover. Let ξ be an ergodic $\mathcal{A}$- pseudo orbit. Then, $d\{j \in \mathbb{Z}^+ : \{f(y_j), y_{j+1}\} \nsubseteq B(z_i, \delta/2)$, for any $i \in \{1, 2, ..., m\}\} = 0$. It implies that ξ is an ergodic $\delta-$pseudo orbit. Therefore, there exists $y \in X$ such that $d(N^c(y, \xi, \epsilon)) = 0$. It implies that $d\{j \in \mathbb{Z}^+ : \{f^j(y), \xi_j\} \nsubseteq C$, for any $C \in \mathcal{C}\} = 0$. Hence (1) is proved. □

We define uniform $\underline{d}$-shadowing property and Hausdorff $\underline{d}$-shadowing property in uniform dynamical systems and Hausdorff dynamical systems respectively.

Definition 3.8. A uniform dynamical system (X, f) is said to have uniform $\underline{d}$-shadowing property if for every entourage E there is an entourage D such that every uniform ergodic $D-$pseudo orbit ξ is E-shadowed by some point y in such a way that $\underline{d}(N_U(y, \xi, E)) > 0$.

Definition 3.9. A Hausdorff dynamical system (X, f) is said to have Hausdorff $\underline{d}$-shadowing property if for every finite open cover $\mathcal{A}$, there is a finite open cover $\mathcal{B}$ such that every ergodic $\mathcal{B}$-pseudo orbit ξ is $\mathcal{A}$-shadowed by some point y in such a way that $\underline{d}(N_H(y, \xi, \mathcal{A})) > 0$.

Theorem 3.10. *Let (X, f) be a Hausdorff dynamical system, where X is a compact Hausdorff space. Then, the following claims are equivalent:*

(1) *(X, f) has Hausdorff $\underline{d}$-shadowing property.*
(2) *(X, f) has Uniform $\underline{d}$-shadowing property.*

If X is metric, then (1) and (2) are equivalent to:

(3) *(X, f) has $\underline{d}$-shadowing property.*

Proof. The proof is similar to Theorem 3.7. □

In [10] Das and Das prove that if a uniform dynamical system (X, f) has uniform ergodic shadowing property then for any natural number k, (X, f^k) has uniform ergodic shadowing property. We want to extend this result to Hausdorff dynamical system. In this theorem, we don't necessarily restrict the space X to be a compact space. When the space is a compact Hausdorff space, due to Theorem 3.7, the result of Das and Das in [10] is equivalent to the following theorem.

Theorem 3.11. *If (X, f) has Hausdorff ergodic shadowing property then for any natural number k, (X, f^k) has Hausdorff ergodic shadowing property.*

Proof. Suppose (X, f) has Hausdorff ergodic shadowing property. Fix $k > 1$, and let $\mathcal{A}$ be a finite open cover of X. Then there exists a finite open cover $\mathcal{B}$ of X such that every ergodic $\mathcal{B}$-pseudo orbit is $\mathcal{A}$-shadowed by some point in X. Let $\xi = \{\xi_j\}_{j\in\mathbb{Z}^+}$ be an ergodic $\mathcal{B}$-pseudo orbit for f^k. Putting $\{f(\xi_i), f^2(\xi_i), f^3(\xi_i), ..., f^{k-1}(\xi_i)\}$ in between ξ_i and ξ_{i+1} for each $i \in \mathbb{Z}^+$, we get an ergodic $\mathcal{B}$-pseudo orbit $\gamma = \{y_j\}_{j\in\mathbb{Z}^+}$ for f such that $y_{ik} = \xi_i$ for all $i \in \mathbb{Z}^+$. Since f has Hausdorff ergodic shadowing and the sequence $\{ik\}_{i\in\mathbb{Z}^+}$ has positive density $1/k$, the fact that γ is being ergodic $\mathcal{A}$-shadowed by some point y with respect to f implies ξ is ergodic $\mathcal{A}$ shadowed by y with respect to f^k. □

Lastly we want to extend the notion of eventual shadowing property to Hausdorff dynamical system and uniform dynamical system. We prove that Hausdorff eventual shadowing property is equivalent to uniform eventual shadowing property when the phase space is a compact Hausdorff space. Further, we prove that they are equivalent to eventual shadowing property when the phase space is a compact metric space.

Definition 3.12. A Hausdorff dynamical system (X, f) is said to have Hausdorff eventual shadowing property on a closed invariant subset Λ of X if for every finite open cover $\mathcal{A}$ of Λ there is a finite open cover $\mathcal{B}$ of Λ such that for each $\mathcal{B}$-pseudo orbit ξ in Λ there exist $N > 0$, and $y \in X$ such that for each $i \geq N$ there exists $A \in \mathcal{A}$ with $\{f^i(y), x_i\} \subset A$. If $\Lambda = X$, then (X, f) is said to have Hausdorff eventual shadowing property on X.

Definition 3.13. A uniform dynamical system (X, f) is said to have uniform eventual shadowing property on a closed invariant subset Λ of X if for each entourage $E \in \mathcal{U}$ there exists an entourage $D \in \mathcal{U}$ such that for each D-pseudo orbit ξ in Λ there exist $N > 0$, and a point $y \in X$ such that $(f^i(y), \xi_i) \in E$ for all $i \geq N$.

Remark 3.14. Let X be a compact Hausdorff space, and $\mathcal{U}$ the unique uniformity of X that induces its topology. Then, for every open cover $\mathcal{A}$ of X there exists $V \in \mathcal{U}$ such that $\mathcal{C}(V)$ refines $\mathcal{A}$ (see [16, Proposition 8.16]).

Theorem 3.15. *Let (X, f) be a Hausdorff dynamical system, where X is a compact Hausdorff space. Then, the following statements are equivalent:*

(1) *(X, f) has Hausdorff eventual shadowing property.*
(2) *(X, f) has Uniform eventual shadowing property.*

If X is metric, then (1) *and* (2) *are equivalent to:*

(3) *(X, f) has eventual shadowing property.*

Proof. (1) $\Rightarrow$ (2): Suppose (X, f) has Hausdorff eventual shadowing property. Let $\mathcal{U}$ be the unique uniformity inducing the topology on X. Let $E, D \in \mathcal{U}$ be symmetric entourages such that $D \circ D \subset E$. $\{D[z] : z \in X\}$ is an open cover of X. Since X is compact, there exist points $z_1, z_2, z_3, ..., z_m$ in X such that $\mathcal{A} = \{D[z_1], D[z_2], ..., D[z_m]\}$ is a finite open cover of X. By (1), there exists a finite open cover $\mathcal{B}$ such that every $\mathcal{B}$-pseudo orbit is Hausdorff eventual shadowed by some point in X. Let $V \in \mathcal{U}$ be a symmetric entourage such that $\mathcal{C}(V)$ refines $\mathcal{B}$ [by Remark 3.14]. Let ξ be a V-pseudo orbit. Then $(f(\xi_i), \xi_{i+1}) \in V$ for all $i \in \mathbb{Z}^+$, that is, $f(\xi_i) \in V[\xi_{i+1}]$ for all $i \in \mathbb{Z}^+$. Since $\mathcal{C}(V)$ refines $\mathcal{B}$, there exists $B \in \mathcal{B}$ such that $V[\xi_{i+1}] \subset B$. It follows that ξ is a $\mathcal{B}$-pseudo orbit. Hence, there exist $y \in X$, and $N > 0$ such that for each $i \geq N$ there exists $j \in \{1, 2, ..., m\}$ with $\{f^i(y), \xi_i\} \subset D[z_j]$. It follows that $(f^i(y), \xi_i) \in E$ for all $i \geq N$.

(2) $\Rightarrow$ (1): Suppose (X, f) has uniform eventual shadowing property. Let $\mathcal{A} = \{A_1, A_2, ..., A_m\}$ be a finite open cover of X. Put $E = \bigcup_{j=1}^{m} A_j \times A_j$, then E is a symmetric entourage in $\mathcal{U}$, where $\mathcal{U}$ is the unique uniformity inducing the topology of X. There is an entourage D such that every D-pseudo orbit is eventual $E-$shadowed by some point. Let V be a symmetric entourage such that $V \circ V \subset D$. Now, $\{V[z] : z \in X\}$ is an open cover of X. Since X is compact, there exist points $z_1, z_2, ..., z_n$ such that $\mathcal{B} = \{V[z_1], ..., V[z_n]\}$ is a finite open cover of X. It is obvious that every $\mathcal{B}$-pseudo orbit is a $D-$pseudo orbit. Let ξ be a $\mathcal{B}-$pseudo orbit. By hypothesis, there exist a point y, and a positive integer N such that $(f^i(y), \xi_i) \in E$ for all $i \geq N$. So, for each $i \geq N$ there exists $j \in \{1, 2, ..., m\}$ with $\{f^i(y), \xi_i\} \subseteq A_j$.

For the metric space X:

(1) $\Rightarrow$ (3): Suppose (X, f) has Hausdorff eventual shadowing property. Let $\epsilon > 0$. Then, $\{B(z, \epsilon/2) : z \in X\}$ is an open cover of X. Since X is compact there exist points $z_1, z_2, ..., z_m$ in X such that $\mathcal{A} = \{B(z_1, \epsilon/2), B(z_2, \epsilon/2), ..., B(z_m, \epsilon/2)\}$ is a finite open cover of X. By hypothesis, there exists a finite open cover $\mathcal{B}$ such that every $\mathcal{B}$-pseudo orbit is eventual $\mathcal{A}-$shadowed by some point. Let $\delta > 0$ be a Lebesgue number of $\mathcal{B}$, and ξ a δ-pseudo orbit. It can be shown that ξ is a $\mathcal{B}$-pseudo orbit. So, there exist a positive integer N, and a point y such that for each $i \geq N$ there exists $j \in \{1, 2, ..., m\}$ with $\{f^i(y), \xi_i\} \subseteq B(z_j, \epsilon/2)$. It is equivalent to say that $\rho(f^i(y), \xi_i) < \epsilon$ for all $i \geq N$.

(3) $\Rightarrow$ (1): Suppose (X, f) has eventual shadowing property. Let $\mathcal{A}$ be a finite open cover of X. Let $\epsilon > 0$ be a Lebesque number of $\mathcal{A}$. By hypothesis, there exists $\delta > 0$ such that every δ-pseudo orbit is eventual $\epsilon-$shadowed by some point. $\{B(z, \delta/2) : z \in X\}$ is an open cover of X. Since X is compact, there exist finite points $z_1, z_2, ..., z_m$ such that $\mathcal{B} = \{B(z_j, \delta/2) : 1 \leq j \leq m\}$ is a finite open cover. It is obvious that every $\mathcal{B}-$pseudo orbit is a $\delta-$pseudo orbit. Let ξ be a $\mathcal{B}$-pseudo orbit. Then, there exist a positive integer N, and a point y such that $\rho(f^i(y), \xi_i) < \epsilon$ for all $i \geq N$. So, for each $i \geq N$, there exists $A_j \in \mathcal{A}$ with $\{f^i(y), \xi_i\} \subseteq A_j$. $\square$

In [18] the author proves that if a continuous map f has eventual shadowing property on X, then f has eventual shadowing property on Λ, where Λ is a closed f-invariant set of X. We extend this result to the case when X is a compact Hausdorff space.

Theorem 3.16. *Let Λ be a closed f-invariant subset of X. If f has Hausdorff eventual shadowing property on X, then f has Hausdorff eventual shadowing property on Λ.*

Proof. Suppose f has Hausdorff eventual shadowing property on X. Let $\mathcal{A}$ be a finite open cover of Λ. Then $\mathcal{A}' = \mathcal{A} \bigcup (X - \Lambda)$ is an open cover of X. There exists a finite open cover $\mathcal{B}$ of X such that every $\mathcal{B}$-pseudo orbit is Hausdorff eventual $\mathcal{A}$-shadowed by some point in X. Let ξ be a $\mathcal{B}$-pseudo orbit in $\Lambda \subset X$. Then there exist $N > 0$, and $y \in X$ such that for each $i \geq N$, there exists $A \in \mathcal{A}'$ with $\{f^i(y), \xi_i\} \subseteq A$. If $A = X - \Lambda$, then it follows that $\xi_i \notin \Lambda$, a contradiction. Therefore, $\{f^i(y), \xi_i\} \subset A$, for some $A \in \mathcal{A}$. $\square$

Remark 3.17. Because of Theorem 3.15, the above theorem can also be stated as: Let Λ be a closed f-invariant subset of X. If f has uniform eventual shadowing property on X, then f has uniform eventual shadowing property on Λ.

Theorem 3.18. *Let Λ be an f-invariant dense subset of X, where X is a compact Hausdorff space. If f has uniform eventual shadowing property on Λ, then f has uniform eventual shadowing property on X.*

Proof. Suppose f has the uniform eventual shadowing property on Λ. Let $E, D \in \mathcal{U}$ be symmetric entourages such that $D \circ D \subset E$. Then there exists $U \in \mathcal{U}$ such that every $U-$pseudo orbit in Λ is uniform eventual $D-$shadowed by some point. Let V be a symmetric entourage such that $V \circ V \circ V \subset U$. Let ξ be a V-pseudo orbit in X. Without loss of generality we can take $V \circ V \circ V \subset U \subset D$. Since f is uniformly continuous, there exists a symmetric entourage $W \in \mathcal{U}$ with $W \subset V$ such that for any $x, y \in X$ if $(x, y) \in W$ then $(f(x), f(y)) \in V$. Since Λ is dense in X, we have $W[\xi_i] \cap \Lambda \neq \phi$ for all $i \geq 0$. Let $y_i \in W[\xi_i] \cap \Lambda$. It follows that $(f(\xi_i), f(y_i)) \in V$ for all $i \geq 0$. Now, $(f(\xi_i), \xi_{i+1})) \in V, (\xi_{i+1}, y_{i+1}) \in W, (f(y_i), f(\xi_i)) \in V$. It follows that $(f(y_i), y_{i+1}) \in V \circ V \circ V \subset U \subset D$. Therefore, $\eta = \{y_0, y_1, ..., y_n, ...\}$ is a U-pseudo orbit in Λ. Since f has uniform eventual shadowing property on Λ, there exist $N > 0$, and a point $z \in X$ such that $(f^i(z), y_i) \in D$ for all $i \geq N$. Therefore, $(f^i(z), \xi_i) \in D \circ D \subset E$. □

Remark 3.19. Because of Theorem 3.15, the above theorem can also be stated as: Let Λ be an f-invariant dense subset of X, where X is a compact Hausdorff space. If f has Hausdorff eventual shadowing property on Λ, then f has Hausdorff eventual shadowing property on X.

Acknowledgments: We are thankful to the anonymous referees for their valuable comments and suggestions towards the improvement of the paper.

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