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SOME INEQUALITIES ON POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let $P(z)$ be a polynomial of degree n. A well-known inequality due to S. Bernstein states that if $P \in P_n$, then

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{|z|=1} |P(z)|.$$

In this paper, we establish some extensions and refinements of the above inequality to polar derivative and some other well-known inequalities concerning the polynomials and their ordinary derivatives.

1. Introduction

Let $P_n(z)$ be the set of complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. According to a well-known classical result due to Bernstein [4], if $P \in P_n$, then

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

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Inequality (1.1) is sharp and equality holds if $p(z)$ has all its zeros at the origin. It was proved by Frappier et al. [5] that, if $P \in P_n$, then

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{1\leq k\leq n} |P(e^{ik\pi n})|. \tag{1.2}$$

It is evident that inequality (1.2) is a refinement of (1.1), since the maximum of $|P(z)|$ on $|z| = 1$ may be larger than the maximum of $|P(z)|$ taken over $(2n)^{th}$ roots of unity, as is shown by the simple example $P(z) = z^n + ia, a > 0$. Aziz [2] improved the bound of inequality (1.2) by proving that, if $P \in P_n$, then for every real α,

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{2}(M_\alpha + M_{\alpha+\pi}), \tag{1.3}$$

where

$$M_\alpha = \max_{1\leq k\leq n} |P(e^{i(\alpha+2k\pi)n})| \tag{1.4}$$

and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha + \pi$.

If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then Erdös conjectured and later Lax [8] proved that

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.5}$$

Inequality (1.5) was improved by Aziz [2] by proving that if $P \in P_n$ and $P(z)$ has no zero in $|z| < 1$, then for every real α,

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{2}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{1.6}$$

where M_α is as defined in (1.4).

In this direction, Rather et al. [12] consider the class $P_{n,\mu}(z)$ of all complex polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \leq \mu \leq n$ of degree n and obtained a generalization of inequality (1.6) by proving that, if $P \in P_{n,\mu}$, has no zero in $|z| < k, k \geq 1$, then for every real α,

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{\sqrt{2(1 + k^{2\mu})}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{1.7}$$

where M_α is as defined in (1.4). They [12] further improved the bound of (1.7) by involving $m = \min_{|z|=k} |P(z)|$ and obtained

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{\sqrt{2(1 + k^{2\mu})}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}}, \tag{1.8}$$

where M_α is as defined in (1.4). Further, by involving some coefficients of the polynomials in $P_{n,\mu}(z)$, Rather et al. [12] improved the bound (1.7) and proved that, if $P \in P_{n,\mu}$, has no zero in $|z| < k$, $k \geq 1$, then for every real α,

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{\sqrt{2\left\{1 + k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\}}}(M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{1.9}$$

where M_α is as defined in (1.4). Rather et al. [12] improved inequality (1.8) under the same hypothesis and obtained,

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{\sqrt{2\left\{1 + k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}\right)^2\right\}}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}}, \tag{1.10}$$

where $m = \min_{|z|=k} |P(z)|$ and M_α is as defined in (1.4).

For a polynomial $P(z)$ of degree n, we now define the polar derivative of $P(z)$ with respect to a real or complex number β as

$$D_\beta p(z) = np(z) + (\beta - z)p^{'}(z).$$

This polynomial $D_\beta P(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative $P^{'}(z)$ in the sense that

$$\lim_{\beta\to\infty} \frac{D_\beta P(z)}{\beta} = P^{'}(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

Aziz [1] was among the first who extended some of the above inequalities to polar versions by replacing the derivative of the polynomial with the polar derivative of the polynomial. He, in fact, extended inequality (1.5) to polar derivative by proving that if $P(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every real or complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta P(z)| \leq \frac{n}{2}(|\beta| + 1) \max_{|z|=1} |P(z)|. \tag{1.11}$$

Dividing both sides of (1.11) by $|\beta|$ and letting $|\beta| \to \infty$, we get inequality (1.5).

Over the last four decades many different authors produced a large number of results concerning the polar derivative of polynomials. More information on classical results and polar derivatives can be found in the books of Milovanović et al. [10] and Marden [9].

2. Lemmas

We need the following important lemmas to prove our theorems. Next lemma is a special case of a result due to Govil and Rahman [7].

Lemma 2.1. *If $P(z)$ is a polynomial of degree n, then on $|z|=1$,*

$$|P^{'}(z)|+|Q^{'}(z)|\leq n\max_{|z|=1}|P(z)|, \tag{2.1}$$

where $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}$.

The following lemma is due to Aziz [2].

Lemma 2.2. *If $P(z)$ is a polynomial of degree n, then for $|z|=1$ and for every real α,*

$$|P^{'}(z)|^2+|nP(z)-zP^{'}(z)|^2\leq\frac{n^2}{2}(M_\alpha^2+M_{\alpha+\pi}^2), \tag{2.2}$$

where M_α is as defined in (1.4).

The following two lemmas are due to Aziz and Rather [3].

Lemma 2.3. *If $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$ is a polynomial of degree at most n and $P(z)\neq 0$ for $|z|<k, k\geq 1$, then for $|z|=1$,*

$$k^\mu|P^{'}(z)|\leq|nP(z)-zP^{'}(z)|. \tag{2.3}$$

Lemma 2.4. *If $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$ is a polynomial of degree at most n and $P(z)\neq 0$ for $|z|<k, k\geq 1$, then for $|z|=1$,*

$$k^\mu|P^{'}(z)|\leq|nP(z)-zP^{'}(z)|-nm, \tag{2.4}$$

where $m=\min_{|z|=k}|P(z)|$.

The next lemma was proved by Qazi [11].

Lemma 2.5. *If $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$ is a polynomial of degree at most n and $P(z)\neq 0$ for $|z|<k$, $k\geq 1$, then for $|z|=1$,*

$$k^{\mu+1}\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}|P^{'}(z)|\leq|nP(z)-zP^{'}(z)|. \tag{2.5}$$

The following lemma is due to Gardner et al. [6].

Lemma 2.6. *If* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree at most* n *and* $P(z) \neq 0$ *for* $|z| < k$, $k \geq 1$, *and* $m = \min_{|z|=k} |P(z)|$, *then for* $|z| = 1$,

$$k^{\mu+1}\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}|P^{'}(z)| \leq |nP(z) - zP^{'}(z)| - nm. \tag{2.6}$$

3. Main results

In this paper, we extend inequalities (1.7), (1.8), (1.9) and (1.10) to polar derivative. In this direction, we first prove the following extensions of inequalities (1.7) and (1.8) to polar derivative.

Theorem 3.1. *If* $P \in P_{n,\mu}$, *has no zero in* $|z| < k$, $k \geq 1$ *and* $m = \min_{|z|=k} |P(z)|$, *then for every real or complex number* β *with* $|\beta| \geq 1$ *and for every real* α,

$$|D_\beta P(z)| \leq n \max_{|z|=1} |P(z)| + (|\beta| - 1)\frac{n}{\sqrt{2(1+k^{2\mu})}}(M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}}, \tag{3.1}$$

where M_α *is as defined in* (1.4).

Proof. If $P(z)$ is a polynomial of degree n, then for $|z| = 1$,

$$|Q^{'}(z)| = |nP(z) - zP^{'}(z)|, \tag{3.2}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

Now, for every real or complex number β, the polar derivative of $P(z)$ with respect to β is

$$D_\beta P(z) = nP(z) + (\beta - z)P^{'}(z),$$

which further implies for $|z| = 1$, from Lemma 2.1, we have

$$\begin{aligned} |D_\beta P(z)| &= |nP(z) - zP^{'}(z)| + |\beta||P^{'}(z)| \\ &= |Q^{'}(z)| + |\beta||P^{'}(z)| \\ &\leq n \max_{|z|=1} |P(z)| + (|\beta| - 1)|P^{'}(z)|. \end{aligned} \tag{3.3}$$

By hypothesis, $P(z)$ does not vanish in $|z| < k, k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, therefore by Lemma 2.4, we have for $|z| = 1$

$$(k^\mu|P^{'}(z)| + nm)^2 \leq |nP(z) - zP^{'}(z)|^2.$$

This gives with the help of Lemma 2.2 for $|z|=1$,

$$\begin{aligned}|P^{'}(z)|^2+(k^{\mu}|P^{'}(z)|+nm)^2 &\leq |P^{'}(z)|^2+|nP(z)-zP^{'}(z)|^2\\ &\leq \frac{n^2}{2}(M_{\alpha}^2+M_{\alpha+\pi}^2).\end{aligned}$$

Since

$$\begin{aligned}(k^{\mu}|P^{'}(z)|+nm)^2 &= k^{2\mu}|P^{'}(z)|^2+n^2m^2+2nmk^{\mu}|P^{'}(z)|\\ &\geq k^{2\mu}|P^{'}(z)|^2+n^2m^2,\end{aligned}$$

it follows that

$$(1+k^{2\mu})|P^{'}(z)|^2+n^2m^2\leq \frac{n^2}{2}(M_{\alpha}^2+M_{\alpha+\pi}^2),$$

which implies, for $|z|=1$

$$|P^{'}(z)|\leq \frac{n}{\sqrt{2(1+k^{2\mu})}}(M_{\alpha}^2+M_{\alpha+\pi}^2-2m^2)^{\frac{1}{2}}. \tag{3.4}$$

Combining (3.3) and (3.4), we have for $|z|=1$,

$$|D_{\beta}P(z)|\leq n\max_{|z|=1}|P(z)|+(|\beta|-1)\frac{n}{\sqrt{2(1+k^{2\mu})}}(M_{\alpha}^2+M_{\alpha+\pi}^2-2m^2)^{\frac{1}{2}}.$$

This completes the proof. □

Remark 3.2. Dividing both sides of inequality (3.1) by $|\beta|$ and letting $|\beta|\to\infty$, we obtain inequality (1.8).

Corollary 3.3. *If $P\in P_{n,\mu}$, has no zero in $|z|<k$, $k\geq 1$, then for every real or complex number β with $|\beta|\geq 1$ and for every real α,*

$$|D_{\beta}P(z)|\leq n\max_{|z|=1}|P(z)|+(|\beta|-1)\frac{n}{\sqrt{2(1+k^{2\mu})}}(M_{\alpha}^2+M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{3.5}$$

where M_{α} is as defined in (1.4).

Proof. The proof of Corollary 3.3 follows on the same lines as that of Theorem 3.1 except that instead of applying Lemma 2.4, we use Lemma 2.3. We omit the details. □

Remark 3.4. Dividing both sides of inequality (3.5) by $|\beta|$ and letting $|\beta|\to\infty$, we get inequality (1.7).

Remark 3.5. For $\mu=1$, Theorem 3.1 gives a refinement in polar derivative of the inequality (1.6).

We next prove the following result, which not only gives extensions of inequalities (1.9) and (1.10) to polar derivative versions but also provides an improvement of Theorem 3.1 under the same hypotheses.

Theorem 3.6. *If $P \in P_{n,\mu}$, has no zero in $|z| < k$, $k \geq 1$, then for every real or complex number β with $|\beta| \geq 1$ and for every real α,*

$$|D_\beta P(z)| \leq n \max_{|z|=1} |P(z)| + (|\beta| - 1) \frac{n}{\sqrt{2\left\{1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\}}} \times (M_\alpha^2 + M_{\alpha+\pi}^2)^{\frac{1}{2}}, \tag{3.6}$$

where M_α is as defined in (1.4).

Proof. Proceeding similarly as in the proof of Theorem 3.1, we get from inequality (3.3),

$$|D_\beta P(z)| \leq n \max_{|z|=1} |P(z)| + (|\beta| - 1)|P'(z)|. \tag{3.7}$$

By hypothesis, $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree at most n and does not vanish in $|z| < k$, $k \geq 1$, therefore by Lemma 2.5, we have for $|z| = 1$

$$k^{\mu+1} \frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1} + 1}{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}} |P'(z)| \leq |nP(z) - zP'(z)|,$$

which implies, for $|z| = 1$

$$k^{2(\mu+1)} \left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1} + 1}{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2 |P'(z)|^2 \leq |nP(z) - zP'(z)|^2.$$

This gives with the help of Lemma 2.2 for $|z| = 1$,

$$\begin{aligned}\left\{1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1} + 1}{1 + \frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\} |P'(z)|^2 &\leq |P'(z)|^2 + |nP(z) - zP'(z)|^2 \\ &\leq \frac{n^2}{2}(M_\alpha^2 + M_{\alpha+\pi}^2)\end{aligned}$$

or

$$|P'(z)| \leq \frac{n}{\sqrt{2\left\{1 + k^{2(\mu+1)} \left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\}}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{\frac{1}{2}}. \tag{3.8}$$

Combining (3.7) and (3.8), we have for $|z|=1$,

$$\begin{aligned}|D_\beta P(z)| \quad &\leq \quad n\max_{|z|=1}|P(z)|+(|\beta|-1)\frac{n}{\sqrt{2\left\{1+k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu-1}+1}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}}\right)^2\right\}}}\\ &\quad \times(M_\alpha^2+M_{\alpha+\pi}^2)^{\frac{1}{2}},\end{aligned}$$

which completes the proof of Theorem 3.6. □

Remark 3.7. Dividing both sides of inequality (3.6) by $|\beta|$ and letting $|\beta|\to\infty$, we obtain inequality (1.9).

Corollary 3.8. *If $P\in P_{n,\mu}$, has no zero in $|z|<k, k\geq 1$ and $m=\min_{|z|=k}|P(z)|$, then for every real or complex number β with $|\beta|\geq 1$ and for every real α,*

$$\begin{aligned}|D_\beta P(z)| \quad &\leq \quad n\max_{|z|=1}|P(z)|+(|\beta|-1)\frac{n}{\sqrt{2\left\{1+k^{2(\mu+1)}\left(\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}}\right)^2\right\}}}\\ &\quad \times(M_\alpha^2+M_{\alpha+\pi}^2-2m^2)^{\frac{1}{2}}, \qquad (3.9)\end{aligned}$$

where M_α is as defined in (1.4).

Proof. The proof of Corollary 3.8 follows on the same lines as that of Theorem 3.6 but instead of applying Lemma 2.5, we use Lemma 2.6. We omit the details. □

Remark 3.9. Dividing both sides of inequality (3.9) by $|\beta|$ and letting $|\beta|\to\infty$, we have inequality (1.10).

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References

[1] A. Aziz, *Inequalities for the polar derivative of a polynomial*, J. Approx. Theory., **55**(2) (1988), 183-193.

[2] A. Aziz, *A refinement of an inequality of S. Bernstein*, J. Math. Anal. Appl., **142**(1) (1989), 226-235.

[3] A. Aziz and N.A. Rather, *New L_q inequalities for polynomial*, Math. Inequal. Appl., **2** (1998), 177-191.

[4] S. Bernstein, *Lecons Sur Les Propriétés extrémales et la meilleure approximation desfunctions analytiques d'une fonctions reele*, Gauthier-Villars, Paris, 1926.

[5] C. Frappier, Q.I. Rahman and S. Ruscheweyh, *New inequalities for polynomials*, Trans. Amer. Math. Soc., **288** (1985), 69-99.
[6] R.B. Gardner, N.K. Govil and A. Weems, *Some results concerning rate of growth of polynomials*, East J. Approx., **10**(3) (2004), 301-312.
[7] N.K. Govil and Q.I. Rahman, *Functions of exponential type not vanishing in a half-plane and related polynomials*, Trans. Amer. Math. Soc., **137** (1969), 501-517.
[8] P.D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509-513.
[9] M. Marden, *Geometry of Polynomials*, Math. Surveys, Vol. 3, Amer. Math. Soc., Providence, **3** (1966).
[10] G.V. Milovanović, D. S. Mitrinović and T. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific., Singapore, 1994.
[11] M.A. Qazi, *On the maximum modulus of polynomials*, Proc. Amer. Math. Soc., **115**(2) (1992), 337-343.
[12] N.A. Rather, S.H. Ahangar and M.A. Shah, *Some inequalities for the derivative of a polynomial*, Int. J. Appl. Math., **26**(2) (2013), 177-185.