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POSITIVE SOLUTIONS FOR A NONLINEAR MATRIX EQUATION USING FIXED POINT RESULTS IN EXTENDED BRANCIARI b-DISTANCE SPACES

Reena Jain[1], Hemant Kumar Nashine[2] and J. K. Kim[3]

[1]Mathematics Division, SASL, VIT Bhopal University, Madhya Pradesh-466114, India

[2]Mathematics Division, SASL, VIT Bhopal University, Madhya Pradesh-466114, India,
Department of Mathematics and Applied Mathematics, University of Johannesburg,
Kingsway Campus, Auckland Park 2006, South Africa
e-mail: `hemant.nashine@vit.ac.in; drhemantnashine@gmail.com`

[3]Department of Mathematics Education, Kyungnam University,
Changwon, Gyeongnam, 51767, Korea
e-mail: `jongkyuk@kyungnam.ac.kr`

Abstract. We consider the nonlinear matrix equation (NMEs) of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U})\mathcal{A}_i$, where $\mathcal{Q}$ is $n \times n$ Hermitian positive definite matrices (HPDS), $\mathcal{A}_1$, $\mathcal{A}_2$, $\ldots$, $\mathcal{A}_m$ are $n \times n$ matrices, and $\hbar$ is a nonlinear self-mappings of the set of all Hermitian matrices which are continuous in the trace norm. We discuss a sufficient condition ensuring the existence of a unique positive definite solution of a given NME and demonstrate this sufficient condition for a NME

$$\mathcal{U} = \mathcal{Q} + \mathcal{A}_1^*(\mathcal{U}^2/900)\mathcal{A}_1 + \mathcal{A}_2^*(\mathcal{U}^2/900)\mathcal{A}_2 + \mathcal{A}_3^*(\mathcal{U}^2/900)\mathcal{A}_3.$$

In order to do this, we define $\mathcal{FG}_w$-contractive conditions and derive fixed points results based on aforesaid contractive condition for a mapping in extended Branciari b-metric distance followed by two suitable examples. In addition, we introduce weak well-posed property, weak limit shadowing property and generalized Ulam-Hyers stability in the underlying space and related results.

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[0]Corresponding author: Hemant Kumar Nashine(`drhemantnashine@gmail.com`),
J. K. Kim(`jongkyuk@kyungnam.ac.kr`).

1. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+ := [0, +\infty)$, $\mathbb{N}$ the set of natural numbers, and $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.

1.1. **Positive definite solution of NME.** The study of nonlinear matrix equations (NME) appeared first in the literature concerned with algebraic Riccati equation. These equations occur in large number of problems in control theory, dynamical programming, ladder network, stochastic filtering, queuing theory, statistics and many other applicable areas.

Let $\mathcal{H}(n)$ (resp. $\mathcal{K}(n)$, $\mathcal{P}(n)$) denote the set of all $n \times n$ Hermitian (resp. positive semi-definite, positive definite) matrices over $\mathbb{C}$ and $\mathcal{M}(n)$ the set of all $n \times n$ matrices over $\mathbb{C}$. In [31], Ran and Reurings discussed the existence of solutions of the following equation:

$$\mathcal{U} + \mathcal{B}^* \hbar(\mathcal{U}) \mathcal{B} = \mathcal{Q} \tag{1.1}$$

in $\mathcal{K}(n)$, where $\mathcal{B} \in \mathcal{M}(n)$, $\mathcal{Q}$ is positive definite and $\hbar$ is a mapping from $\mathcal{K}(n)$ into $\mathcal{M}(n)$. Note that $\mathcal{U}$ is a solution of (1.1) if and only if it is a fixed point of the mapping $\mathcal{G}(\mathcal{U}) = \mathcal{Q} - \mathcal{B}^* \hbar(\mathcal{U}) \mathcal{B}$.

In [32], they used the notion of partial ordering and established a modification of Banach contraction principle, which they applied for solving a class of NMEs of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{B}_i^* \hbar(\mathcal{U}) \mathcal{B}_i$ using the Ky Fan norm in $\mathcal{M}(n)$.

Theorem 1.1. ([32]) *Let $\hbar : \mathcal{H}(n) \to \mathcal{H}(n)$ be an order-preserving, continuous mapping which maps $\mathcal{P}(n)$ into itself and $\mathcal{Q} \in \mathcal{P}(n)$. If $\mathcal{B}_i, \mathcal{B}_i^* \in \mathcal{P}(n)$ and $\sum_{i=1}^{k} \mathcal{B}_i \mathcal{B}_i^* < M \cdot \mathcal{I}_n$ for some $M > 0$ ($\mathcal{I}_n$ -the unit matrix in $\mathcal{M}(n)$) and if*

$$|\operatorname{tr}(\hbar(\mathcal{V}) - \hbar(\mathcal{U}))| \leq \frac{1}{M} |\operatorname{tr}(\mathcal{Y} - \mathcal{X})|,$$

for all $\mathcal{X}, \mathcal{Y} \in \mathcal{H}(n)$ with $\mathcal{U} \leq \mathcal{V}$, then the equation

$$\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{B}_i^* \hbar(\mathcal{U}) \mathcal{B}_i$$

has a unique positive definite solution (PDS).

In [34], Sawangsup and Sintunavarat studied the NME of the form $\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{B}_i^* \hbar(\mathcal{U}) \mathcal{B}_i$ using the spectral norm of a matrix, and applied a generalized contraction condition in metric spaces endowed with a transitive binary relation, they also tested numerically its approximate solutions. In the papers [3, 14, 15], the authors discussed on PDS's of a pair of NMEs. Recently, in [12], Garai and Dey obtained sufficient conditions for the existence and uniqueness of solution for a system of NMEs, using common fixed point results in Banach spaces under conditions with a pair of altering distance functions.

1.2. **Generalized metric spaces.** The distance notion in the metric fixed point theory is introduced and generalized in several different ways by many authors [4, 16, 17, 19, 21, 22]. Bakhtin [2] define the notion of b-metric space which is further used by Czerwik in [7, 8]. Kamran et al. [17] introduced the notion of extended b-metric space while Banciari [5] extended the metric space and introduced the notion of Branciari distance by changing the property of triangle inequality with quadrilateral one.

Definition 1.2. Let $\Xi \neq \emptyset$ be a set and $w : \Xi^2 \to \mathbb{R}_+\backslash(0,1)$. We say that a function $\rho_e : \Xi^2 \to \mathbb{R}_+$ is said to be an extended b-metric (ρ_e-metric, in short) if it satisfies:

(eb1) $\rho_e(\vartheta,\nu) = 0$ if and only if $\vartheta = \nu$;
(eb2) $\rho_e(\vartheta,\nu) = \rho_e(\nu,\vartheta)$ (symmetry);
(eb3) $\rho_e(\vartheta,\nu) \leq w(\vartheta,\nu)[\rho_e(\vartheta,\upsilon) + \rho_e(\upsilon,\nu)]$

for all $\vartheta,\nu,\upsilon \in \Xi$. The symbol (Ξ,ρ_e) denotes a ρ_e-metric space.

Definition 1.3. Let $\Xi \neq \emptyset$ be a set and let $b : \Xi^2 \to \mathbb{R}_+$ such that, for all $\vartheta,\nu \in \Xi$ and all $u,v \in \Xi\backslash\{\vartheta,\nu\}$,

(bd1) $b(\vartheta,\nu) = 0$ if and only if $\vartheta = \nu$ (self-distance/indistancy);
(bd2) $b(\vartheta,\nu) = b(\nu,\vartheta)$ (symmetry);
(bd3) $b(\vartheta,\nu) \leq b(\vartheta,u) + b(u,v) + b(v,\nu)$ (quadrilateral inequality).

The symbol (Ξ,b) denotes Branciari distance space and abbreviated as "BDS".

Recently Abdeljawad et al. [1] define the notion of extended Branciari b-distance space by combining, extended b-metric and Branciari distance.

Definition 1.4. Let $\Xi \neq \emptyset$ be a set and $w : \Xi^2 \to \mathbb{R}_+\backslash(0,1)$. We say that a function $e_b : \Xi^2 \to \mathbb{R}_+$ is an extended Branciari b-metric (e_b-metric, in short) if it satisfies:

(ebb1) $e_b(\vartheta,\upsilon) = 0$ if and only if $\vartheta = \upsilon$,
(ebb2) $e_b(\vartheta,\upsilon) = e_b(\upsilon,\vartheta)$,
(ebb3) $e_b(\vartheta,\upsilon) \leq w(\vartheta,\upsilon)[e_b(\vartheta,\nu) + e_b(\nu,\varrho) + e_b(\varrho,\upsilon)]$

for all $\vartheta,\upsilon \in S$ all distinct $\nu,\varrho \in \Xi\backslash\{\vartheta,\upsilon\}$. The symbol (Ξ,e_b) denotes extended Branciari b-distance space (EBbDS, in short). For $w(\vartheta,\upsilon) = 1$, (Ξ,e_b) will be called a Branciari b-distance space (BbDS, in short).

Example 1.5. Let $\Xi = C([0,1],\ \mathbb{R})$ and define $e_b : \Xi^2 \to \mathbb{R}_+$ by

$$e_b(P,\ Q) = \int_0^1 (P(t) - Q(t))^2 dt$$

with $w(P,Q) = |P(t)| + |Q(t)| + 2$. Note that $e_b(P,\ Q) \geq 0$ for all $P,\ Q \in \Xi$, and $e_b(P,\ Q) = 0$ if and only if $P = Q$. Also $e_b(P,\ Q) = e_b(Q,\ P)$. Hence it is clear that $(\Xi,\ e_b)$ is an EBbDS, but it is neither an BDS nor metric space.

Definition 1.6. ([1]) Let $\Xi \neq \emptyset$ be a set endowed with extended Branciari b-distance e_b.

(a) A sequence $\{\vartheta_n\}$ in Ξ converges to ϑ if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $e_b(\vartheta_n, \vartheta) < \epsilon$ for all $n \geq N$. For this particular case, we write $\lim_{n\to\infty} \vartheta_n = \vartheta$.

(b) A sequence $\{\vartheta_n\}$ in Ξ is called Cauchy if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $e_b(\vartheta_m, \vartheta_n) < \epsilon$ for all $m, n \geq N$.

(c) A e_b-metric space (Ξ, e_b) is complete if every Cauchy sequence in S is convergent.

2. $\mathcal{FG}_w$-contractive mapping and based results

Definition 2.1. ([25]) The collection of all functions $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ satisfying:

($\mathbb{F}_1$) $\mathcal{F}$ is continuous and strictly increasing;

($\mathbb{F}_2$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\lim\limits_{n\to\infty} \xi_n = 0$ iff $\lim\limits_{n\to\infty} \mathcal{F}(\xi_n) = -\infty$,

will be denoted by $\mathbb{F}$.

The collection of all pairs of mappings $(\mathcal{G}, \beta)$, where $\mathcal{G} : \mathbb{R}_+ \to \mathbb{R}$, $\beta : \mathbb{R}_+ \to [0, 1)$, satisfying:

($\mathbb{F}_3$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup\limits_{n\to\infty} \mathcal{G}(\xi_n) \geq 0$ iff $\limsup\limits_{n\to\infty} \xi_n \geq 1$;

($\mathbb{F}_4$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup\limits_{n\to\infty} \beta(\xi_n) = 1$ implies $\lim\limits_{n\to\infty} \xi_n = 0$;

($\mathbb{F}_5$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\sum_{n=1}^{\infty} \mathcal{G}(\beta(\xi_n)) = -\infty$,

will be denoted by $\mathbb{G}_\beta$.

Definition 2.2. Let (Ξ, e_b) be an EBbDS and $\Im : \Xi \to \Xi$ be a mapping. We say $\Im$ is an $\mathcal{FG}_w$-contractive mapping if there exist $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$, such that for all $\vartheta, \nu \in \Xi$ and $e_b(\Im\vartheta, \Im\nu) > 0$,

$$\mathcal{F}(w(\vartheta, \nu)^r e_b(\Im\vartheta, \Im\nu)) \leq \mathcal{F}(\mathfrak{M}_w(\vartheta, \nu)) + \mathcal{G}(\beta(\mathfrak{M}_w(\vartheta, \nu))), \tag{2.1}$$

where $r \geq 2$ and

$$\mathfrak{M}_w(\vartheta, \nu) = \max\left\{e_b(\vartheta, \nu), e_b(\vartheta, \Im\vartheta), e_b(\nu, \Im\nu), \frac{e_b(\nu, \Im\nu)[1 + e_b(\vartheta, \Im\vartheta)]}{w(\vartheta, \nu)[1 + e_b(\vartheta, \nu)]}\right\}. \tag{2.2}$$

The set of all fixed points of a self-mapping $\Im$ on a set $\Xi \neq \emptyset$ will be denoted by $Fix(\Im)$.

Theorem 2.3. *Let (Ξ, e_b) be a complete EBbDS and $\Im : \Xi \to \Xi$ be an $\mathcal{FG}_w$-contractive mapping for $\mathcal{F} \in \mathbb{F}$, $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$. Then $Fix(\Im)$ is a singleton set. Furthermore, for any $\vartheta_0 \in \Xi$, the sequence ϑ_n satisfying $\vartheta_n = \Im\vartheta_{n-1}$ is convergent.*

Proof. We first show that $\Im$ has at most one fixed point. Let us suppose that ϑ and ϑ^* are two different fixed points of $\Im$. That is, $\Im\vartheta^* = \vartheta^* \neq \vartheta = \Im\vartheta$. It follows that

$$e_b(\Im\vartheta, \Im\vartheta^*) = e_b(\vartheta, \vartheta^*) > 0.$$

Since $\Im : \Xi \to \Xi$ be an $\mathcal{FG}$-contractive for $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$, we can write

$$\mathcal{F}(w(\vartheta, \vartheta^*)^r e_b(\Im\vartheta, \Im\vartheta^*)) \leq \mathcal{F}(\mathfrak{M}_w(\vartheta, \vartheta^*)) + \mathcal{G}(\beta(\mathfrak{M}_w(\vartheta, \vartheta^*))), \tag{2.3}$$

where $r \geq 2$ and

$$\begin{aligned} \mathfrak{M}_w(\vartheta, \vartheta^*) &= \max \left\{ \begin{array}{c} e_b(\vartheta, \vartheta^*), e_b(\vartheta, \Im\vartheta), e_b(\vartheta^*, \Im\vartheta^*), \\ \frac{e_b(\vartheta^*, \Im\vartheta^*)[1+e_b(\vartheta, \Im\vartheta)]}{w(\vartheta, \vartheta^*)[1+e_b(\vartheta, \vartheta^*)]} \end{array} \right\} \\ &= \max\{e_b(\vartheta, \vartheta^*), 0, 0, 0\} \\ &= e_b(\vartheta, \vartheta^*). \end{aligned}$$

Therefore,

$$\mathcal{F}(e_b(\vartheta, \vartheta^*)) \leq \mathcal{F}(w(\vartheta, \vartheta^*)^r e_b(\Im\vartheta, \Im\vartheta^*)) \leq \mathcal{F}(e_b(\vartheta, \vartheta^*)) + \mathcal{G}(\beta(e_b(\vartheta, \vartheta^*)))$$

which implies

$$\mathcal{G}(\beta(e_b(\vartheta, \vartheta^*))) \geq 0,$$

that is,

$$\beta(e_b(\vartheta, \vartheta^*) \geq 1,$$

which is a contradiction. Hence $\Im$ has at most one fixed point.

Next, we prove the existence of fixed points of $\Im$. For any $\vartheta_0 \in \Xi$ we set $\vartheta_n = \Im\vartheta_{n-1}$. Now, we discuss the two cases:

Case I. Let there exists $n_0 \in \mathbb{N}$ such that $\vartheta_{n_0} = \vartheta_{n_0-1}$ then we have $\Im\vartheta_{n_0-1} = \vartheta_{n_0-1}$. This shows ϑ_{n_0-1} a fixed point of $\Im$. And the proof is established.

Case II. If $\vartheta_n \neq \vartheta_{n-1}$ for all $n \in \mathbb{N}$ then we have $e_b(\vartheta_n, \vartheta_{n-1}) > 0$ and so we can write

$$\begin{aligned} \mathcal{F}(e_b(\Im\vartheta_n, \Im\vartheta_{n-1})) &\leq \mathcal{F}(w(\vartheta_n, \vartheta_{n-1})^r e_b(\Im\vartheta_n, \Im\vartheta_{n-1})) \\ &\leq \mathcal{F}(\mathfrak{M}_w(\vartheta_n, \vartheta_{n-1})) + \mathcal{G}(\beta(\mathfrak{M}_w(\vartheta_n, \vartheta_{n-1}))), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{M}_w(\vartheta_{n-1}, \vartheta_n) &= \max \left\{ \begin{array}{c} e_b(\vartheta_n, \vartheta_{n-1}), e_b(\vartheta_n, \Im\vartheta_n), e_b(\vartheta_{n-1}, \Im\vartheta_{n-1}), \\ \frac{e_b(\vartheta_n, \Im\vartheta_n)[1+e_b(\vartheta_{n-1}, \Im\vartheta_{n-1})]}{w(\vartheta_{n-1}, \vartheta_n)[1+e_b(\vartheta_{n-1}, \vartheta_n)]} \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} e_b(\vartheta_n, \vartheta_{n-1}), e_b(\vartheta_n, \Im\vartheta_n), e_b(\vartheta_{n-1}, \Im\vartheta_{n-1}), \\ \frac{e_b(\vartheta_n, \vartheta_{n+1})}{w(\vartheta_{n-1}, \vartheta_n)} \end{array} \right\} \\ &\leq \max \left\{ e_b(\vartheta_n, \vartheta_{n-1}), e_b(\vartheta_n, \vartheta_{n+1}) \right\}. \end{aligned}$$

If $\mathfrak{M}_w(\vartheta_n, \vartheta_{n-1}) = e_b(\vartheta_n, \vartheta_{n+1})$, then

$$\mathcal{F}(e_b(\vartheta_n, \vartheta_{n+1})) \leq \mathcal{F}(e_b(\vartheta_n, \vartheta_{n+1})) + \mathcal{G}(\beta(e_b(\vartheta_n, \vartheta_{n+1})))$$

which implies

$$\mathcal{G}(\beta(e_b(\vartheta_n, \vartheta_{n+1}))) \geq 0,$$

that is,

$$\beta(e_b(\vartheta_n, \vartheta_{n+1})) \geq 1,$$

which is a contradiction. Therefore

$$e_b(\vartheta_n, \vartheta_{n+1}) \leq e_b(\vartheta_{n-1}, \vartheta_n), \ \ \forall n \in \mathbb{N}$$

and so

$$\begin{aligned} \mathcal{F}(e_b(\vartheta_n, \vartheta_{n+1})) &\leq \mathcal{F}((w(\vartheta_n, \vartheta_{n-1})^r e_b(\vartheta_n, \vartheta_{n+1})) \\ &\leq \mathcal{F}(e_b(\vartheta_{n-1}, \vartheta_n)) + \mathcal{G}(\beta(e_b(\vartheta_{n-1}, \vartheta_n))) \end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, we have

$$\begin{aligned} \mathcal{F}(e_b(\vartheta_{n-1}, \vartheta_n)) &\leq \mathcal{F}(e_b(\vartheta_{n-2}, \vartheta_{n-1})) + \mathcal{G}(\beta(e_b(\vartheta_{n-2}, \vartheta_{n-1}))) \\ &\vdots \\ &\leq \mathcal{F}(e_b(\vartheta_0, \vartheta_1)) + \sum_{i=1}^{i=n} \mathcal{G}(\beta(e_b(\vartheta_i, \vartheta_{i-1}))). \end{aligned}$$

Letting $n \to \infty$ gives $\lim_{n\to\infty} \mathcal{F}(e_b(\vartheta_{n-1}, \vartheta_n)) = -\infty$ and $\mathcal{F} \in \mathbb{F}$ gives

$$\lim_{n\to\infty} e_b(\vartheta_{n-1}, \vartheta_n)) = 0. \tag{2.4}$$

We will now show that the sequence $\{\vartheta_n\}$ is Cauchy in (Ξ, e_b). On the contrary, we suppose that there exist $\zeta > 0$ and two subsequences $\{\vartheta_{n(j)}\}$ and $\{\vartheta_{m(j)}\}$ of $\{\vartheta_n\}$ such that $n(j)$ is the smallest index for which $n(j) > m(j) > j$ and

$$e_b(\vartheta_{n(j)}, \vartheta_{m(j)}) \geq \zeta. \tag{2.5}$$

This means that $m(j) > n(j) > j$ and

$$e_b(\vartheta_{m(j)}, \vartheta_{n(j)-2}) < \zeta. \tag{2.6}$$

Letting the upper limit as $j \to \infty$, we get

$$\limsup_{j\to\infty} e_b(\vartheta_{m(j)}, \vartheta_{n(j)-2}) \leq \zeta. \tag{2.7}$$

On the other hand

$$e_b(\vartheta_{n(j)}, \vartheta_{m(j)}) \leq w(\vartheta_{m(j)}, \vartheta_{n(j)}) \left[\begin{array}{c} e_b(\vartheta_{m(j)}, \vartheta_{m(j)+1}) \\ +e_b(\vartheta_{m(j)+1}, \vartheta_{n(j)+1}) \\ +e_b(\vartheta_{n(j)+1}, \vartheta_{n(j)}) \end{array} \right].$$

Taking the upper limit as $j \to \infty$, and making use of (2.4) and (2.5), we get

$$\frac{\zeta}{\limsup_{j\to\infty} w(\vartheta_{m(j)}, \vartheta_{n(j)})} \le \limsup_{j\to\infty} e_b(\vartheta_{m(j)+1}, \vartheta_{n(j)+1}). \tag{2.8}$$

Similarly we have

$$\limsup_{j\to\infty} e_b(\vartheta_{m(j)}, \vartheta_{n(j)}) \le \zeta \limsup_{j\to\infty} w(\vartheta_{m(j)}, \vartheta_{n(j)}). \tag{2.9}$$

So we can write

$$\begin{aligned}
&\mathfrak{M}_w(\vartheta_{m(j)}, \vartheta_{n(j)}) \\
&= \max\left\{ \begin{array}{c} e_b(\vartheta_{m(j)}, \vartheta_{n(j)}), e_b(\vartheta_{m(j)}, \Im\vartheta_{m(j)}), e_b(\vartheta_{n(j)}, \Im\vartheta_{n(j)}), \\ \frac{e_b(\vartheta_{n(j)}, \Im\vartheta_{n(j)})[1+e_b(\vartheta_{m(j)}, \Im\vartheta_{m(j)})]}{w(\vartheta_{m(j)}, \vartheta_{n(j)})[1+e_b(\vartheta_{m(j)}, \vartheta_{n(j)})]} \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} e_b(\vartheta_{m(j)}, \vartheta_{n(j)}), e_b(\vartheta_{m(j)}, \vartheta_{m(j)+1}), e_b(\vartheta_{n(j)}, \vartheta_{n(j)+1}), \\ \frac{e_b(\vartheta_{n(j)}, \vartheta_{n(j)+1})[1+e_b(\vartheta_{m(j)}, \vartheta_{m(j)+1}]}{w(\vartheta_{m(j)}, \vartheta_{n(j)})[1+e_b(\vartheta_{m(j)}, \vartheta_{n(j)})]} \end{array} \right\}.
\end{aligned}$$

Taking upper limit as $j \to \infty$ and making use of (2.4), (2.8) and (2.9), we get

$$\begin{aligned}
\limsup_{j\to\infty} \mathfrak{M}_w(\vartheta_{m(j)}, \vartheta_{n(j)}) &= \limsup_{j\to\infty} e_b(\vartheta_{m(j)}, \vartheta_{n(j)}) \\
&< \zeta \limsup_{j\to\infty} w(\vartheta_{m(j)}, \vartheta_{n(j)}).
\end{aligned} \tag{2.10}$$

Therefore, from (2.3), (2.4) and (2.10), we have

$$\begin{aligned}
&\mathcal{F}(\zeta \limsup_{j\to\infty} w(\vartheta_{m(j)}, \vartheta_{n(j)})) \\
&\le \mathcal{F}\left(\limsup_{j\to\infty} w(\vartheta_{m(j)}, \vartheta_{n(j)})^r \frac{\zeta}{\limsup_{j\to\infty} w(\vartheta_{m(j)}, \vartheta_{n(j)})} \right) \\
&\le \mathcal{F}(\limsup_{j\to\infty} w(\vartheta_{m(j)}, \vartheta_{n(j)})^r \limsup_{j\to\infty} e_b(\vartheta_{m(j)+1}, \vartheta_{n(j)+1})) \\
&\le \limsup_{j\to\infty} \mathcal{F}(\mathfrak{M}_w(\vartheta_{m(j)}, \vartheta_{n(j)})) + \limsup_{j\to\infty} \mathcal{G}(\beta(\mathfrak{M}_w(\vartheta_{m(j)}, \vartheta_{n(j)}))) \\
&\le \mathcal{F}(\zeta \limsup_{j\to\infty} w(\vartheta_{m(j)}, \vartheta_{n(j)})) + \limsup_{j\to\infty} \mathcal{G}(\beta(\mathfrak{M}_w(\vartheta_{m(j)}, \vartheta_{n(j)}))),
\end{aligned}$$

which implies that

$$\limsup_{j\to\infty} \mathcal{G}(\beta(\mathfrak{M}_w((\vartheta_{m(j)}, \vartheta_{n(j)}))) \ge 0,$$

which gives

$$\limsup_{j\to\infty} \beta(\mathfrak{M}_w(\vartheta_{m(j)}, \vartheta_{n(j)})) \ge 1,$$

and taking in account that $\beta(\xi) < 1$ for all $\xi \geq 0$, we have

$$\limsup_{\jmath\to\infty} \beta(\mathfrak{M}_w(\vartheta_{m(j)}, \vartheta_{n(j)})) = 1.$$

Therefore, $\limsup_{\jmath\to\infty} \mathfrak{M}_w(\vartheta_{m(j)}, \vartheta_{n(j)}) = 0$, which is a contradiction. Hence, $\{\vartheta_n\}$ is a Cauchy sequence in Ξ. The completeness of Ξ implies that there exists $\vartheta^* \in \Xi$ such that $\lim_{n\to\infty} e_b(\vartheta_n, \vartheta^*) = 0$. On the other hand assuming $e_b(\vartheta_n, \vartheta^*) > 0$ and making use of $(\mathbb{F}_1)$ with (2.3) we have

$$\begin{aligned}\mathcal{F}(e_b(\vartheta_{n+1}, \Im\vartheta^*)) &\leq \mathcal{F}(w(\vartheta_n, \vartheta^*)^r e_b(\Im\vartheta_n, \Im\vartheta^*)) \\ &\leq \mathcal{F}(\mathfrak{M}_w(\vartheta, \vartheta^*)) + \mathcal{G}(\beta(\mathfrak{M}_w(\vartheta, \vartheta^*))),\end{aligned} \tag{2.11}$$

where $r \geq 2$ and

$$\mathfrak{M}_w(\vartheta_n, \vartheta^*) = \max\left\{ \begin{array}{c} e_b(\vartheta_n, \vartheta^*), e_b(\vartheta_n, \Im\vartheta_n), e_b(\vartheta^*, \Im\vartheta^*), \\ \frac{e_b(\vartheta^*, \Im\vartheta^*)[1+e_b(\vartheta_n, \Im\vartheta_n)]}{w(\vartheta_n, \vartheta^*)[1+e_b(\vartheta_n, \vartheta^*)]} \end{array} \right\}, \tag{2.12}$$

that is,

$$\lim_{n\to\infty} \mathfrak{M}_w(\vartheta_n, \vartheta^*) = e_b(\vartheta^*, \Im\vartheta^*).$$

Applying $n \to \infty$ in (2.11) and (2.12), we get

$$\mathcal{F}(e_b(\vartheta, \Im\vartheta^*)) \leq \mathcal{F}(e_b(\vartheta^*, \Im\vartheta^*) + \mathcal{G}(\beta(e_b(\vartheta^*, \Im\vartheta^*)),$$

which gives

$$\mathcal{G}(\limsup_{n\to\infty} \beta(e_b(\vartheta^*, \Im\vartheta^*)) \geq 1,$$

and taking in account that $\beta(\xi) < 1$ for all $\xi \geq 0$, we have

$$\beta(e_b(\vartheta^*, \Im\vartheta^*) \geq 1,$$

which is a contradiction. So $e_b(\vartheta^*, \Im\vartheta^*) = 0$, that is, $\vartheta^* = \Im\vartheta^*$. $\square$

Example 2.4. Let $\Xi = \mathcal{K} \cup \mathcal{L}$, where $\mathcal{K} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $\mathcal{L} = [1, 2]$. Define $e_b : \Xi^2 \to \mathbb{R}_+$ such that $e_b(\vartheta, \nu) = e_b(\nu, \vartheta)$ for all $\vartheta, \nu \in \Xi$, and

$$e_b(\frac{1}{2}, \frac{1}{3}) = 0.06, \quad e_b(\frac{1}{2}, \frac{1}{4}) = 0.02, \quad e_b(\frac{1}{2}, \frac{1}{5}) = 0.02,$$
$$e_b(\frac{1}{3}, \frac{1}{4}) = 0.03, \quad e_b(\frac{1}{3}, \frac{1}{5}) = 0.01, \quad e_b(\frac{1}{5}, \frac{1}{4}) = 0.01,$$

$e_b(\vartheta, \nu) = (\vartheta - \nu)^2$ otherwise. Then (Ξ, e_b) is a EBbDS with $w(\vartheta, \nu) = \vartheta + \nu + 2$ but it is not a BDS (Ξ, b) and metric space (Ξ, d) as well. For instances

$$e_b(\frac{1}{2}, \frac{1}{3}) = 0.06 \not\leq 0.05 = e_b(\frac{1}{2}, \frac{1}{4}) + e_b(\frac{1}{4}, \frac{1}{3})$$

and

$$e_b(\frac{1}{2}, \frac{1}{3}) = 0.06 \not\leq 0.04 = e_b(\frac{1}{2}, \frac{1}{4}) + e_b(\frac{1}{4}, \frac{1}{5}) + e_b(\frac{1}{5}, \frac{1}{3})$$

but

$$e_b(\frac{1}{2},\frac{1}{4}) = 0.06 \le 0.1133 = w(\vartheta,\upsilon)\left[e_b(\frac{1}{2},\frac{1}{4}) + e_b(\frac{1}{4},\frac{1}{5}) + e_b(\frac{1}{5},\frac{1}{3})\right].$$

Consider the self-mapping $\Im$ on Ξ given by

$$\Im(\vartheta) = \begin{cases} \frac{1}{4}, & \text{if } \vartheta \in \mathcal{K}, \\ \frac{1}{5}, & \text{if } \vartheta \in \mathcal{L}. \end{cases}$$

Taking $\mathcal{F}(\zeta) = \ln \zeta$, $\beta(\zeta) = e^{-\tau}$, $\mathcal{G}(\zeta) = \ln \zeta$ $(\zeta > 0)$, $\tau = \ln(\frac{1}{k})$, where $r > 2$ and $k < 1$ are chosen so that (2.3) would be of the form

$$\frac{25}{16}\Big(\frac{9}{2}\Big)^r \frac{1}{50} < k \tag{2.13}$$

(it is easy to see that such coefficients exist). That is, we have to check that

$$\Big(\frac{9}{2}\Big)^r e_b(\Im\nu, \Im\vartheta) < k\mathfrak{M}_w(\nu,\vartheta) \tag{2.14}$$

holds whenever $e_b(\Im\vartheta, \Im\nu) > 0$.

We will check that $\Im$ satisfy (2.14). There are two non-trivial possible cases when $e_b(\Im\nu, \Im\vartheta) > 0$. Here $w(\vartheta,\nu) \in [\frac{16}{5}, \frac{9}{2}]$.

Case 1. $\vartheta = \frac{1}{2}$, $\nu = 2$. Then $e_b(\Im\vartheta, \Im\nu) = 0.01$ and

$$\mathfrak{M}_w(\vartheta,\nu) = \max\left\{\frac{9}{4}, 0.06, \frac{81}{25}, 0.23483\right\} = \frac{81}{25},$$

and it is easily seen that (2.14) is fulfilled.

Case 2. $\vartheta = \frac{1}{2}$, $\nu = 1$. Then $e_b(\Im\vartheta, \Im\nu) = 0.01$ and

$$\mathfrak{M}_w(\vartheta,\nu) = \max\left\{\frac{1}{4}, 0.06, \frac{16}{25}, 0.1206\right\} = \frac{16}{25},$$

and again (2.14) holds true.

Thus, all the conditions are fulfilled and the $\Im$ have a unique fixed point, which is $\vartheta^* = \frac{1}{3}$.

Example 2.5. Consider $\Xi = [0,1]$ and define $e_b : \Xi^2 \to \mathbb{R}_+$ by $e_b(\vartheta,\nu) = |\vartheta - \nu|^2$. Then (Ξ, e_b) is a EBbDS with $w(\vartheta,\nu) = \vartheta + \nu + \frac{5}{2}$ but it is not a BDS (Ξ, b) and metric space (Ξ, d) as well. For instances

$$e_b(0,1) = 1 \not\le \frac{1}{2} = e_b(0,\frac{1}{2}) + e_b(\frac{1}{2},1)$$

and

$$e_b(0,1) = 1 \not\le 0.4902 = e_b(0,\frac{1}{2}) + e_b(\frac{1}{2},0.99) + e_b(0.99,1)$$

but

$$\begin{aligned}
e_b(\vartheta,\nu) &= |\vartheta-\nu|^2\\
&= |\vartheta-\mu+\mu-\upsilon+\upsilon-\nu|^2\\
&\le |\vartheta-\mu|^2+|\mu-\upsilon|^2+|\upsilon-\nu|^2+2|\vartheta-\mu||\mu-\upsilon|\\
&\quad +2|\mu-\upsilon||\upsilon-\nu|+2|\upsilon-\nu||\vartheta-\mu|\\
&\le \left(\vartheta+\nu+\frac{5}{2}\right)[|\vartheta-\mu|^2+|\mu-\upsilon|^2+|\upsilon-\nu|^2]\\
&= w(\vartheta,\nu)[e_b(\vartheta,\mu)+e_b(\mu,\ \upsilon)+e_b(\upsilon,\nu)]
\end{aligned}$$

for all $\vartheta,\nu,\mu,\upsilon\in\Xi$.

Consider the self-mapping $\Im$ on Ξ given by $\Im(\vartheta)=\frac{\vartheta}{5}$. Taking $\beta(\zeta)=e^{-\tau}$, $\mathcal{G}(\zeta)=\ln\zeta$ and $\mathcal{F}(\zeta)=-1/\sqrt{\zeta}$ $(\zeta>0)$ in (2.3), then we have to check that

$$w(\vartheta,\nu)^r e_b(\Im\vartheta,\Im\nu)\le\frac{\mathfrak{M}_w(\vartheta,\nu)}{[1+\tau\mathfrak{M}_w(\vartheta,\nu)]^2} \tag{2.15}$$

holds whenever $e_b(\Im\vartheta,\Im\nu)>0$.

For $\vartheta\ne\nu$, $e_b(\Im\vartheta,\Im\nu)=\frac{|\vartheta-\nu|^2}{25}>0$ and

$$\begin{aligned}
\mathfrak{M}_w(\vartheta,\nu) &= \max\left\{|\vartheta-\nu|^2,|\vartheta-\Im\vartheta|^2,|\nu-\Im\nu|^2,\frac{|\nu-\Im\nu|^2[1+|\vartheta-\Im\vartheta|]}{(\vartheta+\nu+\frac{5}{2})[1+|\vartheta-\nu|^2]}\right\}\\
&= |\vartheta-\nu|^2,\ \text{for all } \vartheta,\nu\in\Xi.
\end{aligned}$$

Then (2.15) will be

$$\left(\vartheta+\nu+\frac{5}{2}\right)^r\frac{|\vartheta-\nu|^2}{25}\le\frac{|\vartheta-\nu|^2}{[1+\tau|\vartheta-\nu|^2]},$$

that is,

$$\left(\vartheta+\nu+\frac{5}{2}\right)^r[1+\tau|\vartheta-\nu|^2]\le 25,$$

which is true for any $\vartheta,\nu\in\Xi$, and for some $r>2$ and $\tau>0$. Thus, all the conditions are fulfilled and the $Fix(\Im)=\{0\}$ is a singleton set.

Several results can be obtained from Theorem 2.3 by taking various possible choices for functions $\mathcal{F}$, $\mathcal{G}$ and β.

Corollary 2.6. *Let all the conditions of Theorem* 2.3 *be satisfied, except that the $\mathcal{FG}_w$-contractive condition is replaced by Geraghty-type* [10, 13] *condition of the form : for all $\nu,\vartheta\in\Xi, e_b(\Im\vartheta,\Im\nu)>0$,*

$$w(\vartheta,\nu)^r e_b(\Im\vartheta,\Im\nu)\le\beta(\mathfrak{M}_w(\vartheta,\nu))\mathfrak{M}_w(\vartheta,\nu) \tag{2.16}$$

where $r\ge 2$, and $\mathfrak{M}_w(\vartheta,\nu)$ is given in (2.4) *holds. Then $Fix(\Im)$ is a singleton.*

Proof. It is enough to take $\mathcal{F}(\xi) = \mathcal{G}(\xi) = \ln \xi$ $(\xi > 0)$ in (2.3). $\square$

Similar type of corollaries can be constructed for Corollary 2.3.

3. Weak well-posed, weak limit shadowing, generalized w-Ulam-Hyers stability

The notion of well-posedness of a fixed point problem (fpp) has evoked much interest of several mathematicians, for example, Popa [27, 28] and other [30]. In the paper [6], authors defined a weak well-posed (wwp) property in BbDS. In the following, we extend this notion to EBbDS.

Definition 3.1. Let (Ξ, e_b) be a complete EBbDS and $\Im : \Xi \to \Xi$ be a mapping. The fpp of $\Im$ is said to be wwp if it satisfies:

(1) $\vartheta^* \in Fix(\Im)$ is a singleton set in Ξ;

(2) for any sequence $\{\vartheta_p\}$ in Ξ with $\lim_{p\to\infty} e_b(\vartheta_p, \Im(\vartheta_p)) = 0$ and $\lim_{p,q\to\infty} e_b(\Im(\vartheta_p), \Im(\vartheta_q)) = 0$, one has

$$\lim_{p\to\infty} e_b(\vartheta_p, \vartheta^*) = 0.$$

Theorem 3.2. *Let (Ξ, e_b) be a complete EBbDS and $\Im : \Xi \to \Xi$ be an $\mathcal{F}\mathcal{G}_w$-contractive mapping for $\mathcal{F} \in \mathbb{F}$, $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ with $\{\vartheta_n\}$ in Ξ such that $\lim_{n\to\infty} e_b(\vartheta_n, \Im\vartheta_n) = 0$, $\lim_{n,m\to\infty} e_b(\Im\vartheta_n, \Im\vartheta_m) = 0$ and $\vartheta^* \in Fix(\Im)$. Then the fpp of $\Im$ is wwp.*

Proof. Let $\{\vartheta_n\}$ be a sequence in Ξ such that $\lim_{n\to\infty} e_b(\vartheta_n, \Im(\vartheta_n)) = 0$ and $\lim_{n,m\to\infty} e_b(\Im\vartheta_n, \Im\vartheta_m) = 0$, for $m > n$ we obtain from (ebb3) that

$$e_b(\vartheta_n, \vartheta^*) \leq w(\vartheta_n, \vartheta^*)\{e_b(\vartheta_n, \Im\vartheta_m) + e_b(\Im\vartheta_m, \Im\vartheta_n) + e_b(\Im\vartheta_n, \vartheta^*)\}.$$

Taking limit $n \to \infty$

$$\lim_{n\to\infty} e_b(\vartheta_n, \vartheta^*) \leq \lim_{n\to\infty} w(\vartheta_n, \vartheta^*)\{e_b(\vartheta_n, \Im\vartheta_m) + e_b(\Im\vartheta_n, \vartheta^*)\}. \tag{3.1}$$

Without loss of generality, we can assume that there exists a distinct subsequence $\{\Im\vartheta_{n_k}\}$ of $\{\Im\vartheta_n\}$. Otherwise, there exists $\vartheta_0 \in \Xi$ and $n_1 \in \mathrm{N}$ such that $\Im\vartheta_n = \vartheta_0$ for $n \geq n_1$. Since $\lim_{n\to\infty} e_b(\vartheta_n, \Im\vartheta_n) = 0$, we get $\lim_{n\to\infty} e_b(\vartheta_n, \vartheta_0) = 0$. If $\vartheta_0 \neq \vartheta^*$, then $\vartheta_0 \neq \Im\vartheta_0$ duo to uniqueness of the fixed point of $\Im$. For $n \geq n_1$, we obtain $\vartheta_0 = \Im\vartheta_n \neq \Im\vartheta_0$. So we have

$$e_b(\vartheta_0, \Im\vartheta_0) = e_b(\Im\vartheta_n, \Im\vartheta_0) \leq w(\vartheta_n, \vartheta_0) e_b(\Im\vartheta_n, \Im\vartheta_0).$$

Since $\mathcal{F}$ is non-dcreasing and continuous, we get

$$\begin{aligned}
\mathcal{F}(e_b(\vartheta_0, \Im\vartheta_0)) &\leq \mathcal{F}(w(\vartheta_n, \vartheta_0) d(\Im\vartheta_n, \Im\vartheta_0)) \\
&\leq \mathcal{F}(w(\vartheta_n, \vartheta_0)^r d(\Im\vartheta_n, \Im\vartheta_0)) \\
&\leq \mathcal{F}(\mathfrak{M}_w(\vartheta_n, \vartheta_0) + \mathcal{G}(\beta(\mathfrak{M}_w(\vartheta_n, \vartheta_0)),
\end{aligned}$$

where

$$\mathfrak{M}_w(\vartheta_n, \vartheta_0) = \max\left\{ \begin{array}{c} e_b(\vartheta_n, \vartheta_0), e_b(\vartheta_n, \Im\vartheta_n), e_b(\vartheta_0, \Im\vartheta_0), \\ \frac{e_b(\vartheta_0, \Im\vartheta_0)[1+e_b(\vartheta_n, \Im\vartheta_n)]}{w(\vartheta_n, \vartheta_0)[1+e_b(\vartheta_n, \vartheta_0)]} \end{array} \right\},$$

implies

$$\lim_{n\to\infty} \mathfrak{M}_w(\vartheta_n, \vartheta_0) = e_b(\vartheta_0, \Im\vartheta_0).$$

Therefore,

$$\mathcal{F}(\lim_{n\to\infty} e_b(\vartheta_0, \Im\vartheta_0)) \leq \mathcal{F}(\lim_{n\to\infty} e_b(\vartheta_0, \Im\vartheta_0)) + \lim_{n\to\infty} \mathcal{G}(\beta(e_b(\vartheta_0, \Im\vartheta_0)) \tag{3.2}$$

which gives

$$\lim_{n\to\infty} \mathcal{G}(\beta(e_b(\vartheta_0, \Im\vartheta_0)) \geq 0$$

or

$$\mathcal{G}(\beta(e_b(\vartheta_0, \Im\vartheta_0)) \geq 0.$$

This yields to

$$\beta(e_b(\vartheta_0, \Im\vartheta_0) \geq 1$$

and since $\beta(\xi) < 1$ for all $\xi \geq 0$, we have

$$\beta(e_b(\vartheta_0, \Im\vartheta_0)) = 1.$$

So $e_b(\vartheta_0, \Im\vartheta_0) = 0$, that, $\vartheta_0 = \Im\vartheta_0$, which is a contradiction. Hence, there exist $m,\ q,\ n > n_0$ $(m > q > n)$ such that

$$\Im\vartheta_m \neq \Im\vartheta_q \neq \Im\vartheta_n \neq \vartheta_n.$$

Then

$$\begin{aligned} e_b(\vartheta_n, \Im\vartheta_m) &\leq w(\vartheta_n, \Im\vartheta_m)\{e_b(\vartheta_n, \Im\vartheta_n) + e_b(\Im\vartheta_n, \Im\vartheta_q) + e_b(\Im\vartheta_q, \Im\vartheta_m)\} \\ &\to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.3}$$

On replacing the value in (3.1), we get

$$\lim_{n\to\infty} e_b(\vartheta_n, \vartheta^*) \leq \lim_{n\to\infty} w(\vartheta_n, \vartheta^*) e_b(\Im\vartheta_n, \vartheta^*). \tag{3.4}$$

Since $\mathcal{F}$ is non-decreasing and continuous, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{F}(e_b(\vartheta_n, \vartheta^*)) &\leq \lim_{n\to\infty} \mathcal{F}(w(\vartheta_n, \vartheta^*) e_b(\Im\vartheta_n, \Im\vartheta^*)) \\ &\leq \lim_{n\to\infty} \mathcal{F}(w(\vartheta_n, \vartheta^*)^r e_b(\Im\vartheta_n, \Im\vartheta^*)) \\ &\leq \lim_{n\to\infty} \mathcal{F}(\mathfrak{M}_w(\vartheta_n, \vartheta^*) + \lim_{n\to\infty} \mathcal{G}(\beta(\mathfrak{M}_w(\vartheta_n, \vartheta^*)), \end{aligned}$$

where

$$\mathfrak{M}_w(\vartheta_n, \vartheta^*) = \max\left\{ \begin{array}{c} e_b(\vartheta_n, \vartheta^*), e_b(\vartheta_n, \Im\vartheta_n), e_b(\vartheta^*, \Im\vartheta^*), \\ \frac{e_b(\vartheta^*, \Im\vartheta^*)[1+e_b(\vartheta_n, \Im\vartheta_n)]}{w(\vartheta_n, \vartheta^*)[1+e_b(\vartheta_n, \vartheta^*)]} \end{array} \right\}, \tag{3.5}$$

implies

$$\lim_{n\to\infty} \mathfrak{M}_w(\vartheta_n, \vartheta^*) = \lim_{n\to\infty} e_b(\vartheta_n, \vartheta^*).$$

Then

$$\mathcal{F}(\lim_{n\to\infty} e_b(\vartheta_n, \vartheta^*)) \leq \mathcal{F}(\lim_{n\to\infty} e_b(\vartheta_n, \vartheta^*)) + \lim_{n\to\infty} \mathcal{G}(\beta(e_b(\vartheta_n, \vartheta^*)), \tag{3.6}$$

which gives

$$\lim_{n\to\infty} \mathcal{G}(\beta(e_b(\vartheta_n, \vartheta^*)) \geq 0.$$

This yields to

$$\lim_{n\to\infty} \beta(e_b(\vartheta_n, \vartheta^*) \geq 1$$

and since $\beta(\xi) < 1$ for all $\xi \geq 0$, we have

$$\lim_{n\to\infty} \beta(e_b(\vartheta_n, \vartheta^*)) = 1.$$

Therefore $\lim_{n\to\infty} e_b(\vartheta_n, \vartheta^*) = 0$. This completes the proof. □

The limit shadowing property of fixed point problems has been discussed in the papers [24, 33]. We define weak limit shadowing property (wlsp) in EBbDS.

Definition 3.3. Let (Ξ, e_b) be a complete EBbDS and $\Im : \Xi \to \Xi$ be a mapping. The fpp of $\Im$ is said to have wlsp in Ξ if assuming that $\{\vartheta_n\}$ in Ξ satisfies $e_b(\vartheta_n, \Im\vartheta_n) \to 0$ as $n \to \infty$ and $e_b(\Im\vartheta_n, \Im\vartheta_m) \to 0$, it follows that there exists $\vartheta \in \Xi$ such that $e_b(\vartheta_n, \Im^n\vartheta) \to 0$ as $n \to \infty$.

Theorem 3.4. *Let (Ξ, e_b) be a complete EBbDS and $\Im : \Xi \to \Xi$ be an $\mathcal{F}\mathcal{G}_w$-contractive mapping for $\mathcal{F} \in \mathbb{F}$, $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ with $\{\vartheta_n\}$ in Ξ such that $\lim_{n\to\infty} e_b(\vartheta_n, \Im\vartheta_n) = 0$, $\lim_{n,m\to\infty} e_b(\Im\vartheta_n, \Im\vartheta_m) = 0$ and $\vartheta^* \in Fix(\Im)$. Then $\Im$ has the wlsp.*

Proof. Since ϑ^* is a fixed point of $\Im$, we have $e_b(\vartheta^*, \Im\vartheta^*) = 0$ and let $\{\vartheta_n\}$ in Ξ such that $\lim_{n\to\infty} e_b(\vartheta_n, \Im\vartheta_n) = 0$, $\lim_{n,m\to\infty} e_b(\Im\vartheta_n, \Im\vartheta_m) = 0$. Then by virtue of Theorem 3.2, we have $\lim_{n\to\infty} e_b(\vartheta_n, \vartheta^*) = 0$ and therefore we can gete $\lim_{n\to\infty} e_b(\vartheta_n, \Im^n\vartheta^*) = 0$. □

Next, we define generalized w-Ulam-Hyers stability (GwUHS) of fpp in EBbDS as an extension of b-metric space discussed in [11, 26] (see also [18, 20, 23]).

Definition 3.5. Let (Ξ, e_b) be a complete EBbDS and $\Im : \Xi \to \Xi$ be a mapping. The fixed point equation (FPE)

$$\vartheta = \Im\vartheta, \ \vartheta \in \Xi \tag{3.7}$$

is called Gw-UHS in the setting of EBbDS if there exists an increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, continuous at 0 with $\phi(0) = 0$, such that for each $\varepsilon > 0$ and an ε-solution $\upsilon \in \Xi$, that is,

$$e_b(\upsilon, \Im\upsilon) \leq \varepsilon,$$

there exists a solution $\vartheta^* \in \Xi$ of (3.7) such that

$$e_b(\upsilon, \vartheta^*) \leq \phi(w(\vartheta^*, \upsilon)\varepsilon). \tag{3.8}$$

If $\phi(\xi) = \alpha\xi$ for all $\xi \in \mathbb{R}_+$, where $\alpha > 0$, then FPE (3.7) is said to be w-UHS in the setting of EBbDS.

Remark 3.6. If $w(\vartheta, \upsilon) = 1$, then Definition 3.5 converted to the notion of GUHS in BDS. Also, if $\phi(\xi) = \alpha\xi$ for all $\alpha \in \mathbb{R}_+$, where $\alpha > 0$, then it converted to the notion of UHS in BDS. Also if $e_b(\vartheta, \upsilon) = |\vartheta - \upsilon|$, then it is reduced to the classical UHS.

Theorem 3.7. *Let* (Ξ, e_b) *be a complete EBbDS and* $\Im : \Xi \to \Xi$ *be an Geraghty-type* (2.16) *contractive mapping for* $\beta \in (\mathcal{G}, \beta) \in \mathbb{G}_\beta$, *and also that the function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by* $\varphi(\xi) := \xi[1 - \beta(\xi)]$ *is strictly increasing and onto. Then the FPE* (3.7) *is Gw-UHS.*

Proof. From Theorem 3.4, we have $\Im\vartheta^* = \vartheta^*$, that is, $\vartheta^* \in \Xi$ is a solution of the FPE (3.7) with $e_b(\vartheta^*, \vartheta^*) = 0$. Let $\varepsilon > 0$ and $\upsilon^* \in \Xi$ be an ε-solution of FPE (3.7), that is,

$$e_b(\upsilon^*, \Im\upsilon^*) \leq \varepsilon.$$

Since $e_b(\vartheta^*, \Im\vartheta^*) = e_b(\vartheta^*, \vartheta^*) = 0 \leq \varepsilon$, ϑ^* and υ^* are ε-solutions. Since we have $w(\vartheta^*, \upsilon^*) \geq 1$ and so

$$\begin{aligned} e_b(\vartheta^*, \upsilon^*) &\leq w(\vartheta^*, \upsilon^*)[e_b(\vartheta^*, \Im\vartheta^*) + e_b(\Im\vartheta^*, \Im\upsilon^*) + e_b(\Im\upsilon^*, \upsilon^*)] \\ &\leq w(\vartheta^*, \upsilon^*)e_b(\Im\vartheta^*, \Im\upsilon^*) + \varepsilon w(\vartheta^*, \upsilon^*) \\ &\leq w(\vartheta^*, \upsilon^*)^r e_b(\Im\vartheta^*, \Im\upsilon^*) + \varepsilon w(\vartheta^*, \upsilon^*) \\ &\leq \beta(\mathfrak{M}_w(\vartheta^*, \upsilon^*))\mathfrak{M}_w(\vartheta^*, \upsilon^*) + \varepsilon w(\vartheta^*, \upsilon^*), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \mathfrak{M}_w(\vartheta^*, \upsilon^*) &= \max\left\{ \begin{array}{c} e_b(\vartheta^*, \upsilon^*), e_b(\vartheta^*, \Im\vartheta^*), e_b(\upsilon^*, \Im\upsilon^*), \\ \frac{e_b(\upsilon^*, \Im\upsilon^*)[1+e_b(\vartheta^*, \Im\vartheta^*)]}{w(\vartheta^*, \upsilon^*)[1+e_b(\vartheta^*, \upsilon^*)]} \end{array} \right\} \\ &\leq \max\left\{ e_b(\vartheta^*, \upsilon^*), 0, \varepsilon, \frac{\varepsilon}{w(\vartheta^*, \upsilon^*)[1+e_b(\vartheta^*, \upsilon^*)]} \right\} \\ &= \max\left\{ e_b(\vartheta^*, \upsilon^*), \varepsilon \right\}. \end{aligned}$$

Let us discuss the two possible cases.

Case I. If $\mathfrak{M}_w(\vartheta^*, \upsilon^*) = e_b(\vartheta^*, \upsilon^*)$, then we get

$$e_b(\vartheta^*, \upsilon^*) \leq \beta(e_b(\vartheta^*, \upsilon^*))e_b(\vartheta^*, \upsilon^*) + w(\vartheta^*, \upsilon^*)\varepsilon,$$

that is,

$$e_b(\vartheta^*, \upsilon^*)[1 - \beta(e_b(\vartheta^*, \upsilon^*))] \leq w(\vartheta^*, \upsilon^*)\varepsilon.$$

Since $\varphi(\xi) = \xi[1-\beta(\xi)]$, we have

$$\varphi(e_b(\vartheta^*, \upsilon^*)) \leq w(\vartheta^*, \upsilon^*)\varepsilon,$$

which implies that

$$e_b(\vartheta^*, \upsilon^*) \leq \phi(w(\vartheta^*, \upsilon^*)\varepsilon),$$

where $\phi := \varphi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ exists, is increasing, continuous at 0 and $\varphi^{-1}(0) = 0$. Since $0 \leq \beta(\xi) < 1$, it is $0 \leq \varphi(\xi) \leq \xi$, and so $\phi(\xi) \geq \xi$ for $\xi \in \mathbb{R}_+$.
Case II. If $\mathfrak{M}_w(\vartheta^*, \upsilon^*) = \varepsilon$, then (3.9) gives that

$$e_b(\vartheta^*, \upsilon^*) \leq \varepsilon \leq w(\vartheta^*, \upsilon^*)\varepsilon \leq \phi(w(\vartheta^*, \upsilon^*)\varepsilon).$$

It shows that the inequality (3.8) is true for all cases and, thus the FPE (3.7) is Gw-UHS. This completes the proof. □

4. Application to nonlinear matrix equations

Denote $s(\mathcal{U})$, any singular value of a matrix $\mathcal{U}$, and the trace norm of $\mathcal{U}$ will be denoted by $s^+(\mathcal{U}) = \|\mathcal{U}\|$. We will use the standard partial order on $\mathcal{H}(n)$ given by $\mathcal{U} \succeq \mathcal{V}$ if and only if $\mathcal{U} - \mathcal{V}$ is a positive semi-definite matrix.

Theorem 4.1. *Consider the system*

$$\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i, \tag{4.1}$$

where $\mathcal{Q} \in \mathcal{P}(n)$, $\mathcal{A}_i \in M(n)$, $i = 1, \ldots, k$, and the operator $\hbar \colon \mathcal{P}(n) \to \mathcal{P}(n)$ is continuous in the trace norm. Let, for some $M, N_1 \in \mathbb{R}$, and for any $\mathcal{U} \in \mathcal{P}(n)$ with $\|\mathcal{U}\| \leq M$, $s(\hbar(\mathcal{U})) \leq N_1$ hold for all singular values of $\hbar(\mathcal{U})$.
Assume that:

1. $\|\mathcal{Q}\|, \leq M - NN_1 n$, *where* $\sum\limits_{i=1}^{k} \|\mathcal{A}_i^*\| \, \|\mathcal{A}_i\| = N$;
2. *for any* $\mathcal{W} \in \mathcal{P}(n)$ *with* $\|\mathcal{W}\| \leq M$, $\sum_{i=1}^k \mathcal{A}_i^* \hbar(\mathcal{W}) \mathcal{A}_i \succeq O$ *holds;*
3. *for any* $\mathcal{W} \in \mathcal{P}(n)$ *with* $\|\mathcal{W}\| \leq M$, $\mathcal{W} \preceq \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^* \hbar(\mathcal{W}) \mathcal{A}_i$ *holds;*
4. *there exist* $w(\mathcal{U}, \mathcal{V}) \geq 1$, $p > 1$, $r \geq 2$ *and* $\tau > 0$,

$$2NN_1 \leq \frac{1}{nw(\mathcal{U}, \mathcal{V})^{r/p}} \Upsilon(\mathcal{U}, \mathcal{V}) \tag{4.2}$$

holds, where

$$\Upsilon(\mathcal{U},\mathcal{V}) = \max\left\{\begin{array}{l} \frac{\left|s^+(\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{U})\mathcal{A}_i+\mathcal{Q}-\mathcal{U})\right|}{\left[1+\tau\left|s^+(\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{U})\mathcal{A}_i+\mathcal{Q}-\mathcal{U})\right|^{p/2}\right]^{2/p}}, \\ \frac{\left|s^+(\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{V})\mathcal{A}_i+\mathcal{Q}-\mathcal{V})\right|}{\left[1+\tau\left|s^+(\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{V})\mathcal{A}_i+\mathcal{Q}-\mathcal{V})\right|^{p/2}\right]^{2/p}}, \\ \frac{s^+(\mathcal{X}-\mathcal{Y})}{\left[1+\tau|s^+(\mathcal{X}-\mathcal{Y})|^{p/2}\right]^{2/p}}, \\ \frac{\frac{\left|s^+(\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{V})\mathcal{A}_i+\mathcal{Q}-\mathcal{V})\right|\left[1+\left|s^+(\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{U})\mathcal{A}_i+\mathcal{Q}-\mathcal{U})\right|\right]}{w(\mathcal{U},\mathcal{V})^{r/p}[1+|s^+(\mathcal{X}-\mathcal{Y})|]}}{\left[1+\tau\left|\frac{\left|s^+(\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{V})\mathcal{A}_i+\mathcal{Q}-\mathcal{V})\right|\left[1+\left|s^+(\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{U})\mathcal{A}_i+\mathcal{Q}-\mathcal{U})\right|\right]}{w(\mathcal{U},\mathcal{V})^{r/p}[1+|s^+(\mathcal{X}-\mathcal{Y})|]}\right|^{p/2}\right]^{2/p}} \end{array}\right\}, \tag{4.3}$$

for all $\mathcal{U},\mathcal{V}\in\mathcal{P}(n)$ *with* $\|\mathcal{U}\|,\|\mathcal{V}\|\le M$, $\mathcal{U}\preceq\mathcal{V}$ *and* $\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{U})\mathcal{A}_i \ne \sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{V})\mathcal{A}_i$.

Then the system (4.1) *has a unique solution* $\widehat{\mathcal{U}}\in\mathcal{P}(n)$ *with* $\|\widehat{\mathcal{U}}\|\le M$. *Further, the solution can be obtained as the limit of the iterative sequence* $\{\mathcal{U}_n\}$, *where for* $j\ge 0$,

$$\mathcal{U}_{j+1} = \mathcal{Q} + \sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{U}_j)\mathcal{A}_i \tag{4.4}$$

and $\mathcal{U}_0$ *is an arbitrary element of* $\mathcal{P}(n)$ *satisfying* $\|\mathcal{U}_0\|\le M$.

Proof. Denote $\Lambda := \{\mathcal{U}\in\mathcal{P}(n) : \|\mathcal{U}\|\le M\}$, being a closed subset of $\mathcal{P}(n)$. According to (2), any solution of (4.1) in Λ has to be positive definite. We have, for any $\mathcal{U}\in\Lambda$,

$$\begin{aligned} &\|\mathcal{Q}+\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{U})\mathcal{A}_i\| \\ &\le \|\mathcal{Q}\| + \|\sum_{i=1}^k \mathcal{A}_i^*\hbar(\mathcal{U})\mathcal{A}_i\| \\ &\le \|\mathcal{Q}\| + \sum_{i=1}^k \|\mathcal{A}_i^*\|\,\|\mathcal{A}_i\|\,\|\hbar(\mathcal{U})\| \\ &= \|\mathcal{Q}\| + N\|\hbar(\mathcal{U})\|. \end{aligned} \tag{4.5}$$

Since, for all $s(\hbar(\mathcal{U})) \leq N_1$, it follows that $\|\hbar(\mathcal{U})\| \leq N_1 n$. Thus, (4.5) implies

$$\begin{aligned}\|\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i\| &\leq \|\mathcal{Q}\| + NN_1 n \\ &\leq M - NN_1 n + NN_1 n = M.\end{aligned}$$

Define now an operator $\Im : \Lambda \to \Lambda$ by

$$\Im(\mathcal{U}) = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i,$$

for $\mathcal{U} \in \Lambda$. Then it is clear that finding positive definite solution(s) of the equation (4.1) is equivalent to finding fixed point(s) of $\Im$.

Now, for any $\mathcal{U},\ \mathcal{V} \in \Lambda$ with $\mathcal{U} \preceq \mathcal{V}$, we have

$$\begin{aligned}\|\Im(\mathcal{U}) - \Im(\mathcal{V})\| &= \|\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i - \mathcal{Q} - \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i\| \\ &\leq \|\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i - \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i\| \\ &\leq \sum_{i=1}^{k} \|\mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i - \mathcal{A}_i^* \hbar(\mathcal{V}) \mathcal{A}_i\| \\ &\leq \sum_{i=1}^{k} \|\mathcal{A}_i^*\| \, \|\mathcal{A}_i\| \, \|\hbar(\mathcal{U}) - \hbar(\mathcal{V})\| \\ &\leq N(\|\hbar(\mathcal{U})\| + \|\hbar(\mathcal{V})\|) \\ &\leq N(N_1 n + N_1 n) \\ &= 2NN_1 n.\end{aligned}$$

Thus, for any $\mathcal{U},\ \mathcal{V} \in \Lambda$ with $\mathcal{U} \preceq \mathcal{V}$, we have

$$\|\Im(\mathcal{U}) - \Im(\mathcal{V})\| \leq 2NN_1 n. \tag{4.6}$$

For some fixed $\mathcal{U}, \mathcal{V} \in \Lambda$ with $\mathcal{U} \preceq \mathcal{V}$, from (4.2) and (4.3), if

$$\Upsilon(\mathcal{U}, \mathcal{V}) = \frac{\left| s^{+}(\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i + \mathcal{Q} - \mathcal{U}) \right|}{\left[1 + \tau \left| s^{+}(\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U}) \mathcal{A}_i + \mathcal{Q} - \mathcal{U}) \right|^{p/2} \right]^{2/p}},$$

then we have

$$\begin{aligned}
2NN_1 &\leq \frac{1}{nw(\mathcal{U},\mathcal{V})^{r/p}} \cdot \frac{\left| s^{+}(\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U})\mathcal{A}_i + \mathcal{Q} - \mathcal{U}) \right|}{\left[1 + \tau \left| s^{+}(\sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U})\mathcal{A}_i + \mathcal{Q}) - \mathcal{U} \right|^{p/2}\right]^{2/p}} \\
&= \frac{1}{nw(\mathcal{U},\mathcal{V})^{r/p}} \cdot \frac{\left\| \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U})\mathcal{A}_i + \mathcal{Q} - \mathcal{U} \right\|}{\left[1 + \tau \left\| \sum_{i=1}^{k} \mathcal{A}_i^* \hbar(\mathcal{U})\mathcal{A}_i + \mathcal{Q} - \mathcal{U} \right\|^{p/2}\right]^{2/p}} \\
&= \frac{1}{nw(\mathcal{U},\mathcal{V})^{r/p}} \cdot \frac{\|\Im(\mathcal{U}) - \mathcal{U}\|}{\left[1 + \tau \|\Im(\mathcal{U}) - \mathcal{U}\|^{p/2}\right]^{2/p}},
\end{aligned}$$

that is,

$$2NN_1 n \leq \frac{1}{w(\mathcal{U},\mathcal{V})^{r/p}} \cdot \frac{\|\Im(\mathcal{U}) - \mathcal{U}\|}{\left[1 + \tau \|\Im(\mathcal{U}) - \mathcal{U}\|^{p/2}\right]^{2/p}}.$$

Therefore, from (4.6) we have

$$\|\Im(\mathcal{U}) - \Im(\mathcal{V})\| \leq \frac{1}{w(\mathcal{U},\mathcal{V})^{r/p}} \cdot \frac{\|\Im(\mathcal{U}) - \mathcal{U}\|}{\left[1 + \tau \|\Im(\mathcal{U}) - \mathcal{U}\|^{p/2}\right]^{2/p}},$$

that is,

$$w(\mathcal{U},\mathcal{V})^r \|\Im(\mathcal{U}) - \Im(\mathcal{V})\|^p \leq \frac{\|\Im(\mathcal{U}) - \mathcal{U}\|^p}{\left[1 + \tau \|\Im(\mathcal{U}) - \mathcal{U}\|^{p/2}\right]^2}. \tag{4.7}$$

By similar arguments, using (4.2), (4.3), (4.6), we have, for any $\mathcal{U},\mathcal{V} \in \Lambda$ with $\mathcal{U} \preceq \mathcal{V}$,

$$w(\mathcal{U},\mathcal{V})^r \|\Im(\mathcal{U}) - \Im(\mathcal{V})\|^p \leq \frac{\|\Im(\mathcal{V}) - \mathcal{V}\|^p}{\left[1 + \tau \|\Im(\mathcal{V}) - \mathcal{V}\|^{p/2}\right]^2}, \tag{4.8}$$

$$w(\mathcal{U},\mathcal{V})^r \|\Im(\mathcal{U}) - \Im(\mathcal{V})\|^p \leq \frac{\|\mathcal{U} - \mathcal{V}\|^p}{\left[1 + \tau \|\mathcal{U} - \mathcal{V}\|^{p/2}\right]^2} \tag{4.9}$$

and

$$w(\mathcal{U},\mathcal{V})^r \|\Im(\mathcal{U}) - \Im(\mathcal{V})\|^p \leq \frac{\frac{\|\mathcal{V} - \Im(\mathcal{V})\|^p [1 + \|\mathcal{U} - \Im(\mathcal{U})\|^p]}{w(\mathcal{U},\mathcal{V})[1 + \|\mathcal{U} - \mathcal{V}\|^p]}}{\left[1 + \tau \left[\frac{\|\mathcal{V} - \Im(\mathcal{V})\|^p [1 + \|\mathcal{U} - \Im(\mathcal{U})\|^p]}{w(\mathcal{U},\mathcal{V})[1 + \|\mathcal{U} - \mathcal{V}\|^p]}\right]^{p/2}\right]^2}. \tag{4.10}$$

Let $e_b \colon \mathcal{P}(n) \times \mathcal{P}(n) \to \mathbb{R}_+$ be defined by

$$e_b(\mathcal{U},\mathcal{V}) = \|\mathcal{U} - \mathcal{V}\|^p \quad \text{for all} \quad \mathcal{U},\mathcal{V} \in \mathcal{P}(n).$$

Then $(\mathcal{P}(n), e_b)$ is a complete extended Branciari b-distance space with coefficient $w(\mathcal{U},\mathcal{V}) = \|\mathcal{U}\| + \|\mathcal{V}\| + 3^{p-1}$. It follows from (4.7)–(4.9) that

$$\begin{aligned} &w(\mathcal{U},\mathcal{V})^r e_b(\Im(\mathcal{U}),\Im(\mathcal{V})) \\ &\le \max\left\{\begin{array}{l} \frac{e_b(\mathcal{U},\mathcal{V})}{\left[1+\tau\sqrt{e_b(\mathcal{U},\mathcal{V})}\right]^2}, \frac{e_b(\Im(\mathcal{U}),\mathcal{U})}{\left[1+\tau\sqrt{e_b(\Im(\mathcal{U}),\mathcal{U})}\right]^2}, \\ \frac{e_b(\Im(\mathcal{V}),\mathcal{V})}{\left[1+\tau\sqrt{e_b(\Im(\mathcal{V}),\mathcal{V})}\right]^2}, \frac{\frac{e_b(\mathcal{V},\Im(\mathcal{V}))[1+e_b(\mathcal{U},\Im(\mathcal{U}))]}{w(\mathcal{U},\mathcal{V})[1+e_b(\mathcal{U},\mathcal{V})]}}{\left[1+\tau\sqrt{\frac{e_b(\mathcal{V},\Im(\mathcal{V}))[1+e_b(\mathcal{U},\Im(\mathcal{U}))]}{w(\mathcal{U},\mathcal{V})[1+e_b(\mathcal{U},\mathcal{V})]}}\right]^2} \end{array}\right\}, \end{aligned}$$

that is,

$$w(\mathcal{U},\mathcal{V})^r e_b(\Im(\mathcal{U}),\Im(\mathcal{V})) \le \frac{\mathfrak{M}_w(\mathcal{U},\mathcal{V})}{[1+\tau\sqrt{\mathfrak{M}_w(\mathcal{U},\mathcal{V})}]^2}, \tag{4.11}$$

where

$$\mathfrak{M}_w(\mathcal{U},\mathcal{V}) := \max\left\{\begin{array}{l} \frac{e_b(\mathcal{U},\mathcal{V})}{\left[1+\tau\sqrt{e_b(\mathcal{U},\mathcal{V})}\right]^2}, \frac{e_b(\Im(\mathcal{U}),\mathcal{U})}{\left[1+\tau\sqrt{e_b(\Im(\mathcal{U}),\mathcal{U})}\right]^2}, \\ \frac{e_b(\Im(\mathcal{V}),\mathcal{V})}{\left[1+\tau\sqrt{e_b(\Im(\mathcal{V}),\mathcal{V})}\right]^2}, \\ \frac{\frac{e_b(\mathcal{V},\Im(\mathcal{V}))[1+e_b(\mathcal{U},\Im(\mathcal{U}))]}{w(\mathcal{U},\mathcal{V})[1+e_b(\mathcal{U},\mathcal{V})]}}{\left[1+\tau\sqrt{\frac{e_b(\mathcal{V},\Im(\mathcal{V}))[1+e_b(\mathcal{U},\Im(\mathcal{U}))]}{w(\mathcal{U},\mathcal{V})[1+e_b(\mathcal{U},\mathcal{V})]}}\right]^2} \end{array}\right\}.$$

Let $\beta(\zeta) = e^{-\tau}(\tau > 0)$, $\mathcal{G}(\zeta) = \ln\zeta$ and $\mathcal{F}(\zeta) = -1/\sqrt{\zeta}$ $(\zeta > 0)$ in (4.11). Then the formulated results follow from Theorem 2.3. $\square$

Example 4.2. Consider the system (4.1) for $k = 3$, $n = 3$, with $\hbar(\mathcal{U}) = \mathcal{U}^2/900$, that is,

$$\mathcal{U} = \mathcal{Q}_1 + \mathcal{A}_1^*(\mathcal{U}^2/900)\mathcal{A}_1 + \mathcal{A}_2^*(\mathcal{U}^2/900)\mathcal{A}_2 + \mathcal{A}_3^*(\mathcal{U}^2/900)\mathcal{A}_3, \tag{4.12}$$

where

$$\mathcal{A}_1 = \begin{bmatrix} 2.25 & 0 & 9.98 \\ -6.32 & 0.469 & -5.85 \\ 3.25 & -1.87 & 0.896 \end{bmatrix}, \ \mathcal{A}_2 = \begin{bmatrix} -2.66 & 0 & 8.59 \\ 0.279 & -2.45 & 8.25 \\ -3.62 & 0.98 & -5.36 \end{bmatrix},$$
$$\mathcal{A}_3 = \begin{bmatrix} -5.45 & -0.586 & 0.827 \\ 0 & 0.324 & -0.552 \\ 0.996 & -2.36 & 0.256 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

After calculations, we get $\|\mathcal{Q}_1\| = 3$, $N = 8.079768372577067 \times 10^{02}$, $N_1 = 1.5625 \times 10^{-05}$.

Let $M = 1.030042042174708 \times 10^2$, $r = 3$, $\tau = 0.001$, $p = 3$, and $w(\mathcal{U},\mathcal{V}) = \|\mathcal{U}\| + \|\mathcal{V}\| + 3^{p-1}$. The conditions of Theorem 4.1 can be checked numerically, taking various special values for matrices involved. For example, they can be tested (and verified to be true) for

$$\mathcal{U}=\begin{bmatrix}0.125 & 0 & 0\\ 0 & 0.125 & 0\\ 0 & 0 & 0.125\end{bmatrix},\quad \mathcal{V}=10^{02}\times\begin{bmatrix}0.705164 & 0.236584 & -0.520859\\ 0.236584 & 1.303675 & 0.348708\\ -0.520859 & 0.348708 & 1.142139\end{bmatrix},$$

$$\mathcal{W}=\begin{bmatrix}0.040 & 0 & 0\\ 0 & 0.040 & 0\\ 0 & 0 & 0.040\end{bmatrix},$$

where

$$\|\mathcal{U}\| = 0.375, \quad \|\mathcal{V}\| = 315.0977999999999, \quad \|\mathcal{W}\| = 0.12.$$

To see the convergence of the sequence $\{\mathcal{U}_n\}$ defined in (4.4), we start with initial value

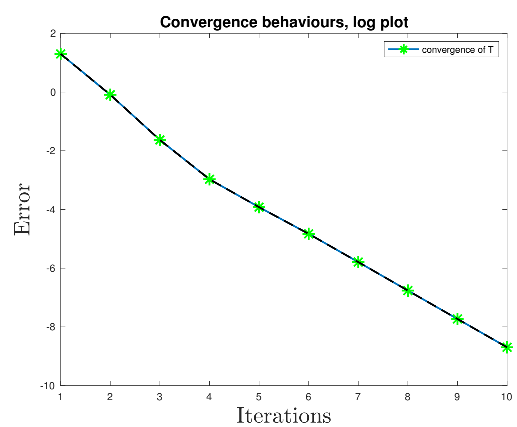
$$\mathcal{U}_0=\begin{bmatrix}1.87561 & 0.791132 & -0.220904\\ 0.791132 & 0.990034 & -0.068062\\ -0.220904 & -0.068062 & 0.17869\end{bmatrix} \text{ with } \|\mathcal{U}_0\|=3.044334000000001,$$

and after 10 iterations, we have the following approximation of the unique PDS of NME (4.12)

$$\begin{aligned}\widehat{\mathcal{U}} &\approx \mathcal{U}_{10}\\ &= \begin{bmatrix}1.8977068424646789 & -0.29908302956523 & 0.11479043330130\\ -0.29908302956523 & 1.034166978513196 & -0.054450928984136\\ 0.11479043330130 & -0.054450928984136 & 1.457646671310517\end{bmatrix}\end{aligned}$$

with $\|\widehat{\mathcal{U}}\| = 3.681584334070510$.

The graphical representation of convergence of $\{\mathcal{U}_n\}$ is shown below:



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References

[1] T. Abdeljawad, E. Karapinar, S.K. Panda and N. Mlaiki, *Solutions of boundary value problems on extended-Branciari b-distance*, J. Ineq. Appl., **2020** (2020), Art. number 103, 1–16.

[2] I.A. Bakhtin, *The contraction mapping principle in quasi metric spaces*, Funkc. Anal. Ulianowsk Gos. Ped. Inst., **30** (1999), 243–253.

[3] S. Bose, Sk.M. Hossein and K. Paul, *Solution of a class of nonlinear matrix equations*, Linear Algebra Appl., **530** (2017), 109–126.

[4] R. Bouhafs, A. Tallafha and W. Shatanawi, *Fixed point theorems in ordered b-metric spaces with alternating distance functions*, Nonlinear Funct. Anal. Appl., **26**(3) (2021), 581–600.

[5] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **29**(9) (2002), 531-536.

[6] L. Chen, S. Huang, C. Li and Y. Zhao, *Several fixed-point theorems for F-contractions in complete Branciari b-metrics, and applications*, J. Funct. Spaces, **2020** (2020), Art. Number 7963242, 1–10.

[7] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis, **5** (1993), 5–11.

[8] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 263–276.

[9] S. Ding, M. Imdad, S. Radenović and J. Vujakovic, *On some fixed point results in b-metric, rectangular and b-rectangular metric spaces*, Arab J. Math. Sci., **22**(2) (2016), 151–164.

[10] D. Dukić, Z. Kadelburg and S. Radenović, *Fixed points of Geraghty-type mappings in various generalized metric spaces*, Abstract Appl. Anal., **2011**, Art. Number 561245, 1–13.

[11] A. Felhi, S. Sahmim and H. Aydi, *Ulam-Hyers stability and well-posedness of fixed point problems for α-λ-contractions on quasi b-metric spaces*, Fixed Point Theory Appl., **2016** (2016), Art. Numner 1, 1–20.

[12] H. Garai and L.K. Dey, *Common solution to a pair of nonlinear matrix equations via fixed point results*, J. Fixed Point Theory Appl., **21**(61) (2019), https://doi.org/10.1007/s11784-019-0698-7.

[13] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608.

[14] Sk.M. Hossein, S. Bose and K. Paul, *On positive definite solution of a nonlinear matrix equations*, Linear Mult. Alg., **66**(5) (2018), 881–893.

[15] Sk.M. Hossein, S. Bose and K. Paul, *Solution of a pair of nonlinear matrix equations*, Fixed Point Theory, **19**(1) (2018), 265–274.

[16] M. Jleli and B. Samet, *A new generalization of the Banach contraction principle*, J. Ineq. Appl., **2014**(2014), Art. Number ID 38, 1–8.

[17] T. Kamran, M. Samreen and O.U. Ain, *A generalization of b-metric space and some fixed point theorems*, Mathematics, **5**(2) (2017), Art. Number 19.

[18] C. Kausika, K. Balachandran, N. Annapoorani and J.K. Kim, *Existence and stability results of generalized fractional integrodifferential equations,* Nonlinear Funct. Anal. Appl., **26**(4) (2021), 793–809.

[19] F. Khojasteh, *A new approach to the study of fixed point theorems via simulation functions*, Filomat, **96** (2015), 1189–1194.

[20] N. Limpanukorn, P. Sa Ngiamsunthorn, D. Songsanga and A. Suechoei, *On the stability of differential systems involving ψ-Hilfer fractional derivative,* Nonlinear Funct. Anal. Appl., **27**(3) (2022), 513–532.

[21] Z.D. Mitrovi¸ H. Isk and S. Radenović, *The new results in extended b-metric spaces and applications*, Int. J. Nonlinear Anal. Appl., **11**(1) (2020), 473–482.

[22] Z. Mustafa, V. Parvaneh, J.R. Roshan and Z. Kadelburg, *b2-Metric spaces and some fixed point theorems*, Fixed Point Theory Appl., **2014** (2014), Art. Number 144, 1–23.

[23] H.K. Nashine and Z. Kadelburg, *Existence of solutions of cantilever beam problem via $\alpha - \beta - FG$-contractions in b-metric-like spaces*, Filomat, **31**(11) (2017), 3057–3074.

[24] M. Păcurar and I.A. Rus, *Fixed point theory for cyclic ϕ-contractions*, Nonlinear Anal., **72** (2010), 1181–1187.

[25] V. Parvaneh, N. Hussain and Z. Kadelburg, *Generalized Wardowski type fixed point theorems via α-admissible FG-contractions in b-metric spaces*, Acta Math. Scientia, **36**(5) (2016), 1445–1456.

[26] S. Phiangsungnoen, W. Sintunavarat and P. Kumam, *Fixed point results, generalized Ulam-Hyers stability and well-posedness via α-admissible mappings in b-metric spaces*, Fixed Point Theory Appl., **2014** (2012), Art. Number 188, 1–17.

[27] V. Popa, *Well-posedness of fixed point problems in orbitally complete metric spaces*, Stud. Cerc. St. Ser. Mat. Univ., **16** (2006), Supplement. Proc. ICMI 45, Bacau, Sept. 18-20 (2006), 209–214.

[28] V. Popa, *Well-posedness of fixed point problems in compact metric spaces*, Bul. Univ. Petrol-Gaze, Ploiesti, Sec. Mat. Inform. Fiz., **60**(1) (2008), 1–4.

[29] D. Rakić, A. Mukheimer, T. Doenović and Z.D. Mitrović, and S. Radenović, *On some new fixed point results in fuzzy b-metric spaces*, J. Ineq. Appl., **2020** (2020), Art. Number 99, 1–14.

[30] T. Ram, J.K. Kim and R. Kour, *On optimal solutions of well-posed problems and variational inequalities*, Nonlinear Funct. Anal. Appl., **26**(4) (2021), 781–792.

[31] A.C.M. Ran and M.C.B. Reurings, *On the matrix equation $X + A^*F(X)A = Q$: solutions and perturbation theory*, Linear Alg. Appl., **346** (2002), 15–26.

[32] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443.

[33] I.A. Rus, *The theory of a metrical fixed point theorem: Theoretical and applicative relevances*, Fixed Point Theory, **9** (2008), 541–559.

[34] K. Sawangsup and W. Sintunavarat, *Fixed point and multidimensional fixed point theorems with applications to nonlinear matrix equations in terms of weak altering distance functions*, Open Math., **15** (2017), 111–125.

[35] S.M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1964.