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SYMBOLICALLY EXPANSIVE DYNAMICAL SYSTEMS

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Abstract. In this article, we consider the notion of expansiveness on compact metric spaces for symbolically point of view. And we show that a homeomorphism is symbolically countably expansive if and only if it is symbolically measure expansive. Moreover, we prove that a homeomorphism is symbolically N-expansive if and only if it is symbolically measure N-expanding.

1. Introduction

Let (X, d) be a compact metric space and f be a homeomorphism on X. The notion of expansiveness for f on X introduced by Utz [7] is very important to study the qualitative theory of the dynamical systems. A homeomorphism $f : X \to X$ is called *expansive* if there is $\delta > 0$ such that for any $x \neq y \in X$ there exists $i \in \mathbb{Z}$ such that $d(f^i(x), f^i(y)) > \delta$. Simply speaking, a system is expansive if two orbits cannot remain close to each other under the action of the system. This notion is closely related with the orbit behavior, so, it plays a key role in the study of the stability of the dynamics.

It is natural to consider various type of expansiveness such as N-expansiveness[3], measure expansiveness[5], countably expansiveness[5], and so on. Artigue and Carrasco-Olivera [1] consider a relationship between the measure expansiveness and the countably expansiveness.

Remark 1.1. [1, Theorem 2.1] Let f be a homeomorphism on a compact metric space X. The following statements are equivalent :

(1) f is countably-expansive,
(2) f is measure-expansive.

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And [4] consider the new notion of expansiveness which is a symbolic viewpoint, called by *symbolically expansiveness*. So, we have a following problem that

"What is the relationship between the symbolically countable expansiveness and the symbolically measure expansiveness ?

For solving this problem, we consider the notion of symbolically expansiveness, symbolically N-expansiveness, symbolically countably expansiveness on compact metric spaces. And we show that a homeomorphism is symbolically countably expansive if and only if it is symbolically measure expansive. Moreover, we prove that a homeomorphism is symbolically N-expansive if and only if it is symbolically measure N-expanding.

2. Symbolically expansive homeomorphisms

Let X be a compact metric space, and $f : X \to X$ be a homeomorphism. We say that $\boldsymbol{\epsilon} = \{A_i \subset X : 1 \leq i \leq n\}$ is a *finite partition of x* if it satisfies the following three conditions :

- Each $A_i \neq \emptyset$: Borel set for all $i = 1, \cdots, n$,
- $A_i \cap A_j = \emptyset$ if $i \neq j$,
- $\bigcup_{i=1}^{n} A_i = X$.

And we denote $\mathcal{F}(X)$ by the collection of all finite partition of X.

For $\forall \boldsymbol{\epsilon} \in \mathcal{F}(X)$, $\forall x \in X$, denote $\boldsymbol{\epsilon}(x)$ by the element in $\boldsymbol{\epsilon}$ containing x. If $\forall \boldsymbol{\epsilon} \in \mathcal{F}(X)$ and $\forall x \in X$, the following set called by the *dynamic* $\boldsymbol{\epsilon}$*-ball at* x

$$\begin{aligned}\Gamma_{\boldsymbol{\epsilon}}(x) &= \{y \in X : f^n(y) \in \boldsymbol{\epsilon}(f^n(x)),\ \forall n \in \mathbb{Z}\} \\ &= \bigcap_{n \in \mathbb{Z}} f^{-n}[\boldsymbol{\epsilon}(f^n(x))].\end{aligned}$$

From the above concept of partition, we define the symbolically expansiveness as following.

Definition 2.1. Let X be a compact metric space, and $f : X \to X$ be a homeomorphism.

(1) f is *symbolically expansive* if there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that $\Gamma_{\boldsymbol{\epsilon}}(x) = \{x\}$ for all $x \in X$.

(2) f is *symbolically N-expansive* if there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that $\Gamma_{\boldsymbol{\epsilon}}(x)$ has at most N-elements for all $x \in X$.

(3) f is *symbolically countably expansive* if there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that $\Gamma_{\boldsymbol{\epsilon}}(x)$ has at countably many elements for all $x \in X$.

Based on the above definitions, it is natural to consider the relationship between expansiveness and symbolically expansiveness on a compact metric space. There is an example that it is symbolically expansive but is not expansive.

EXAMPLE 2.2. *Let S^1 be a unit circle and $f : S^1 \to S^1$ be an irrational rotation map. Regard S^1 as $[0, 2\pi)$ for convenience. Then*

(1) *It is well known that there is no expansive map on S^1. Therefore, f is not expansive.*
(2) *f is symbolically expansive.*

Proof. For each $i = 1, 2, 3, 4$, let $A_i = [\frac{(i-1)\pi}{2}, \frac{i\pi}{2})$ and $\boldsymbol{\epsilon} = \{A_1, A_2, A_3, A_4\}$. Then $\boldsymbol{\epsilon}$ is a finite partition of S^1 with a diameter $\frac{\pi}{2}$. For any $x \in S^1$, choose A_i containing x and take a point $y \in A_i$ with $x \neq y$. Suppose $x < y$. Since f is a minimal isometry, there exists $k \in \mathbb{Z}$ such that

$$f^k(x) \in A_{i-1} \text{ and } f^k(y) \in A_i,$$

where $A_0 = A_4$. This means that $y \notin \Gamma^f_{\boldsymbol{\epsilon}}(x)$ and so $\Gamma^f_{\boldsymbol{\epsilon}}(x) = \{x\}$. Therefore, we conclude that f is symbolically expansive. □

Clearly, by the definitions of Definition 2.1, we can see that symbolically 1-expansive $\Rightarrow$ symbolically 2-expansive $\cdots \Rightarrow$ symbolically N-expansive $\Rightarrow$ symbolically countably expansive.

But the converses are not true. That is, we can construct an example which is symbolically N-expansive but is not N-expansive on a compact metric space (for more details, see [2]).

Now we will characterize the symbolically expansiveness for measure-theoretic point of view.

Let X be a compact metric space, and $f : X \to X$ be a homeomorphism. And let $\beta(X)$ be the Borel σ-algebra on X endowed with weak* topology, denote by

- $\mathcal{M}(X)$: the set of Borel probability measures on X,
- $\mathcal{M}^*(X)$: the set of all nonatomic measures on X.

Definition 2.3. Let X be a compact metric space, and $f : X \to X$ be a homeomorphism.

(1) f is *symbolically measure expansive* if there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that $\mu(\Gamma_{\boldsymbol{\epsilon}}(x)) = 0$ for $\forall \mu \in \mathcal{M}^*(X)$, $\forall x \in X$.
(2) f is *symbolically measure expanding* if there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that $\mu\big(\Gamma_{\boldsymbol{\epsilon}}(x) \setminus \{x\}\big) = 0$ for $\forall \mu \in \mathcal{M}(X)$, $\forall x \in X$.
(3) f is *symbolically measure N-expanding* if there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that there is a set B_x containing x satisfying the following two conditions
   · $\#B_x \leq N$,
   · $\mu\big(\Gamma_{\boldsymbol{\epsilon}}(x) \setminus B_x\big) = 0$ for $\forall \mu \in \mathcal{M}(X)$, $\forall x \in X$.
(4) f is *symbolically invariant measure N-expanding* if there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that there is a set B_x containing x satisfying the following two conditions
   · $\#B_x \leq N$, where $\#B$ is the cardinality of the set B,
   · $\mu\big(\Gamma_{\boldsymbol{\epsilon}}(x) \setminus B_x\big) = 0$ for $\forall \mu \in \mathcal{M}(X)$, $\forall x \in X$.

So, we can check that easily,
symbolically measure expansive = symbolically measure 1-expanding $\Rightarrow$ symbolically measure 2-expanding $\cdots$ $\Rightarrow$ symbolically measure N-expanding.

We are ready to prove that symbolically countably expansiveness is equivalent to the symbolically measure expansiveness as following.

Theorem 2.4. *f is symbolically countably expansive if and only if f is symbolically measure expansive.*

Proof. ($\Rightarrow$) clear.

($\Leftarrow$) Suppose that f is symbolically measure expansive. Then there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that $\mu(\Gamma_{\boldsymbol{\epsilon}}(x)) = 0$ for $\forall \mu \in \mathcal{M}^*(X)$, $\forall x \in X. \cdots\cdots (*)$

Assume that f is not symbolically countably expansive. Then given $\boldsymbol{\epsilon} \in \mathcal{F}(X)$, there is $\bar{x}(\boldsymbol{\epsilon}) = \bar{x} \in X$ such that $\Gamma_{\boldsymbol{\epsilon}}(\bar{x})$ is uncountable. By Theorem 8.1. of [6], there is $\mu \in \mathcal{M}^*(X)$ such that

$$\mu(\Gamma_{\boldsymbol{\epsilon}}(\bar{x})) > 0.$$

This contradicts to $(*)$. Therefore, we complete the proof. □

Moreover, we prove that symbolically N-expansiveness is equivalent to symbolically measure N-expanding as following.

THEOREM 2.5. *f is symbolically N-expansive if and only if f is symbolically measure N-expanding.*

Proof. ($\Rightarrow$) clear.

($\Leftarrow$) Suppose that f is symbolically measure N-expanding. Then there exists $\boldsymbol{\epsilon} \in \mathcal{F}(X)$ such that there is a set B_x containing x satisfying that

$$\text{(i) } \#B_x \leq N,$$
$$\text{(ii) } \mu\big(\Gamma_{\boldsymbol{\epsilon}}(x) \setminus B_x\big) = 0 \ \ \forall \mu \in \mathcal{M}(X), \ \forall x \in X.$$

Assume that f is not symbolically N-expansive. Given $\boldsymbol{\epsilon} \in \mathcal{F}(X)$, there exists $\bar{x} \in X$, $\#\Gamma_{\boldsymbol{\epsilon}}(\bar{x}) \geq N+1$. Choose a set $B \subset \Gamma_{\boldsymbol{\epsilon}}(\bar{x}) \setminus \{\bar{x}\}$ such that $\#B = N$.

Say $B = \{y_1, y_2, \cdots, y_N\}$. For each $i = 1, 2, \cdots, N$, let δ_{y_i} be the Dirac measure at y_i. That is, for all $A \in \beta(X)$,

$$\delta_{y_i}(A) = \begin{cases} 0, & \text{if } y_i \notin A \\ 1, & \text{if } y_i \in A. \end{cases}$$

Define $\mu = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \delta_{y_i}$, then it is a Borel probability measure. For every set $B_{\bar{x}}$ containing $\bar{x}$ with $\#B_{\bar{x}} \leq N$,

$$(\Gamma_{\boldsymbol{\epsilon}}(\bar{x}) \setminus B_{\bar{x}}) \cap B = (\Gamma_{\boldsymbol{\epsilon}}(\bar{x}) \cap B) - (B_{\bar{x}} \cap B) \neq \emptyset.$$

This set contains at least one element, put y_i. For this y_i,

$$\mu(\Gamma_{\boldsymbol{\epsilon}}(\bar{x}) \setminus B_{\bar{x}}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{y_i}(\Gamma_{\boldsymbol{\epsilon}}(\bar{x}) \setminus B_{\bar{x}})) \geq \frac{1}{N} \neq 0.$$

This contradicts to complete the proof. □

In fact, if $N = 1$, it is easy to see that f is symbolically 1-expansive if and only if it is symbolically measure 1-expanding.

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