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ON STABILITY OF EXPANSIVE INDUCED HOMEOMORPHISMS ON HYPERSPACES

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Abstract. In this paper we investigate the topological stability of induced homeomorphisms on a hyperspace. More precisely, we show that an expansive induced homeomorphism on a hyperspace is topologically stable. We also give examples and a diagram about implications to illustrate our results.

1. Introduction and preliminaries

Variant notions of expansiveness and the shadowing property for homeomorphisms are usually playing an important role in the investigation of the stability theory and spectral decomposition property in dynamical systems(see [1, 9, 10, 12]).

Walters [14] introduced the notion of topological stability for homeomorphisms of a compact metric space in which continuous perturbations are allowed, and showed that every expansive homeomorphism with the shadowing property on a compact metric space is topologically stable. Then, Lee and Morales [10] introduced the concepts of topological stability and the shadowing property for Borel measures on a compact metric space, and showed that any expansive measure with the shadowing property is topologically stable. Thereafter, Chung and Lee [3] extended the notion of topological stability from homeomorphisms to group actions on a compact metric space, and proved that if every finite generated group action is expansive and has the shadowing property, then it is topologically stable. Recently, Koo et al. [7] introduced the some pointwise notions for homeomorphisms and studied the relationship between the pointwise topological stability and shadowableness for expansive homeomorphisms.

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Furthermore, Artigue [2] defined the strong concept of expansiveness on a hyperspace and proved sufficient and necessary conditions for satisfying expansiveness on a hyperspace. Fernández and Good [4] studied the various notions of the shadowing for the induced maps on hyperspaces via a general result on shadowing properties in dense subspaces and proved that a continuous function $f : X \to X$ has shadowing if and only if the induced map $2^f : 2^X \to 2^X$ has shadowing. Recently, Koo et al. [8] studied the connection between the hyper-expansivity and the shadowing property to investigate Walters' topological stability for induced maps on a hyperspace.

For study of dynamical properties of induced maps on a hyperspace, we recall some basic notions concerning dynamical systems which are used in this paper. For the basic concepts and results concerning hyperspaces, we mainly refer to excellent monographs [5, 11].

Throughout this paper, let $f : X \to X$ be a homeomorphism on a compact metric space X equipped with a metric d and let 2^X denote the collection of all nonempty closed subsets of X.

We recall the hyperspace 2^X of X with the *Hausdorff metric* d_H defined by

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subseteq B_d(B, \varepsilon) \text{ and } B \subseteq B_d(A, \epsilon)\},\ A, B \in 2^X,$$

where $B_d(A, r)$ is the generalized open d-ball in X about $A \in 2^X$ with radius $r > 0$ given by

$$B_d(A, r) = \{x \in X \mid d(x, A) < r\}.$$

It is well known that a homeomorphism $f : X \to X$ induces the *induced homeomorphism* $2^f : 2^X \to 2^X$ defined by $2^f(A) = f(A)$ for each $A \in 2^X$(see [11, Theorem 0.52]).

We denote by $C(X)$ and $H(X)$ the set of all continuous self-maps of X and the set of all homeomorphisms of X, respectively. We define two metrics d_{C_0} on $C(X)$ and $\bar{d}_{C_0}$ on $H(X)$ by

$$d_{C_0}(f, g) = \sup_{x \in X} d(f(x), g(x)),\ f, g \in C(X),$$

and

$$\bar{d}_{C_0}(f, g) = \max\{d_{C_0}(f, g), d_{C_0}(f^{-1}, g^{-1})\},\ f, g \in H(X),$$

respectively. As $H(X) \subset C(X)$, d_{C_0} gives another metric on $H(X)$, but we know that d_{C_0} and $\bar{d}_{C_0}$ are equivalent metrics on $H(X)$.

Let $\mathbb{Z}$ be the set of all integers, and $\mathbb{Z}_+$ be the set of all nonnegative integers. We recall the definitions of expansivity and shadowing of homeomorphisms. We say that a homeomorphism $f : X \to X$ is *expansive* if there exists a constant $\delta > 0$ such that $d(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$ implies $x = y$. This means that $\Gamma^f_\delta(x) = \{x\}$ for each $x \in X$, where $\Gamma^f_\delta(x)$ is the *dynamical δ-ball* of f centered at x given by

$$\Gamma^f_\delta(x) = \{y \in X \mid d(f^n(x), f^n(y)) \leq \delta, \ \forall n \in \mathbb{Z}\}.$$

We remark that if a homeomorphism $f : X \to X$ on a compact metric space is expansive, then $h \circ f \circ h^{-1}$ is expansive, where $h : X \to X$ is a homeomorphism. Let $\delta > 0$. We say that a bi-infinite sequence $\{x_n\}_{n\in\mathbb{Z}}$ of X is a *δ-pseudo orbit* if $d(f(x_n), x_{n+1}) \leq \delta$ for all $n \in \mathbb{Z}$. Given $\epsilon > 0$, we say that $\{x_n\}_{n\in\mathbb{Z}}$ can be *ϵ-shadowed* if there is $x \in X$ such that $d(f^n(x), x_n) \leq \epsilon$ for all $n \in \mathbb{Z}$. A homeomorphism $f : X \to X$ is said to have the *shadowing property* if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit can be ϵ-shadowed.

In this paper we obtain the topological stability of induced homeomorphisms on a hyperspace. More precisely, we show that an expansive induced homeomorphism on a hyperspace is topologically stable in the class of induced homeomorphisms. We state our main result.

THEOREM 1.1. *Let $f : X \to X$ be a homeomorphism on a compact metric space X. If the induced homeomorphism $2^f : 2^X \to 2^X$ is expansive, then it is topologically stable.*

2. Proof of Theorem 1.1

In this section we introduce the notion of topological stability for the induced homeomorphism $2^f : 2^X \to 2^X$ on a hyperspace 2^X by a homeomorphism $f : X \to X$ on a compact metric space X. Then we obtain the topological stability for induced homeomorphisms on a hyperspace by using Walters' stability theorem for a homeomorphism of a compact metric space(see [14, Theorem 4]).

DEFINITION 2.1. Let $f : X \to X$ be a homeomorphism on a compact metric space X. We say that the induced homeomorphism $2^f : 2^X \to 2^X$ is *topologically stable* in $IH(2^X)$(abbrev. *topologically stable*) if given $\epsilon > 0$ there exists $\delta > 0$ such that for each $2^g \in IH(2^X)$ with $d^{C_0}_H(2^f, 2^g) < \delta$ there is a continuous map $h \in C(X)$ satisfying $2^f \circ 2^h = 2^h \circ 2^g$ and $d^{C_0}_H(2^h, \mathrm{Id}) < \epsilon$, where $\mathrm{Id} : 2^X \to 2^X$ is the identity map and $IH(2^X) = \{2^f \mid f \in H(X)\}$. Also, the metric $d^{C_0}_H$ on $H(2^X)$ is given by

$$d^{C_0}_H(2^f, 2^g) = \sup\{d_H(2^f(A), 2^g(A)) \mid A \in 2^X\}$$

for all $2^f, 2^g \in H(2^X)$.

For the proof of Theorem 1.1, we need the following result.

LEMMA 2.2. *[4, Theorem 3.4] Let X be a compact metric space and let $f : X \to X$ be a homeomorphism. Then f has the shadowing property if and only if 2^f has the shadowing property.*

For a sufficient condition for topological stability of an induced homeomorphism on a hyyperspace, we recall the notion of expansivity for the induced homeomorphism $2^f : 2^X \to 2^X$(see [2]).

DEFINITION 2.3. We say that the induced homeomorphism $2^f : 2^X \to 2^X$ is *expansive* if there is $\delta > 0$ such that $d_H((2^f)^n(A), (2^f)^n(B)) \leq \delta$ for all $n \in \mathbb{Z}$ with $A, B \in 2^X$ implies $A = B$.

We remark that if the induced homeomorphism $2^f : 2^X \to 2^X$ is expansive, then the homeomorphism $f : X \to X$ is expansive but the converse is not true as we can see in the following example.

EXAMPLE 2.4. *Let $\sum_2 = \{x = (x_i) \mid x_i \in \{0, 1\}, i \in \mathbb{Z}\}$ be the compact metric space with a metric d given by*

$$d(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}},\ x = (x_i), y = (y_i) \in \Sigma_2.$$

Then the shift map $\sigma : \sum_2 \to \sum_2$ defined by $\sigma(x_i) = x_{i+1}$ is expansive but the induced shift map $2^\sigma : 2^{\Sigma_2} \to 2^{\Sigma_2}$ is not expansive.

Proof. It is easy to see that the shift map $\sigma : \Sigma_2 \to \Sigma_2$ is expansive. For each $n \in \mathbb{N}$, let $p_n \in \Sigma_2$ be the periodic point of σ with period $n+1$ given by

$$p_n = (\cdots, 0, 1, \underbrace{0, 0, 0, \cdots, 0}_{\text{n}-\text{times}}, 1, \underbrace{0, 0, 0, \cdots, 0}_{\text{n}-\text{times}}, 1, 0, \cdots).$$

Then, it follows from [5, Proposition 8.3] that each $\{p_n\} \in 2^{\Sigma_2}$ is an isolated fixed point of 2^σ for each $n \in \mathbb{N}$. Thus the induced shift map 2^σ has infinitely many isolated fixed points. This contradicts the fact that an expansive homeomorphism on a compact metric space has a finite number of isolated fixed points(see [13]). Hence 2^σ is not expansive. □

LEMMA 2.5. *[8, Theorem 3.8] Let $f : X \to X$ be a homeomorphism on a compact metric space X. If the induced homeomorphism $2^f : 2^X \to 2^X$ is expansive, then the homeomorphism $f : X \to X$ has the shadowing property.*

Thus, we obtain the following result by Lemmas 2.2 and 2.5.

COROLLARY 2.6. *Let $f : X \to X$ be a homeomorphism on a compact metric space X. If the induced homeomorphism $2^f : 2^X \to 2^X$ is expansive, then it has the shadowing property.*

LEMMA 2.7. *[14, Theorem 4] Let $f : X \to X$ be a homeomorphism on a compact metric space. If f is expansive with the shadowing property, then it is topologically stable.*

Note that if the induced homeomorphism $2^f : 2^X \to 2^X$ is expansive, then the homeomorphism $f : X \to X$ is expansive. From Lemma 2.5 and Lemma 2.7, we can obtain the following result.

COROLLARY 2.8. *Let $f : X \to X$ be a homeomorphism on a compact metric space X. If the induced homeomorphism $2^f : 2^X \to 2^X$ is expansive, then the homeomorphism $f : X \to X$ is topologically stable.*

Now, we give a proof of our main result.

Proof of Theorem 1.1. Suppose that the induced homeomorphism $2^f : 2^X \to 2^X$ is expansive. Then the homeomorphism $f : X \to X$ has the shadowing property by Lemmas 2.5. Also, 2^f has the shadowing property by Lemma 2.2. Hence an expansive induced homeomorphism 2^f with the shadowing property is topologically stable by Lemma 2.7. □

We give an example to explain our main result.

EXAMPLE 2.9. *Let $X = \{0, 2\} \cup \{a_i = 2^{-i} \mid i \in \mathbb{Z}_+\} \cup \{a_{-i} = 2 - 2^{-i} \mid i \in \mathbb{Z}_+\}$ be a countable compact metric subspace of $\mathbb{R}$. Then the homeomorphism $f : X \to X$ defined by*

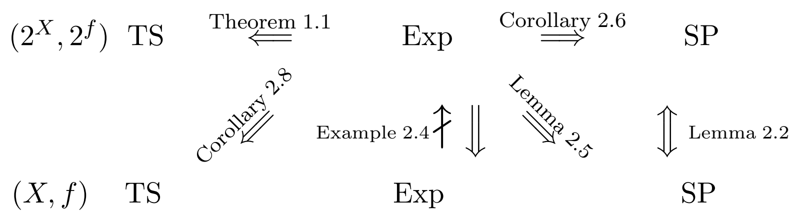
$$f(x) = \begin{cases} x, & \text{if } x = 0, 2, \\ a_{i+1}, & \text{if } x = a_i \text{ for all } i \in \mathbb{Z}, \end{cases}$$

is expansive(see [6, Remark 2.5]). It is easy to see that the nonwandering set $\Omega(f)$ of f is given by

$$\Omega(f) = \mathrm{Per}_a \cup \mathrm{Per}_r = \{0, 2\},$$

where $\mathrm{Per}_a = \{0\}$ and $\mathrm{Per}_r = \{2\}$. Thus the induced homeomorphism $2^f : 2^X \to 2^X$ is expansive(see [2, Theorem 2.2]). Hence, we see that the expansive induced homeomorphism 2^f is topologically stable by Theorem 1.1.

REMARK 2.10. From our results on hyper-dynamics obtained in this section, we give a diagram below to describe implications and relations among three notions of expansiveness(Exp), shadowing property(SP), and topological stability(TS) for homeomorphisms and their induced homeomorphisms, respectively.



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