

## A WEIGHTED EULER METHOD FOR SOLVING STIFF INITIAL VALUE PROBLEMS

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**ABSTRACT.** For an initial value problem, using a weighted average between two adjacent approximates, we propose a simple one-step method based on the Euler method. This method is useful for solving stiff initial value problem, even when the step size is not very small. Moreover, it can be seen that the proposed method with some selected weights results in improved approximation errors.

### 1. INTRODUCTION

For an initial value problem,  $y'(t) = f(t, y)$ ,  $a \leq t \leq b$ , with  $y(a) = \alpha$ , we can find lots of standard numerical methods in the literature (see [1, 2, 3, 4, 5], for example). These methods provide splendid approximations when they are applied to regular initial value problems. But, for the initial value problem having a so-called stiff differential equation where the solution contains transient terms decaying rapidly, most methods are unstable when the step size is not small enough [6, 7, 8, 9]. To overcome this problem one can use implicit methods (or backward methods). Recently, the author proposed an improved second order implicit method based on the weighted average between two approximates on the adjacent nodes [10]. Although this method leads to improved errors and is available for stiff problems, it still has the fundamental problem that the derivative of the given function  $f$  and additional iterations are required to solve the resulting nonlinear equation.

In this paper, we develop a new simple explicit method based on the standard Euler method. We use the weighted average between two approximates on adjacent nodes with a weight  $0 < \omega < 1$ . It is proved that the convergence order of the resulting method becomes around 2 for the weight  $\omega$  nearby  $\frac{1}{2}$ . In fact, the proposed method with  $\omega = \frac{1}{2}$  corresponds to the well-known midpoint method (or Runge-Kutta method of order 2). This means that the proposed method can be regarded as a generalized version of the midpoint method including the parameter  $\omega$ .

From the numerical result of the proposed method applied to some selected test examples of stiff equations, we can find the usefulness of the method. Moreover, it can be seen that the

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Received by the editors December 8 2022; Revised December 18 2022; Accepted in revised form December 19 2022; Published online December 25 2022.

2000 *Mathematics Subject Classification.* 65L04.

*Key words and phrases.* Euler method, weighted average, stiff initial value problem.

optimal value of the weight  $\omega$  for each used step size  $h$  lies to the right of the midpoint  $\omega = \frac{1}{2}$  and it becomes close to  $\frac{1}{2}$  as  $h$  decreases.

## 2. DERIVATION OF THE METHOD

For the numerical solution of the initial value problem, we take the equidistant grid points

$$t_i = a + ih, \quad i = 0, 1, 2, \dots, N,$$

with step size  $h = (b - a)/N$ . In this section we develop a new one-step explicit method for finding the approximate  $y_{i+1} \approx y(t_{i+1})$  using the previous approximate  $y_i \approx y(t_i)$ .

For each  $i$  fixed, we define weighted averages between adjacent nodes  $t_i$  and  $t_{i+1}$  and corresponding approximates  $y_i$  and  $y_{i+1}$  as

$$t_i^{[\omega]} = \omega t_i + (1 - \omega)t_{i+1}, \quad y_i^{[\omega]} = \omega y_i + (1 - \omega)y_{i+1}, \quad (2.1)$$

where  $0 < \omega < 1$ .

Employing the Euler method associated with the weighted averages  $t_i^{[\omega]}$  and  $y_i^{[\omega]}$  above, with  $t_{i+1} - t_i^{[\omega]} = \omega h$ , we have the following equation to determine  $y_{i+1}$ .

$$y_{i+1} = y_i^{[\omega]} + \omega h f\left(t_i^{[\omega]}, y_i^{[\omega]}\right). \quad (2.2)$$

If the first term  $y_i^{[\omega]}$  in the right hand side of (2.2) is replaced by  $\omega y_i + (1 - \omega)y_{i+1}$  as given in the Eq. (2.1), then it follows that

$$y_{i+1} = y_i + h f\left(t_i^{[\omega]}, y_i^{[\omega]}\right). \quad (2.3)$$

This equation becomes the implicit method proposed in the literature [10] if  $y_i^{[\omega]}$  in  $f\left(t_i^{[\omega]}, y_i^{[\omega]}\right)$  is also replaced by  $\omega y_i + (1 - \omega)y_{i+1}$ . However, in order to develop an explicit method that overcomes the drawback of the implicit method, in this work we replace  $y_i^{[\omega]}$  in  $f\left(t_i^{[\omega]}, y_i^{[\omega]}\right)$  by the formula based on the Euler method as

$$y_i^{[\omega]} = y_i + (1 - \omega)h f(t_i, y_i), \quad (2.4)$$

noting that  $t_i^{[\omega]} - t_i = (1 - \omega)h$ . Combining the formulas (2.3) and (2.4), we have

$$y_{i+1} = y_i + h f\left(t_i^{[\omega]}, y_i + (1 - \omega)h f(t_i, y_i)\right). \quad (2.5)$$

The special case of  $\omega = \frac{1}{2}$  becomes the well-known midpoint method below.

$$y_{i+1} = y_i + h f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\right).$$

The following theorem shows that the convergence order of the proposed method becomes 2 when the weight  $\omega$  goes to  $\frac{1}{2}$ . The resulting statement and its proof are similar to those provided in the literature [10] which deals with an implicit version of the main idea of this work.

**Theorem 2.1.** For the initial value problem  $y'(t) = f(t, y)$ ,  $a \leq t \leq b$ , with  $y(a) = \alpha$ , suppose  $f(t, y)$  satisfies a Lipschitz condition in the variable  $y$  with a Lipschitz constant  $L$  on a set  $D = \{(t, y) \mid a \leq t \leq b, -\infty < y < \infty\}$ . Furthermore, let  $f$  be twice continuously differentiable in  $D$  and let the exact solution  $y(t)$  satisfy

$$|y''(t)| \leq M, \quad (t \in [a, b])$$

for a constant  $M > 0$ . Then, for each  $i = 0, 1, 2, \dots, N - 1$  the approximate solution  $y_{i+1}$  to  $y(t_{i+1})$  obtained by the formula (2.5) with  $0 < \omega < 1$  satisfies

$$|y(t_{i+1}) - y_{i+1}| \leq \left( \frac{M}{L} \left| \omega - \frac{1}{2} \right| h + \frac{C}{L} h^2 \right) \left\{ e^{(i+1)hL} - 1 \right\},$$

for step size  $h$  and a constant  $C > 0$ .

*Proof.* The exact solutions  $y(t_i)$  and  $y(t_{i+1})$  at the nodes  $t_i$  and  $t_{i+1}$ , respectively, satisfy

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + C_1 h^3 \tag{2.6}$$

for some constant  $C_1$ . Therein,  $f'$  is a total derivative with respect to  $t$  such as

$$f'(t, y(t)) = f_t(t, y(t)) + f_y(t, y(t))f(t, y(t)).$$

Then, from (2.5) and (2.6),

$$y(t_{i+1}) - y_{i+1} = y(t_i) - y_i + h \left\{ f(t_i, y(t_i)) - f(t_i^{[\omega]}, y_i^{[\omega]}) \right\} + \frac{h^2}{2} f'(t_i, y(t_i)) + C_1 h^3. \tag{2.7}$$

But,

$$t_i^{[\omega]} = t_i + (1 - \omega)h$$

and from (2.4)

$$y_i^{[\omega]} = y_i + (1 - \omega)hf(t_i, y_i).$$

Taylor's theorem implies

$$\begin{aligned} f(t_i^{[\omega]}, y_i^{[\omega]}) &= f(t_i, y_i) + (1 - \omega)hf_t(t_i, y_i) + (1 - \omega)hf(t_i, y_i) f_y(t_i, y_i) + C_2 h^2 \\ &= f(t_i, y_i) + (1 - \omega)hf'(t_i, y_i) + C_2 h^2 \end{aligned}$$

for some constant  $C_2$ . The last equality results from  $f_t(t_i, y_i) + f(t_i, y_i) f_y(t_i, y_i) = f'(t_i, y_i)$ .

Then we have

$$f(t_i, y(t_i)) - f(t_i^{[\omega]}, y_i^{[\omega]}) = \{f(t_i, y(t_i)) - f(t_i, y_i)\} - (1 - \omega)hf'(t_i, y_i) - C_2 h^2.$$

Therefore, from the Eq. (2.7)

$$\begin{aligned} |y(t_{i+1}) - y_{i+1}| &\leq |y(t_i) - y_i| + h|f(t_i, y(t_i)) - f(t_i, y_i)| \\ &\quad + \frac{h^2}{2} |f'(t_i, y(t_i)) - 2(1 - \omega)f'(t_i, y_i)| + |C_1 - C_2|h^3. \end{aligned}$$

By the assumptions,

$$|f(t_i, y(t_i)) - f(t_i, y_i)| \leq L |y(t_i) - y_i|$$

and

$$f'(t_i, y_i) = f'(t_i, y(t_i)) + C' (y_i - y(t_i)).$$

for some constant  $C'$ .

Thus we have

$$\begin{aligned} |y(t_{i+1}) - y_{i+1}| &\leq (1 + hL) |y(t_i) - y_i| \\ &\quad + \frac{h^2}{2} \{ |(2\omega - 1)M + 2(1 - \omega)C'| |y_i - y(t_i)| \} + |C_1 - C_2|h^3 \\ &\leq (1 + hL) |y(t_i) - y_i| + M \left| \omega - \frac{1}{2} \right| h^2 + Ch^3 \end{aligned}$$

for some constant  $C > 0$ . Referring to Lemma 5.8 in [2], if we set  $a_i = |y(t_i) - y_i|$  with  $s = hL$  and  $t = M \left| \omega - \frac{1}{2} \right| h^2 + Ch^3$ , then

$$\frac{t}{s} = \frac{M}{L} \left| \omega - \frac{1}{2} \right| h + \frac{C}{L} h^2$$

and we have

$$|y(t_{i+1}) - y_{i+1}| \leq \frac{t}{s} \left\{ e^{(i+1)s} - 1 \right\}.$$

□

The stability of the numerical method for solving stiff problems is represented by the region of absolute stability [11, 12] which is defined by

$$\{z \in \mathbb{C} \mid |\phi(z)| < 1\}$$

for the so-called stability function  $\phi(z)$ , with  $z = \lambda h$ , of the method. Therein,  $h$  is a step size and  $\lambda \in \mathbb{C}$ ,  $\text{Re}(\lambda) < 0$ , indicates the stiffness of the test equation  $y'(t) = f(t, y) = \lambda y(t)$  having the solution  $y(t) = e^{\lambda t}$ . In fact, it can be seen that the region of absolute stability of the proposed method (2.5) is

$$\{z \in \mathbb{C} \mid |1 + z + (1 - \omega)z^2| < 1\}.$$

Fig. 1 demonstrates the stability regions of the proposed method with respect to some selected weights  $\omega \geq \frac{1}{2}$ . The case of  $\omega = \frac{1}{2}$  corresponds to the midpoint method. We can see that the area of the stability region is increasing as the weight  $w$  goes to 1. Therefore, we may expect that the proposed method with  $\omega > \frac{1}{2}$  will assure stable approximations for wider range of the step-size  $h$ , compared with the existing midpoint method.

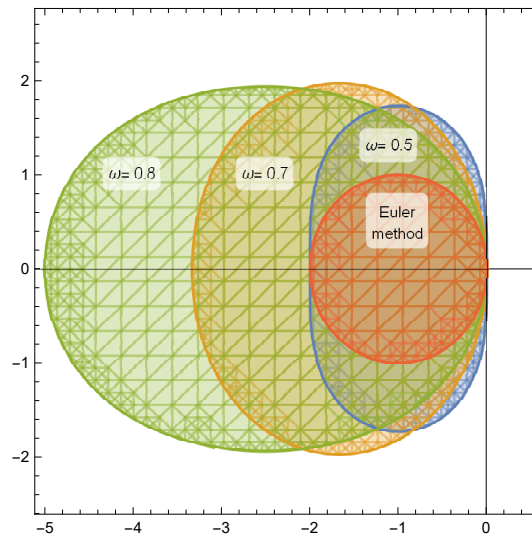


FIGURE 1. The stability regions of the method (2.5) for the weights  $\omega = 0.5$ (midpoint method),  $\omega = 0.7$  and  $\omega = 0.8$ , compared with the Euler method.

### 3. NUMERICAL EXAMPLE

To explore the availability of the proposed method, we select a typical test problem given in the literature [2] as

$$\begin{cases} y'(t) = 5e^{5t}(t - y)^2 + 1, & 0 \leq t \leq 5 \\ y(0) = -1 \end{cases} \tag{3.1}$$

whose exact solution is  $y = t - e^{-5t}$ .

Numerical errors of the approximates  $\{y_{i+1}\}_{i=0}^{N-1}$  obtained by the proposed method (2.5), for various step-sizes, are illustrated over the range of the weights  $0.35 < \omega < 0.75$  in Fig.2. The error is the  $l_2$ -norm error defined by  $E_{2,h} := \sqrt{\sum_{i=0}^{N-1} |y_{i+1} - y(t_{i+1})|^2}$  associated with the step-size  $h$ . It can be seen that the optimal value of the weight  $\omega$  for each  $h$  lies on the right hand side of the midpoint,  $\frac{1}{2}$  and it approaches to  $\frac{1}{2}$  as  $h$  goes to 0. Moreover, Table 1 includes the  $l_2$ -norm errors  $E_{2,h}$  for the weights  $\omega = \frac{1}{2}, \omega_h, \omega_h^*$ , where  $\omega_h := \frac{1}{2} + h$  and  $\omega_h^*$  denotes the optimal weight, found between 0 and 1 at 0.01 interval, with which the proposed method results in the best error. In practice, for each  $h = 0.1, 0.05, 0.025, 0.00625$  the optimal weight  $\omega_h^*$  is indicated by the dotted vertical line in Fig. 2. The errors of the existing midpoint method(or the case of  $\omega = \frac{1}{2}$  in the proposed method) blow up for  $h \geq 0.1$  whereas the proposed method with  $\omega = \omega_h$ , or  $\omega_h^*$  shows stable error tendency.

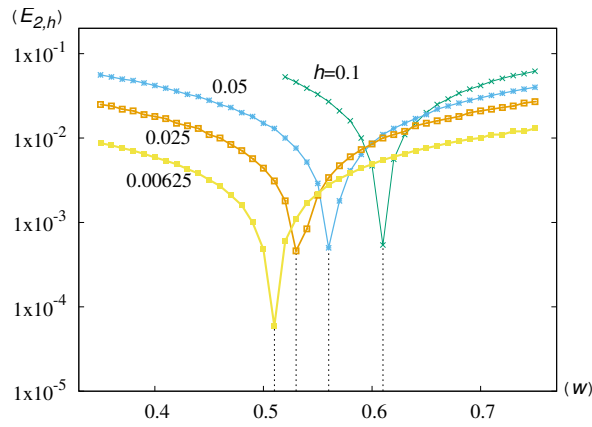


FIGURE 2. The  $l_2$ -norm errors  $E_{2,h}$  of the proposed method (2.5), with various step-sizes  $h$ , for the test example (3.1) with respect to the weights over the range  $0.35 < \omega < 0.75$ .

Figure 3 shows difference errors  $|y_{i+1} - y(t_{i+1})|$  of the approximates  $\{y_{i+1}\}_{i=0}^{N-1}$  obtained by the proposed method (2.5) with  $\omega = \omega_h^*$ . The errors are compared with those of the midpoint method indicated by the thin lines. It can be seen that the proposed method with  $\omega = \omega_h^*$  gives suitable errors even when the step-size is not very small (in (a)) and it provides about  $\frac{1}{10}$  times the errors of the midpoint method, over the whole range, for a small step-size (in (b)).

TABLE 1. Optimal weights  $\omega_h^*$  and the  $l_2$ -norm errors of the proposed method (2.5) with the selected weights  $\omega = \frac{1}{2}$ ,  $\omega_h$ ,  $\omega_h^*$  for step-size  $0 < h \leq \frac{1}{3}$  ( $\omega_h = \frac{1}{2} + h$ ).

step-size $h (N)$	$\omega_h^*$	$l_2$ -norm errors with the weights		
		$\omega = \frac{1}{2}$	$\omega = \omega_h$	$\omega = \omega_h^*$
0.33333(15)	0.74	–	–	$1.8 \times 10^{-2}$
0.25000(20)	0.71	–	$9.8 \times 10^{-2}$	$2.0 \times 10^{-2}$
0.16670(30)	0.66	–	$1.1 \times 10^{-2}$	$3.8 \times 10^{-3}$
0.12500(40)	0.63	–	$2.5 \times 10^{-3}$	$1.1 \times 10^{-3}$
0.10000(50)	0.61	–	$4.7 \times 10^{-3}$	$5.4 \times 10^{-4}$
0.05000(100)	0.56	$1.5 \times 10^{-2}$	$2.9 \times 10^{-3}$	$5.0 \times 10^{-4}$
0.02500(200)	0.53	$4.4 \times 10^{-3}$	$1.1 \times 10^{-3}$	$4.6 \times 10^{-4}$
0.00625(800)	0.51	$4.8 \times 10^{-4}$	$1.5 \times 10^{-4}$	$5.9 \times 10^{-5}$

(The symbol "–" indicates that the  $l_2$ -norm error blows up.)

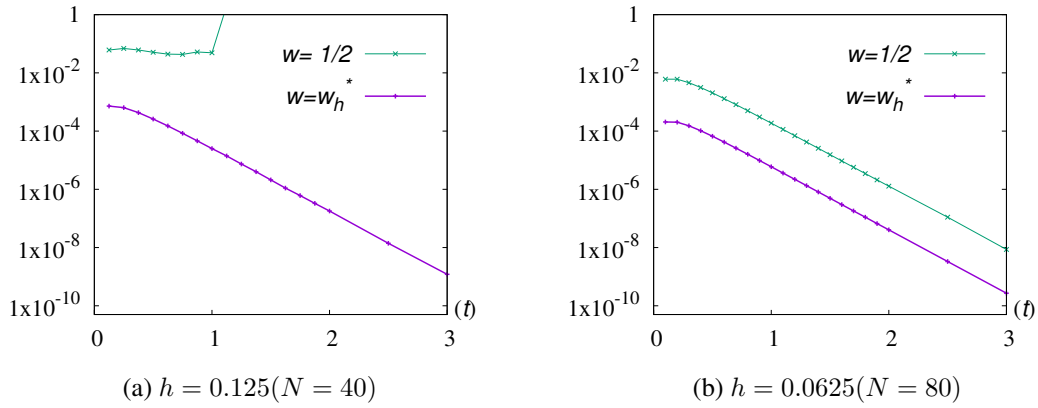


FIGURE 3. Difference errors,  $|y_{i+1} - y(t_{i+1})|$  of the proposed method with  $\omega = \omega_h^*$  compared with the midpoint method ( $\omega = \frac{1}{2}$ ) for the test example (3.1).

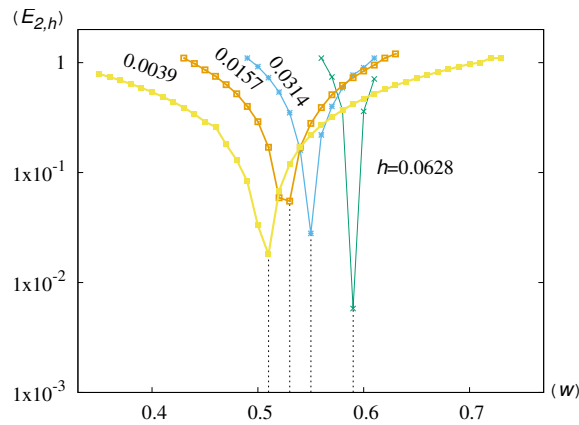


FIGURE 4. The  $l_2$ -norm errors  $E_{2,h}$  of the proposed method (2.5), with various step-sizes  $h$ , for the test example (3.2) with respect to the weights over the range  $0.35 < \omega < 0.75$ .

Additionally, we consider another example of the stiff equation introduced in [13] as follows.

$$\begin{cases} y'(t) = \lambda(y - \cos t) - \sin t, & 0 \leq t \leq \pi \\ y(0) = \eta \end{cases} \quad (3.2)$$

whose exact solution is  $y(t) = (\eta - 1)e^{\lambda t} + \cos t$ . ( $\eta = 50, \lambda = -10$ )

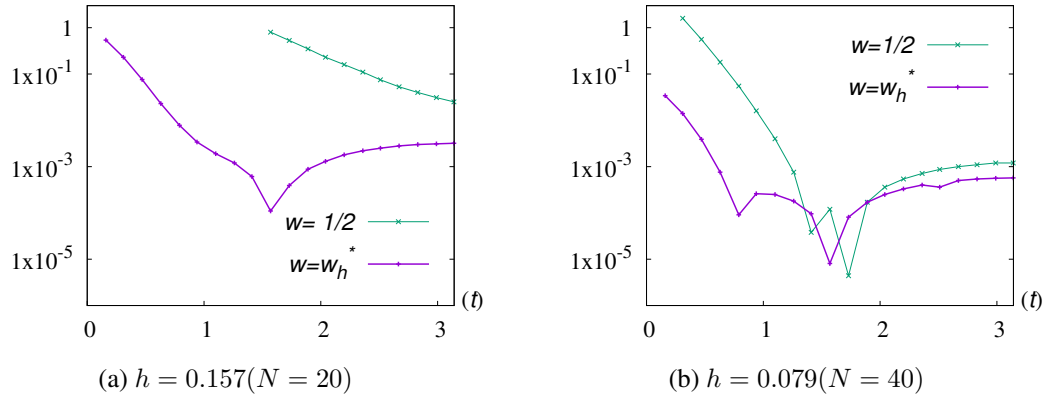


FIGURE 5. Difference errors,  $|y_{i+1} - y(t_{i+1})|$  of the proposed method with  $\omega = \omega_h^*$  compared with the midpoint method ( $\omega = \frac{1}{2}$ ) for the test example (3.2).

Figure 4 shows numerical  $l_2$ -norm errors of the proposed method (2.5) for various step-sizes. Like the case of the previous example in (3.1), given in Fig. 2, we can see that the optimal value of the weight  $\omega$  for each  $h$  is located on the right hand side of the midpoint and it approaches to  $\frac{1}{2}$  as  $h$  decreases to 0.

Difference errors of the proposed method (2.5) with  $\omega = \omega_h^*$  are illustrated in Fig. 5, which are compared with those of the midpoint method. It can be seen that the proposed method with  $\omega = \omega_h^*$  is useful even when the step-size is not very small and it provides better errors than the midpoint method over the whole range.

#### 4. CONCLUSIONS

In this study, we developed a one-step method with a weight  $0 < \omega < 1$  based on the Euler method to solve the initial value problem. The description of the proposed method is summarized as follows.

- i. When the proposed method is applied to a stiff equation, it gives stable approximation errors even for a large step-size whereas existing standard explicit methods such as the midpoint method blow up.
- ii. Approximation error can be significantly improved by choosing appropriate values for the weight  $\omega$  around  $\frac{1}{2}$ . In fact, for a given step-size  $h$  we can find the optimal weight  $\omega_h^* > \frac{1}{2}$ , and it can be seen that the optimal weight approaches  $\frac{1}{2}$  as  $h$  decreases. Therefore, we can conclude that the superiority of the proposed method will be emphasized even when the step size  $h$  is not very small.



## ACKNOWLEDGMENTS

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) [No. 2021R1F1A1047343].

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