Korean J. Math. **30** (2022), No. 4, pp. 615–628
http://dx.doi.org/10.11568/kjm.2022.30.4.615

SOME EXTENSIONS OF ENESTRÖM-KAKEYA THEOREM FOR QUATERNIONIC POLYNOMIALS

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Abstract. In this paper, we will prove some extensions of the Eneström-Kakeya theorem to quaternionic polynomials which were already valid for the classical Eneström-Kakeya theorem to complex polynomials. Our kind of extensions have considerably improved the bounds by relaxing and weakening the hypothesis in some cases.

1. Introduction

Although the Fundamental Theorem of Algebra gives the guarantee of existence of as many zeros of a complex polynomial as its degree in the complex plane. But the impossibility of algebraically solving in general a polynomial equation of degree greater than four is an important problem in the history of mathematics. This motivated the study of identifying a suitable region in the complex plane containing some or all the zeros of a given polynomial. The first result concerning the location of zeros of a polynomial was probably due to by Gauss [5]. However, regarding the condition on the coefficients of a polynomial was initially put by Eneström and Kakeya independently. The Eneström-Kakeya theorem for a complex polynomial with real coefficients also gives the location of zeroes of a polynomial in a particular region and is defined as follows:

Theorem 1.1. *If $p(z) = \sum_{s=0}^{n} a_s z^s$ is a polynomial of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq ... \geq a_0 > 0$, then all the zeros of $p(z)$ lie in $|z| \leq 1$.*

In the literature [2–4, 8–13], several generalisations of Theorem 1.1 have been obtained. In 1967, Joyal et al. [10] extended Theorem 1.1 to those complex polynomials whose coefficients are monotonic and relaxing the non-negativity condition by proving the following result:

Theorem 1.2. *If $p(z) = \sum_{s=0}^{n} a_s z^s$ is a polynomial of degree n with real coefficients satisfying $a_n \geq a_{n-1} \geq ... \geq a_0$, then all the zeros of $p(z)$ lie in $|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$.*

Received August 10, 2022. Revised October 1, 2022. Accepted October 7, 2022.
2010 Mathematics Subject Classification: 30D10, 30C10, 30C15.
Key words and phrases: Quaternionic Polynomial, Eneström-Kakeya Theorem.

2. Preliminaries

Quaternions were invented and developed by Irish mathematician William Rowan Hamilton in 1843 and are essentially a generalisation of Complex numbers to four dimensions. The set of quaternions is denoted by $\mathbb{H}$ in honour of Sir Hamilton and they form a non-commutative division ring as multiplication of quaternions is not commutative in general. Quaternions are generally represented in the form: $q = \alpha + i\beta + j\gamma + k\delta \in \mathbb{H}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, and i, j and k are the unit vectors along the three spatial axes and satisfying $i^2 = j^2 = k^2 = ijk = -1$. The part $i\beta + j\gamma + k\delta$ of q is called the vector part (or sometimes imaginary part) and α is the scalar part (or sometimes real part) of q. Since the real numbers is isomorphic to a commutative sub-division ring of the quaternions. The interest with the quaternions lies, in part, with the fact that they are a division ring. Ferdinand Georg Frobenius proved in 1878 that only three such real associative division algebras exist: real numbers, complex numbers and quaternions. Moreover the set of quaternions forms a four dimensional vector space over $\mathbb{R}$ with $\{1, i, j, k\}$ as a basis. The conjugate of a quaternion $q = \alpha + i\beta + j\gamma + k\delta$ is denoted by q^* and is defined as $q^* = \alpha - i\beta - j\gamma - k\delta$ and hence the norm (or length) of a quaternion $q = \alpha + i\beta + j\gamma + k\delta$ is given by

$$||q|| = \sqrt{qq^*} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}.$$

A quaternion with a unit norm is called a normalised quaternion.
Let us define the angle θ between two quaternions $q_1 = \alpha_1 + i\beta_1 + j\gamma_1 + k\delta_1$ and $q_2 = \alpha_2 + i\beta_2 + j\gamma_2 + k\delta_2$ as

$$\angle(q_1, q_2) = \cos^{-1}\left(\frac{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 + \delta_1\delta_2}{||q_1||||q_2||}\right)$$

and the class of all n^{th} degree quaternionic polynomials by

$$P_n = \left\{p(q); p(q) = \sum_{s=0}^{n} q^s a_s\right\}.$$

In 2020, Carney et al. [2] proved the following extension of Theorem 1.1 to the quaternionic polynomial p(q).

Theorem 2.1. *All the zeros of the polynomial $p \in \mathbb{P}_n$ of degree n with real coefficients, such that $a_n \geq a_{n-1} \geq ... \geq a_0 \geq 0$ lie in $|q| \leq 1$.*

In the same paper, they also proved the following refinement of Theorem 2.1 by removing the positivity condition on the coefficients of $p(q)$, which in turn yields in the generalization of Theorem 1.2 for $p \in \mathbb{P}_n$ with quaternionic coefficients.

Theorem 2.2. *All the zeros of the polynomial $p \in \mathbb{P}_n$ of degree n with quaternionic coefficients $a_s = \alpha_s + i\beta_s + j\gamma_s + k\delta_s \in \mathbb{H}$, $0 \leq s \leq n$, such that $\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_0$, $\beta_n \geq \beta_{n-1} \geq ... \geq \beta_0$, $\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_0$, $\delta_n \geq \delta_{n-1} \geq ... \geq \delta_0$ lie in:*

$$|q| \leq \frac{1}{|a_n|}[(|\alpha_0| - \alpha_0 + \alpha_n) + (|\beta_0| - \beta_0 + \beta_n) + (|\gamma_0| - \gamma_0 + \gamma_n) + (|\delta_0| - \delta_0 + \delta_n)].$$

They also proved the following two results in the same paper:

Theorem 2.3. *Let* $p(q) = \sum_{s=0}^{n} q^s a_s$ *be a quaternionic polynomial of degree* n *satisfying* $\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_0 \geq 0, \alpha_n \neq 0$*, then all the zeros of p lie in:*

$$|q| \leq 1 + \frac{2}{\alpha_n} \sum_{s=0}^{n} \Big(|\beta_s| + |\gamma_s| + |\delta_s| \Big).$$

Theorem 2.4. *Let* $p(q) = \sum_{s=0}^{n} q^s a_s$ *be a polynomial of degree* n *with quaternionic coefficients and quaternionic variable. Let* b *be a nonzero quaternion and suppose* $\angle(a_s, b) \leq \theta \leq \frac{\pi}{2}$ *for some* θ *and* $s = 0, 1, 2, ..., n$*. Assume* $|a_n| \geq |a_{n-1}| \geq ... \geq |a_0|$*. Then all the zeros of p lie in:*

$$|q| \leq \cos\theta + \sin\theta + \frac{2\sin\theta}{|a_n|} \sum_{s=0}^{n-1} |a_s|.$$

Recently, Dinesh Tripathi [3] relaxed the condition on the coefficients of Theorem 2.2 and proved the following result.

Theorem 2.5. *If* $p(q) = \sum_{s=0}^{n} q^s a_s$ *is a polynomial of degree* n *with quaternionic coefficients* $a_s \in \mathbb{H}, 0 \leq s \leq n$ *such that:*
$\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_l$, $\beta_n \geq \beta_{n-1} \geq ... \geq \beta_l$, $\gamma_n \geq \gamma_{n-1} \geq ... \geq \gamma_l$, $\delta_n \geq \delta_{n-1} \geq ... \geq \delta_l, 0 \leq l \leq n$*, then all the zeros of* $p(q)$ *lie in*

$$|q| \leq \frac{1}{|a_n|}[|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + (\alpha_n - \alpha_l) + (\beta_n - \beta_l) + (\gamma_n - \gamma_l) + (\delta_n - \delta_l) + M_l]$$

where

$$M_l = \sum_{s=1}^{l} \big[|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}| \big].$$

We too relaxed the conditions on the coefficients of the quaternionic polynomial $p \in \mathbb{P}_n$ in some other ways and obtained the following desired results that are valid in a bigger class of quaternionic polynomials.

3. Main Results

In this direction, we first prove the following result which gives the generalisation of Theorem 2.5 and hence the generalisation of Theorem 2.2.

Theorem 3.1. *If* $p(q) = \sum_{s=0}^{n} q^s a_s$ *is a polynomial of degree* n *with quaternionic coefficients and quaternionic variable where* $a_s = \alpha_s + i\beta_s + j\gamma_s + k\delta_s$ *for s= 0, 1,..., n such that for some* $k \geq 1$ *and for some* $\lambda > 0$*, we have:*
$k\alpha_n \geq \alpha_{n-1} \geq ... \geq \lambda\alpha_l$, $k\beta_n \geq \beta_{n-1} \geq ... \geq \lambda\beta_l$, $k\gamma_n \geq \gamma_{n-1} \geq ... \geq \lambda\gamma_l$, $k\delta_n \geq$

$\delta_{n-1} \geq ... \geq \lambda\delta_l$, $0 \leq l \leq n-1$, *then all the zeros of p(q) lie in:*

$$\begin{aligned} |q| &\leq \frac{1}{|a_n|}\{[|\alpha_0| + (2k-1)\alpha_n + (1-\lambda)|\alpha_l| - \lambda\alpha_l] + [|\beta_0| + (2k-1)\beta_n \\ &\quad +(1-\lambda)|\beta_l| - \lambda\beta_l] + [|\gamma_0| + (2k-1)\gamma_n + (1-\lambda)|\gamma_l| - \lambda\gamma_l] + \\ &\quad [|\delta_0| + (2k-1)\delta_n + (1-\lambda)|\delta_l| - \lambda\delta_l] + M_l\} \quad \text{if} \quad \lambda < 1 \end{aligned}$$

or

$$\begin{aligned} |q| &\leq \frac{1}{|a_n|}\{[|\alpha_0| + (2k-1)\alpha_n + (\lambda-1)|\alpha_l| - \lambda\alpha_l] + [|\beta_0| + (2k-1)\beta_n \\ &\quad +(\lambda-1)|\beta_l| - \lambda\beta_l] + [|\gamma_0| + (2k-1)\gamma_n + (\lambda-1)|\gamma_l| - \lambda\gamma_l] + \\ &\quad [|\delta_0| + (2k-1)\delta_n + (\lambda-1)|\delta_l| - \lambda\delta_l] + M_l\} \quad \text{if} \quad \lambda \geq 1 \end{aligned}$$

where

$$M_l = \sum_{s=1}^{l} \Big(|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}| \Big).$$

Applying Theorem 3.1 for the polynomial $p(q)$ having real coefficients, i.e., $\beta = \gamma = \delta = 0$, the following result is a consequence.

Corollary 1. *All the zeros of the polynomial $p \in \mathbb{P}_n$ with real coefficients a_s, $0 \leq s \leq n$, such that $ka_n \geq a_{n-1} \geq ... \geq \lambda a_l$, $0 \leq l \leq n-1$, $k \geq 1$ and $\lambda > 0$ lie in:*

$$|q| \leq \tfrac{1}{|a_n|}[|a_0| + (2k-1)a_n + (1-\lambda)|a_l| - \lambda a_l] + \textstyle\sum_{m=1}^{l} |a_m - a_{m-1}|] \quad \text{if} \quad \lambda < 1$$

or

$$|q| \leq \tfrac{1}{|a_n|}[|a_0| + (2k-1)a_n + (\lambda-1)|a_l| - \lambda a_l] + \textstyle\sum_{m=1}^{l} |a_m - a_{m-1}|] \quad \text{if} \quad \lambda \geq 1.$$

If we assume $l = 0$, then the following result obtains from corollary 1.

Corollary 2. *All the zeros of the polynomial $p \in \mathbb{P}_n$ with real coefficients a_s, $0 \leq s \leq n$, such that $ka_n \geq a_{n-1} \geq ... \geq \lambda a_0$, $k \geq 1$ and $\lambda > 0$ lie in:*

$$|q| \leq \frac{1}{|a_n|}\Bigg[(2-\lambda)|a_0| + (2k-1)a_n + |a_0| - \lambda a_0\Bigg] \quad \text{if} \quad \lambda < 1$$

or

$$|q| \leq \frac{1}{|a_n|}\Bigg[\lambda(|a_0| - a_0) + (2k-1)a_n\Bigg] \quad \text{if} \quad \lambda \geq 1.$$

Remark 1. Theorem 2.2 is a special case of Theorem 3.1 by taking $k = 1$, $l = 0$ and $\lambda = 1$.

Remark 2. Theorem 2.5 is also special case of Theorem 3.1 by taking $k = 1$ and $\lambda = 1$.

Theorem 3.2. *If $p(q) = \sum_{s=0}^{n} q^s a_s$ is a quaternionic polynomial of degree n satisfying $\alpha_n + \lambda \geq \alpha_{n-1} \geq ... \geq \alpha_l$, $\alpha_n \neq 0$ with $\lambda \geq 0$ and $0 \leq l \leq n$, then all the zeros of p lie in:*

$$|q| \leq \frac{1}{|\alpha_n|}\left\{\alpha_n + 2\lambda - \alpha_l + |\alpha_0| + N_l + 2\sum_{s=0}^{n}[|\beta_s| + |\gamma_s| + |\delta_s|]\right\}$$

where

$$N_l = \sum_{s=1}^{l} |\alpha_s - \alpha_{s-1}|.$$

If we put $l = 0$, we have the following result.

Corollary 3. *If $p(q) = \sum_{s=0}^{n} q^s a_s$ is a quaternionic polynomial of degree n satisfying $\alpha_n + \lambda \geq \alpha_{n-1} \geq .. \geq \alpha_0, \alpha_n \neq 0$ with $\lambda \geq 0$, then all the zeros of p lie in:*

$$|q| \leq \frac{1}{|\alpha_n|}\left(\alpha_n + 2\lambda - \alpha_0 + |\alpha_0| + 2\sum_{s=0}^{n}\left[|\beta_s| + |\gamma_s| + |\delta_s|\right]\right).$$

Also by taking $\lambda = (k-1)\alpha_n$, $\alpha_n \neq 0$ and $k \geq 1$ in Theorem 3.2, we have the following corollary.

Corollary 4. *If $p(q) = \sum_{s=0}^{n} q^s a_s$ is a quaternionic polynomial of degree n satisfying $k\alpha_n \geq \alpha_{n-1} \geq .. \geq \alpha_0$, $\alpha_n \neq 0$ with $k \geq 1$ then all the zeros of p lie in:*

$$|q| \leq \frac{1}{\alpha_n}\left((2k-1)\alpha_n - \alpha_0 + |\alpha_0| + N_l + 2\sum_{s=0}^{n}\left[|\beta_s| + |\gamma_s| + |\delta_s|\right]\right)$$

where N_l is defined already in Theorem 3.2.

Note 1. Though not mentioned in the above statement that unless $k = 1$, corollary 4 makes sense only when both α_n and α_{n-1} are positive because otherwise it might not be possible to find $k > 1$ that would satisfy the hypothesis of this Corollary.

Remark 3. Theorem 2.3 is a special case of corollary 3 by taking $\lambda = 0$ and $\alpha_0 \geq 0$.

Theorem 3.3. *Let $p(q) = \sum_{s=0}^{n} q^s a_s$ be a quaternionic polynomial of degree n. Let b be a nonzero quaternion and suppose $\angle(a_s, b) \leq \theta \leq \frac{\pi}{2}$ for some θ and for $s = l, l+1, ..., n$. If $k|a_n| \geq |a_{n-1}| \geq ... \geq |a_l|$, $0 \leq l \leq n$, and $k \geq 1$, then all the zeros of p lie in:*

$$\begin{aligned} |q| \leq & \frac{1}{|a_n|}\{(k-1)|a_n| + |a_0| + k|a_n|(\cos\theta + \sin\theta) - |a_l|(\sin\theta + \cos\theta) + \\ & 2\sin\theta\sum_{s=l}^{n-1}|a_s| + \sum_{s=1}^{l}|a_s - a_{s-1}|\}. \end{aligned}$$

If we put $l = 0$ in Theorem 3.3, we have the following corollary.

Corollary 5. *Let $p(q) = \sum_{s=0}^{n} q^s a_s$ be a quaternionic polynomial of degree n. Let b be a nonzero quaternion and suppose $\angle(a_s, b) \leq \theta \leq \frac{\pi}{2}$ for some θ and for $s = 0, 1, ..., n$. If $k|a_n| \geq |a_{n-1}| \geq ... \geq |a_0|$ with $k \geq 1$, then all the zeros of p lie in:*

$$\begin{aligned} |q| \leq & \frac{1}{|a_n|}\{(k-1)|a_n| + k|a_n|(\cos\theta + \sin\theta) + |a_0|(1 - \sin\theta - \cos\theta) + \\ & 2\sin\theta\sum_{s=0}^{n-1}|a_s|\}. \end{aligned}$$

which can be written in more modified form as:

$$|q| \leq \frac{1}{|a_n|}\left((k-1)|a_n| + k|a_n|(\cos\theta + \sin\theta) + 2\sin\theta\sum_{s=0}^{n-1}|a_s|\right)$$

$$\left(\text{using} \quad \cos\theta + \sin\theta \geq 1 \quad \text{when} \quad \theta \in [0, \tfrac{\pi}{2}]\right)$$

Remark 4. Theorem 2.4 is a special case of Corollary 5 for $k = 1$.

4. Lemmas

We use the following lemmas in the proof of our results.

Lemma 1. [2] *Let $f(q) = \sum_{s=0}^{\infty} q^s a_s$ and $g(q) = \sum_{s=0}^{\infty} q^s b_s$ be two given quaternionic power series with radii of convergence greater than R. The regular product of $f(q)$ and $g(q)$ is defined as $(f * g)(q) = \sum_{s=0}^{\infty} q^s c_s$, where $c_s = \sum_{l=0}^{s} a_l b_{s-l}$. Let $|q_0| < R$, then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g(f(q_0)^{-1} q_0 f(q_0)) = 0$.*

Lemma 2. [2] *Let $q_1, q_2 \in \mathbb{H}$ where $q_1 = \alpha_1 + i\beta_1 + j\gamma_1 + k\delta_1$ and $q_2 = \alpha_2 + i\beta_2 + j\gamma_2 + k\delta_2$, such that $\angle(q_1, q_2) = 2\theta' \leq 2\theta$, and $|q_1| \leq |q_2|$. Then*

$$|q_2 - q_1| \leq \big(|q_2| - |q_1|\big)\cos\theta + \big(|q_2| + |q_1|\big)\sin\theta.$$

5. Proofs of Theorems

Proof. $\big($*of Theorem* 3.1$\big)$ Consider the polynomial $f(q) = \sum_{s=1}^{n} q^s(a_s - a_{s-1}) + a_0$. Let $p(q) * (1-q) = f(q) - q^{n+1}a_n$, then by lemma 1, $p(q) * (1-q) = 0$ if and only if either $p(q) = 0$ or $p(q) \neq 0$ implies $p(q)^{-1}qp(q) - 1 = 0$, that is, $p(q)^{-1}qp(q) = 1$. If $p(q) \neq 1$, then $q = 1$. Therefore, the only zeros of $p(q) * (1-q)$ are $q = 1$ and the zeros of $p(q)$. Therefore for $|q| = 1$, we get:

$$\begin{aligned}
|f(q)| &\leq |a_0| + \sum_{s=1}^{n}|a_s - a_{s-1}| \\
&= |\alpha_0 + i\beta_0 + j\gamma_0 + k\delta_0| + \sum_{s=1}^{n}|(\alpha_s - \alpha_{s-1}) + i(\beta_s - \beta_{s-1}) + \\
&\quad j(\gamma_s - \gamma_{s-1}) + k(\delta_s - \delta_{s-1})| \\
&\leq |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{s=1}^{n}[|\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + \\
&\quad |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|]
\end{aligned}$$

$$\begin{aligned}
&= |\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+|\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+...+\\
&\quad |\alpha_{l+1}-\alpha_l|+|\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|+...+|\beta_{l+1}-\beta_l|+\\
&\quad |\gamma_n-\gamma_{n-1}|+|\gamma_{n-1}-\gamma_{n-2}|+...+|\gamma_{l+1}-\gamma_l|+|\delta_n-\delta_{n-1}|+\\
&\quad |\delta_{n-1}-\delta_{n-2}|+...+|\delta_{l+1}-\delta_l|+\sum_{s=1}^{l}\big(|\alpha_s-\alpha_{s-1}|+|\beta_s-\beta_{s-1}|+\\
&\quad |\gamma_s-\gamma_{s-1}|+|\delta_s-\delta_{s-1}|\big)\\
&= \big(|\alpha_0|+|k\alpha_n-\alpha_{n-1}+\alpha_n-k\alpha_n|+|\alpha_{n-1}-\alpha_{n-2}|+...+|\lambda\alpha_l-\alpha_l+\alpha_{l+1}-\lambda\alpha_l|\big)\\
&\quad +\big(|\beta_0|+|k\beta_n-\beta_{n-1}+\beta_n-k\beta_n|+|\beta_{n-1}-\beta_{n-2}|+...+|\lambda\beta_l-\beta_l+\beta_{l+1}-\lambda\beta_l|\big)\\
&\quad +\big(|\gamma_0|+|k\gamma_n-\gamma_{n-1}+\gamma_n-k\gamma_n|+|\gamma_{n-1}-\gamma_{n-2}|+...+|\lambda\gamma_l-\gamma_l+\gamma_{l+1}-\lambda\gamma_l|\big)\\
&\quad +\big(|\delta_0|+|k\delta_n-\delta_{n-1}+\delta_n-k\delta_n|+|\delta_{n-1}-\delta_{n-2}|+...+|\lambda\delta_l-\delta_l+\delta_{l+1}-\lambda\delta_l|\big)+M_l,
\end{aligned}$$

where

$$M_l=\sum_{s=1}^{l}\big(|\alpha_s-\alpha_{s-1}|+|\beta_s-\beta_{s-1}|+|\gamma_s-\gamma_{s-1}|+|\delta_s-\delta_{s-1}|\big).$$

This implies

$$\begin{aligned}
|f(q)| &\le \big(|\alpha_0|+(2k-1)\alpha_n+(1-\lambda)|\alpha_l|-\lambda\alpha_l\big)+\big(|\beta_0|+(2k-1)\beta_n+\\
&\quad (1-\lambda)|\beta_l|-\lambda\beta_l\big)+\big(|\gamma_0|+(2k-1)\gamma_n+(1-\lambda)|\gamma_l|-\lambda\gamma_l\big)+\\
&\quad \big(|\delta_0|+(2k-1)\delta_n+(1-\lambda)|\delta_l|-\lambda\delta_l\big)+M_l \qquad \text{if} \quad \lambda<1
\end{aligned}$$

or

$$\begin{aligned}
|f(q)| &\le \big(|\alpha_0|+(2k-1)\alpha_n+(\lambda-1)|\alpha_l|-\lambda\alpha_l\big)+\big(|\beta_0|+(2k-1)\beta_n+\\
&\quad (\lambda-1)|\beta_l|-\lambda\beta_l\big)+\big(|\gamma_0|+(2k-1)\gamma_n+(\lambda-1)|\gamma_l|-\lambda\gamma_l\big)+\\
&\quad \big(|\delta_0|+(2k-1)\delta_n+(\lambda-1)|\delta_l|-\lambda\delta_l\big)+M_l \qquad \text{if} \quad \lambda\ge 1.
\end{aligned}$$

Since

$$\max_{|q|=1}|q^n*f(\frac{1}{q})|=\max_{|q|=1}|f(\frac{1}{q})|=\max_{|q|=1}|f(q)|.$$

Therefore, $q^n*f(\frac{1}{q})$ has the same bound on $|q|=1$ as $f(q)$, that is:

$$\begin{aligned}
\big|q^n*f\big(\frac{1}{q}\big)\big| &\le \big(|\alpha_0|+(2k-1)\alpha_n+(1-\lambda)|\alpha_l|-\lambda\alpha_l\big)+\big(|\beta_0|+(2k-1)\beta_n\\
&\quad +(1-\lambda)|\beta_l|-\lambda\beta_l\big)+\big(|\gamma_0|+(2k-1)\gamma_n+(1-\lambda)|\gamma_l|-\lambda\gamma_l\big)\\
&\quad +\big(|\delta_0|+(2k-1)\delta_n+(1-\lambda)|\delta_l|-\lambda\delta_l\big)+M_l \quad \text{for} \quad |q|=1;\\
&\quad \text{when} \quad \lambda<1
\end{aligned}$$

or

$$\begin{aligned}
\big|q^n*f\big(\frac{1}{q}\big)\big| &\le \big(|\alpha_0|+(2k-1)\alpha_n+(\lambda-1)|\alpha_l|-\lambda\alpha_l\big)+\big(|\beta_0|+(2k-1)\beta_n\\
&\quad +(\lambda-1)|\beta_l|-\lambda\beta_l\big)+\big(|\gamma_0|+(2k-1)\gamma_n+(\lambda-1)|\gamma_l|-\lambda\gamma_l\big)\\
&\quad +\big(|\delta_0|+(2k-1)\delta_n+(\lambda-1)|\delta_l|-\lambda\delta_l\big)+M_l \quad \text{for} \quad |q|=1;\\
&\quad \text{when} \quad \lambda\ge 1
\end{aligned}$$

Applying maximum modulus theorem ([7] Theorem 3.4), it follows that

$$\begin{aligned}\left|q^n * f(\frac{1}{q})\right| \leq\ & \big(|\alpha_0| + (2k-1)\alpha_n + (1-\lambda)|\alpha_l| - \lambda\alpha_l\big) + \big(|\beta_0| + (2k-1)\beta_n \\ & +(1-\lambda)|\beta_l| - \lambda\beta_l\big) + \big(|\gamma_0| + (2k-1)\gamma_n + (1-\lambda)|\gamma_l| - \lambda\gamma_l\big) \\ & +\big(|\delta_0| + (2k-1)\delta_n + (1-\lambda)|\delta_l| - \lambda\delta_l\big) + M_l \quad \text{for} \quad |q| \leq 1; \\ & \text{when} \quad \lambda < 1\end{aligned}$$

or

$$\begin{aligned}\left|q^n * f(\frac{1}{q})\right| \leq\ & \big(|\alpha_0| + (2k-1)\alpha_n + (\lambda-1)|\alpha_l| - \lambda\alpha_l\big) + \big(|\beta_0| + (2k-1)\beta_n \\ & +(\lambda-1)|\beta_l| - \lambda\beta_l\big) + \big(|\gamma_0| + (2k-1)\gamma_n + (\lambda-1)|\gamma_l| - \lambda\gamma_l\big) \\ & +\big(|\delta_0| + (2k-1)\delta_n + (\lambda-1)|\delta_l| - \lambda\delta_l\big) + M_l \quad \text{for} \quad |q| \leq 1; \\ & \text{when} \quad \lambda \geq 1\end{aligned}$$

That is:

$$\begin{aligned}|f(\frac{1}{q})| \leq\ & \frac{1}{|q|^n}\Big(\big(|\alpha_0| + (2k-1)\alpha_n + (1-\lambda)|\alpha_l| - \lambda\alpha_l\big) + \big(|\beta_0| + (2k-1)\beta_n \\ & +(1-\lambda)|\beta_l| - \lambda\beta_l\big) + \big(|\gamma_0| + (2k-1)\gamma_n + (1-\lambda)|\gamma_l| - \lambda\gamma_l\big) \\ & +\big(|\delta_0| + (2k-1)\delta_n + (1-\lambda)|\delta_l| - \lambda\delta_l\big) + M_l\Big) \quad \text{for} \quad |q| \leq 1; \\ & \text{if} \quad \lambda < 1\end{aligned}$$

or

$$\begin{aligned}|f(\frac{1}{q})| \leq\ & \frac{1}{|q|^n}\Big(\big(|\alpha_0| + (2k-1)\alpha_n + (\lambda-1)|\alpha_l| - \lambda\alpha_l\big) + \big(|\beta_0| + (2k-1)\beta_n \\ & +(\lambda-1)|\beta_l| - \lambda\beta_l\big) + \big(|\gamma_0| + (2k-1)\gamma_n + (\lambda-1)|\gamma_l| - \lambda\gamma_l\big) \\ & +\big(|\delta_0| + (2k-1)\delta_n + (\lambda-1)|\delta_l| - \lambda\delta_l\big) + M_l\Big) \quad \text{for} \quad |q| \leq 1; \\ & \text{if} \quad \lambda \geq 1.\end{aligned}$$

Replacing q by $\frac{1}{q}$, we get for $|q| \geq 1$:

$$\begin{aligned}|f(q)| \leq\ & \Big(\big(|\alpha_0| + (2k-1)\alpha_n + (1-\lambda)|\alpha_l| - \lambda\alpha_l\big) + \big(|\beta_0| + (2k-1)\beta_n + \\ & (1-\lambda)|\beta_l| - \lambda\beta_l\big) + \big(|\gamma_0| + (2k-1)\gamma_n + (1-\lambda)|\gamma_l| - \lambda\gamma_l\big) + \\ & \big(|\delta_0| + (2k-1)\delta_n + (1-\lambda)|\delta_l| - \lambda\delta_l\big) + M_l\Big)|q|^n \quad \text{if} \quad \lambda < 1\end{aligned} \tag{1}$$

or

$$\begin{aligned}|f(q)| \leq\ & \Big(\big(|\alpha_0| + (2k-1)\alpha_n + (\lambda-1)|\alpha_l| - \lambda\alpha_l\big) + \big(|\beta_0| + (2k-1)\beta_n \\ & +(\lambda-1)|\beta_l| - \lambda\beta_l\big)\big(|\gamma_0| + (2k-1)\gamma_n + (\lambda-1)|\gamma_l| - \lambda\gamma_l\big) \\ & +\big(|\delta_0| + (2k-1)\delta_n + (\lambda-1)|\delta_l| - \lambda\delta_l\big) + M_l\Big)|q|^n \quad \text{if} \quad \lambda \geq 1\end{aligned} \tag{2}$$

But

$$\begin{aligned} |p(q)*(1-q)| &= |f(q)-q^{n+1}a_n| \\ &\geq |a_n||q|^{n+1}-|f(q)|. \end{aligned}$$

Using (1) and (2), we have for $|q|\geq 1$,

$$\begin{aligned} |p(q)*(1-q)| \;\geq\; & \Big(|a_n||q| - \{[|\alpha_0|+(2k-1)\alpha_n+(1-\lambda)|\alpha_l|-\lambda\alpha_l]+[|\beta_0| \\ & +(2k-1)\beta_n+(1-\lambda)|\beta_l|-\lambda\beta_l]+[|\gamma_0|+(2k-1)\gamma_n \\ & +(1-\lambda)|\gamma_l|-\lambda\gamma_l]+[|\delta_0|+(2k-1)\delta_n+(1-\lambda)|\delta_l|-\lambda\delta_l] \\ & +M_l\}\Big)|q|^n \quad \text{if} \quad \lambda<1 \end{aligned}$$

or

$$\begin{aligned} |p(q)*(1-q)| \;\geq\; & \Big(|a_n||q| - \{[|\alpha_0|+(2k-1)\alpha_n+(\lambda-1)|\alpha_l|-\lambda\alpha_l]+[|\beta_0| \\ & +(2k-1)\beta_n+(\lambda-1)|\beta_l|-\lambda\beta_l]+[|\gamma_0|+(2k-1)\gamma_n \\ & +(\lambda-1)|\gamma_l|-\lambda\gamma_l]+[|\delta_0|+(2k-1)\delta_n+(\lambda-1)|\delta_l|-\lambda\delta_l] \\ & +M_l\}\Big)|q|^n \quad \text{if} \quad \lambda\geq 1. \end{aligned}$$

This implies that $|p(q)*(1-q)|>0$, i.e., $p(q)*(1-q)\neq 0$ if:

$$\begin{aligned} |q| \;>\; & \frac{1}{|a_n|}\Big([|\alpha_0|+(2k-1)\alpha_n+(1-\lambda)|\alpha_l|-\lambda\alpha_l]+[|\beta_0|+(2k-1)\beta_n \\ & +(1-\lambda)|\beta_l|-\lambda\beta_l]+[|\gamma_0|+(2k-1)\gamma_n+(1-\lambda)|\gamma_l|-\lambda\gamma_l] \\ & +[|\delta_0|+(2k-1)\delta_n+(1-\lambda)\delta_l|)-\lambda\delta_l]+M_l\Big) \qquad \text{when} \quad \lambda<1 \end{aligned}$$

or

$$\begin{aligned} |q| \;>\; & \frac{1}{|a_n|}\Big([|\alpha_0|+(2k-1)\alpha_n+(\lambda-1)|\alpha_l|-\lambda\alpha_l]+[|\beta_0|+(2k-1)\beta_n \\ & +(\lambda-1)|\beta_l|-\lambda\beta_l]+[|\gamma_0|+(2k-1)\gamma_n+(\lambda-1)|\gamma_l|-\lambda\gamma_l] \\ & +[|\delta_0|+(2k-1)\delta_n+(\lambda-1)|\delta_l|-\lambda\delta_l]+M_l\Big) \qquad \text{when} \quad \lambda\geq 1. \end{aligned}$$

Since the only zeros of $p(q)*(1-q)$ are $q=1$ and the zeros of $p(q)$. Therefore, $p(q)\neq 0$ for:

$$\begin{aligned} |q| \;>\; & \frac{1}{|a_n|}\Big([|\alpha_0|+(2k-1)\alpha_n+(1-\lambda)|\alpha_l|-\lambda\alpha_l]+[|\beta_0|+(2k-1)\beta_n+ \\ & (1-\lambda)|\beta_l|-\lambda\beta_l]+[|\gamma_0|+(2k-1)\gamma_n+(1-\lambda)|\gamma_l|-\lambda\gamma_l]+[|\delta_0|+ \\ & (2k-1)\delta_n+(1-\lambda)|\delta_l|-\lambda\delta_l]+M_l\Big) \quad \text{if} \quad \lambda<1 \end{aligned}$$

or

$$\begin{aligned} |q| \;>\; & \frac{1}{|a_n|}\Big([|\alpha_0|+(2k-1)\alpha_n+(\lambda-1)|\alpha_l|-\lambda\alpha_l]+[|\beta_0|+(2k-1)\beta_n+ \\ & (\lambda-1)|\beta_l|-\lambda\beta_l]+[|\gamma_0|+(2k-1)\gamma_n+(\lambda-1)|\gamma_l|-\lambda\gamma_l]+[|\delta_0|+ \\ & (2k-1)\delta_n+(\lambda-1)|\delta_l|-\lambda\delta_l]+M_l\Big) \quad \text{if} \quad \lambda\geq 1. \end{aligned}$$

Hence all the zeros of $p(q)$ lie in :

$$\begin{aligned}|q| \leq & \frac{1}{|a_n|}\Big([|\alpha_0| + (2k-1)\alpha_n + (1-\lambda)|\alpha_l| - \lambda\alpha_l] + [|\beta_0| + (2k-1)\beta_n + \\ & (1-\lambda)|\beta_l| - \lambda\beta_l] + [|\gamma_0| + (2k-1)\gamma_n + (1-\lambda)|\gamma_l| - \lambda\gamma_l] + [|\delta_0| + \\ & (2k-1)\delta_n + (1-\lambda)|\delta_l| - \lambda\delta_l] + M_l\Big) \quad \text{if} \quad \lambda < 1\end{aligned}$$

or

$$\begin{aligned}|q| \leq & \frac{1}{|a_n|}\Big([|\alpha_0| + (2k-1)\alpha_n + (\lambda-1)|\alpha_l| - \lambda\alpha_l] + [|\beta_0| + (2k-1)\beta_n + \\ & (\lambda-1)|\beta_l| - \lambda\beta_l] + [|\gamma_0| + (2k-1)\gamma_n + (\lambda-1)|\gamma_l| - \lambda\gamma_l] + [|\delta_0| + \\ & (2k-1)\delta_n + (\lambda-1)|\delta_l| - \lambda\delta_l] + M_l\Big) \quad \text{if} \quad \lambda \geq 1\end{aligned}$$

as claimed. □

Proof. (*of Theorem* 3.2): Consider the polynomial

$$f(q) = \sum_{s=1}^{n} q^s(a_s - a_{s-1}) + a_0$$

and let $p(q) * (1-q) = f(q) - q^{n+1}\alpha_n$.

Now

$$\begin{aligned}\sum_{s=1}^{n}(|a_s - a_{s-1}|) &= \sum_{s=1}^{n}\big[|(\alpha_s + i\beta_s + j\gamma_s + k\delta_s) \\ &\quad -(\alpha_{s-1} + i\beta_{s-1} + j\gamma_{s-1} + k\delta_{s-1})|\big] \\ &= \sum_{s=1}^{n}\big[|(\alpha_s - \alpha_{s-1}) + i(\beta_s - \beta_{s-1} + j(\gamma_s - \gamma_{s-1}) \\ &\quad +k(\delta_s - \delta_{s-1})|\big] \\ &\leq \sum_{s=1}^{n}(|\alpha_s - \alpha_{s-1}|) + \sum_{s=1}^{n}\big[|\beta_s| + |\beta_{s-1}| + |\gamma_s| + |\gamma_{s-1}| \\ &\quad +|\delta_s| + |\delta_{s-1}|\big] \\ &= |\alpha_n + \lambda - \alpha_{n-1} - \lambda| + |\alpha_{n-1} - \alpha_{n-2}| + \dots + |\alpha_{l+1} - \alpha_l| \\ &\quad + \sum_{s=1}^{l}(|\alpha_s - \alpha_{s-1}|) + \sum_{s=1}^{n}\big[|\beta_s| + |\beta_{s-1}| + |\gamma_s| + |\gamma_{s-1}| \\ &\quad +|\delta_s| + |\delta_{s-1}|\big] \\ &\leq (\alpha_n + 2\lambda - \alpha_l) + N_l + \sum_{s=1}^{n}\big[|\beta_s| + |\beta_{s-1}| + |\gamma_s| + |\gamma_{s-1}| \\ &\quad +|\delta_s| + |\delta_{s-1}|\big] \qquad (3)\end{aligned}$$

where

$$N_l = \sum_{s=1}^{l}(|\alpha_s - \alpha_{s-1}|).$$

Since

$$f(q) = \sum_{s=l}^{n} q^s(a_s - a_{s-1}) + a_0 + q^{n+1}(i\beta_n + j\gamma_n + k\delta_n).$$

Therefore for $|q| = 1$, we get

$$|f(q)| \leq \sum_{s=1}^{n}(|\alpha_s - \alpha_{s-1}|) + |a_0| + |\beta_n| + |\gamma_n| + |\delta_n|.$$

Using (3), we get

$$\begin{aligned} |f(q)| &\leq (\alpha_n + 2\lambda - \alpha_l) + N_l + \sum_{s=1}^{n}\big[|\beta_s| + |\beta_{s-1}| + |\gamma_s| + |\gamma_{s-1}| + \\ &\quad |\delta_s| + |\delta_{s-1}|\big] + |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + |\beta_n| + |\gamma_n| + |\delta_n| \\ &= (\alpha_n + 2\lambda - \alpha_l + |\alpha_0|) + N_l + 2\sum_{s=0}^{n}\big[|\beta_s| + |\gamma_s| + |\delta_s|\big]. \end{aligned}$$

But

$$\max_{|q|=1}\big|q^n * f(\frac{1}{q})\big| = \max_{|q|=1}\big|f(\frac{1}{q})\big| = \max_{|q|=1}\big|f(q)\big|.$$

This implies

$$\Big|q^n * f(\frac{1}{q})\Big| \leq (\alpha_n + 2\lambda - \alpha_l + |\alpha_0|) + N_l + 2\sum_{s=0}^{n}[|\beta_s| + |\gamma_s| + |\delta_s|] \quad \text{for} \quad |q| = 1.$$

Applying maximum modulus theorem [7] for quaternionic polynomials, it follows that:

$$\Big|q^n * f(\frac{1}{q})\Big| \leq (\alpha_n + 2\lambda - \alpha_l + |\alpha_0|) + N_l + 2\sum_{s=0}^{n}\big[|\beta_s| + |\gamma_s| + |\delta_s|\big] \quad \text{for} \quad |q| \leq 1.$$

Replacing q by $\frac{1}{q}$, it yields that:

$$|f(\frac{1}{q})| \leq \Big((\alpha_n + 2\lambda - \alpha_l + |\alpha_0|) + N_l + 2\sum_{s=0}^{n}[|\beta_s| + |\gamma_s| + |\delta_s|]\Big)|q|^n \quad \text{for} \quad |q| \geq 1.$$

Again, for $|q| \geq 1$,

$$\begin{aligned} |p(q) * (1-q)| &= |q^{n+1}\alpha_n - f(q)| \\ &\geq |q^{n+1}||\alpha_n| - |f(q)| \\ &\geq |q|^{n+1}|\alpha_n| - |q|^n\{(\alpha_n + 2\lambda - \alpha_l + |\alpha_0|) + N_l \\ &\quad +2\sum_{s=0}^{n}[|\beta_s| + |\gamma_s| + |\delta_s|]\} \\ &= \Big(|q||\alpha_n| - [(\alpha_n + 2\lambda - \alpha_l + |\alpha_0|) + N_l \\ &\quad +2\sum_{s=0}^{n}[|\beta_s| + |\gamma_s| + |\delta_s|]\Big)|q|^n. \end{aligned}$$

On similar lines as done in proof of Theorems 3.1, we finally conclude that all the zeros of p lie in

$$|q| \leq \frac{1}{|\alpha_n|}\Big\{(\alpha_n + 2\lambda - \alpha_l + |\alpha_0|) + N_l + 2\sum_{s=0}^{n}[|\beta_s| + |\gamma_s| + |\delta_s|]\Big\}.$$

This completes the proof of Theorem 3.2 □

Proof. (*of Theorem* 3.3): Let $f(q) = p(q) * (1-q) + q^{n+1}a_n$. Then for $|q| = 1$, we have

$$\begin{aligned} |f(q)| &= |\sum_{s=1}^{n} q^s(a_s - a_{s-1}) + a_0| \\ &\leq |a_0| + \sum_{s=1}^{n}|a_s - a_{s-1}| \\ &\leq |a_0| + \sum_{s=l+1}^{n}|a_s - a_{s-1}| + \sum_{s=1}^{l}|a_s - a_{s-1}| \\ &= \Big[|a_0| + |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + ... + |a_{l+1} - a_l| \\ &\quad + \sum_{s=1}^{l}|a_s - a_{s-1}|\Big] \\ &= |a_0| + |ka_n - a_{n-1} - ka_n + a_n| + |a_{n-1} - a_{n-2}| + ... + \\ &\quad |a_{l+1} - a_l| + \sum_{s=1}^{l}|a_s - a_{s-1}| \end{aligned}$$

That is

$$\begin{aligned}|f(q)| &\leq \Big[|a_n| + |ka_n - a_{n-1}| + (k-1)|a_n| + |a_{n-1} - a_{n-2}| + ... + \\ &\quad |a_{l+1} - a_l| + \sum_{s=1}^{l} |a_s - a_{s-1}|\Big] \\ &\leq (k-1)|a_n| + |a_0| + k|a_n|(\cos\theta + \sin\theta) - |a_l|(\sin\theta + \cos\theta) \\ &\quad +2\sin\theta \sum_{s=l}^{n-1} |a_s| + \sum_{s=1}^{l} |a_s - a_{s-1}| \qquad \text{(by lemma 2)}.\end{aligned}$$

Proceeding likewise as in the proof of Theorem 3.1, we finally arrive at:

$$\begin{aligned}|f(q)| &\leq \Big\{(k-1)|a_n| + |a_0| + k|a_n|(\cos\theta + \sin\theta) - |a_l|(\sin\theta + \cos\theta) + \\ &\quad 2\sin\theta \sum_{s=l}^{n-1} |a_s| + \sum_{s=1}^{l} |a_s - a_{s-1}|\Big\} \qquad \text{for} \quad |q| \geq 1\end{aligned}$$

Since

$$\begin{aligned}|p(q) * (1-q)| &\geq |a_n||q^{n+1} - |f(q)| \\ &\geq |a_n||q|^{n+1} - \big[(k-1)|a_n| + |a_0| + k|a_n|(\cos\theta + \sin\theta) \\ &\quad -|a_l|(\sin\theta + \cos\theta) + 2\sin\theta \sum_{s=l}^{n-1} |a_s| + \sum_{s=1}^{l} |a_s - a_{s-1}|\big]|q|^n \\ &\quad \text{for} \quad |q| \geq 1 \\ &= \Big(|a_n||q| - \{(k-1)|a_n| + |a_0| + k|a_n|(\cos\theta + \sin\theta) \\ &\quad -|a_l|(\sin\theta + \cos\theta) + 2\sin\theta \sum_{s=l}^{n-1} |a_s| + \sum_{s=1}^{l} |a_s - a_{s-1}|\}\Big)|q|^n \\ &\quad \text{for} \quad |q| \geq 1.\end{aligned}$$

This implies that $|p(q) * (1-q)| > 0$, i.e., $p(q) * (1-q) \neq 0$ if:

$$\begin{aligned}|q| &> \frac{1}{|a_n|}\big\{(k-1)|a_n| + |a_0| + k|a_n|(\cos\theta + \sin\theta) - |a_l|(\sin\theta + \cos\theta) \\ &\quad +2\sin\theta \sum_{s=l}^{n-1} |a_s| + \sum_{s=1}^{l} |a_s - a_{s-1}|\big\}.\end{aligned}$$

But by lemma 1, $p(q) * (1-q) = 0$ if and only if either $q = 1$ or $p(q) = 0$. Hence all the zeros of $p(q)$ lie in:

$$|q| \leq \frac{1}{|a_n|}\Big((k-1)|a_n| + |a_0| + k|a_n|(\cos\theta + \sin\theta) - |a_l|(\sin\theta + \cos\theta) + 2\sin\theta \sum_{s=l}^{n-1} |a_s| + \sum_{s=1}^{l} |a_s - a_{s-1}|\Big)$$

as claimed.

□

Conflicts of interest: The author declares no conflict of interest.

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