

## ON THE HYERS-ULAM SOLUTION AND STABILITY PROBLEM FOR GENERAL SET-VALUED EULER-LAGRANGE QUADRATIC FUNCTIONAL EQUATIONS

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ABSTRACT. By established a Banach space with the Hausdorff distance, we introduce the alternative fixed-point theorem to explore the existence and uniqueness of a fixed subset of  $Y$  and investigate the stability of set-valued Euler-Lagrange functional equations in this space. Some properties of the Hausdorff distance are furthermore explored by a short and simple way.

### 1. Introduction

In the autumn of 1940, the stability problem of group homomorphisms was originally proposed by Ulam [27] in an international conference. More precisely, Ulam proposed about the problem that these sets  $(X, +)$  and  $(Y, \cdot)$  are two groups with a metric  $d : X \times Y \rightarrow [0, \infty)$ . For a given  $\delta > 0$ , if a function  $g : G_1 \rightarrow G_2$  fulfils an approximate inequality  $d(g(x + y), g(x)g(y)) \leq \delta$  for all  $x, y \in G_1$ , then we find out a positive constant  $\varepsilon > 0$  and a unique determined homomorphism  $f : G_1 \rightarrow G_2$  satisfying the condition  $d(f(x), g(x)) \leq \varepsilon$  for every  $x \in G_1$ .

One year later, a very interesting method for solving the problem restricted on the framework of Banach spaces was derived by Hyers [4]. At the present time, this method has a great influence on the study of functional equations and it is still the main tool for solving the problem. Thirty-seven years later, Rassias [31] extended Hyers' theorem to a more general case solved by introducing additive mapping which is to deal with an unbounded Cauchy difference. From then on, a lot of generalizations and several classes of stability problems for functional equations have been widely studied by many scholars by using different methods in different directions.

In particular, the Hyers-Ulam-Rassias stability problem for the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

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due to given an unknown operator  $f$  mapping a normed space  $X$  into a classical Banach space  $Y$ , has been studied by the author [8, 28]. Moreover, it is natural to define a quadratic mapping if it is an arbitrary function satisfying quadratic functional equation.

Cholewa [24] found out the above assertions are still established even though the above results of the domain  $X$  can be relaxed to a group. Furthermore, in [5], the domain of the operator for functional equations can be extended to a nonempty set to solve the stability problems for a simple variable equation.

Lee et.al [13] proved the Hyers-Ulam-Rassias stability problem for Jensen-quadratic functional equation

$$2f(x+y) + 2f(x-y) = f(2x) + f(2y).$$

In inspired by the above works, we investigate the Euler-Lagrange functional equation [26]

$$f(\alpha x + \beta y) + f(\alpha x - \beta y) = 2\alpha^2 f(x) + 2\beta^2 f(y)$$

by introducing the fixed-point alternative theorem. It shall show that the Euler-Lagrange functional equation is a general form of the quadratic functional equation and also is a general form of Jensen quadratic functional equation by setting  $(x, y) = (x+y, x-y)$  and  $\alpha = \beta = 1$ , respectively. In fact, the so called equation Euler-Lagrange functional equation has some misleads seen in [26]. However, the terminology can be kept in mind to stop possible misunderstanding of the validity of uniformity.

Another similar research work, due to solved by direct method, can be dated back to Baak [3] studying stability problem about the following set-valued Cauchy-Jensen functional equations:

$$\begin{aligned} f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x-y}{2} + z\right) &= f(x) + 2f(z), \\ f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x-y}{2} + z\right) &= f(y) \end{aligned}$$

or

$$2f\left(\frac{x+y}{2} + z\right) = f(x) + f(y) + 2f(z)$$

for all  $x, y, z \in X$ , where  $X$  denotes a Banach space. It is worth noting that the stability problem about these three functional equations can also be derived by using another abstract direct method in [5].

There is another fact that the Banach contraction principles have been used to derive the stability problem about functional equations by Baak [14] for the first time. However, the fixed-point alternative theorem was originally used to prove Hyers-Ulam-Rassias stability problem about functional equations which are contributed to the works [32], [15]. From then on, the fixed-point theorem produces a lot of influence in the research field (see [1, 9, 17, 23, 34, 35]).

The investigator [38] studied the generalized Cauchy-Jensen functional equation

$$nf\left(\frac{x+y}{n} + z\right) = f(x) + f(y) + nf(z).$$

The authors [25], due to introducing the fixed point alternative theorem, studied the Hyers-Ulam-Rassias stability problem about the generalization of set-valued Cauchy-Jensen functional equation

$$\alpha f\left(\frac{x+y}{\alpha} + z\right) = f(x) \oplus f(y) \oplus \alpha f(z)$$

for all  $x, y, z \in X$  and  $\alpha \geq 2$  on the classical Banach space. Based on this analysis, there is still an open problem leaving to be solved. Therefore, we investigate the Hyers-Ulam-Rassias stability problem about the following generalized set-valued Cauchy-Jensen functional equation

$$\alpha f\left(\frac{x+y}{\alpha} + z\right) = \alpha f\left(\frac{x-y}{\alpha} + z\right) \oplus f(y)$$

on Banach spaces and the problem can be solved by introducing the alternative fixed point theorem.

In section 2, some basic knowledge will be presented and some properties of the Hausdorff distance are furthermore explored by a short and simple way.

In section 3, the general solution and Hyers-Ulam stability problems about the Euler-Lagrange-Jensen cubic equation will be described by the fixed point alternative theorem in the Banach spaces. At last, the problems of this topic about the relations between Euler-Lagrange functional equation and radical-type functional equations will be discussed in detail.

## 2. Preliminaries and basic knowledge

First of all, the stability problems about various functional equations have already been widely studied and there have appeared many interesting results concerning the problem (see [7, 18, 29, 30, 36, 37]).

**DEFINITION 1.** ([15], [33]) Assume that  $X$  is a nonempty set and let the operator  $d : X \times X \rightarrow [0, \infty]$  be a generalized metric on  $X$  if the operator  $d$  fulfils the assumptions

- I** (Positive definiteness)  $d(x, y) > 0$  if and only if  $x \neq y$ ;
- II** (Symmetrical characteristic)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- III** (Triangle inequality)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

We have already stated the definition that  $(X, d)$  is a generalized metric space. The general metric is different from the usual notion of distance or metric space from which two arbitrary elements from  $X$  do not need a fixed finite value in  $[0, \infty)$ . A complete space with a generalized metric is called a generalized complete metric space.

A mapping  $J : X \rightarrow X$  is called a Lipschitz mapping with a Lipschitz constant  $L \in [0, \infty)$  if it satisfies the condition  $d(Jx, Jy) \leq Ld(x, y)$  for every  $x, y \in X$ . In a word, we introduce the fixed-point alternative theorem.

**THEOREM 2.** ([15], [25], [33]) *Suppose that  $(X, d)$  is a complete generalized metric space and  $J : X \rightarrow X$  is a Lipschitz continuous self-mapping on  $X$  via  $L \in [0, 1)$ . Then for any arbitrary element  $x$  from  $X$ , either there has for all positive integers  $n$*

$$d(J^n x, J^{n+1} x) = \infty$$

or there has for some positive integer  $n_0$ ,  $d(J^n x, J^{n+1} x) < \infty$ ,  $n \geq n_0$  and the successive sequence  $\{J^n x\}$  converges to a unique fixed point  $y^*$  of  $J$  in the set

$$Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$$

and  $d(x, y^*) \leq \frac{1}{1-L} d(x, Jx)$  for all  $x \in Y$ .

*Proof.* For simplicity in notation, we can assume that  $n_0 = 0$ . Since  $J$  is a strictly contractive self-mapping satisfying the condition

$$d(J^n x, J^{n+1} x) < \infty,$$

then the successive sequence  $\{J^n x\}_{n=0}^\infty$  is a d-convergent sequence (We shall  $d(x_n, x_{n+1}) \rightarrow 0$ , as  $n \rightarrow \infty$  by setting  $x_n = J^n x$ ) in  $X$  and thus it converges to a unique element  $y^*$  in the set  $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$  by the completeness of  $X$  and the property **I** of the distance function. From property **III** of the distance function, we can get  $d(x, y^*) \leq (1 + L + L^2 + \dots) d(x, Jx) = \frac{1}{1-L} d(x, Jx)$  for all  $x \in Y$ . The desired assertion can be obtained.  $\square$

**REMARK 1.** If we only consider a unique determined fixed-point on the set  $Y$ , then there does not necessarily require the function  $d$  satisfying the property of **III**. This property can be used to prove a more precisely approximate distance  $\frac{1}{1-L}$ .

In 1996, Isac et al [11] are the first investigators to prove the stability problem about functional equations by introducing a new notion of fixed point theorems. Via introducing several types of fixed-point approaches, many results about the Hyers-Ulam-Rassias stability problems about various functional equations appeared in the research field (see [12, 19–21]). Suppose that  $Y$  is a Banach space. Several classes of subsets of  $Y$  are given in following:

$$\begin{aligned} 2^Y &= \{A : A \subseteq Y\}; \\ C_b(Y) &= \{A \subseteq Y : A \text{ is bounded}\}; \\ C_c(Y) &= \{A \subseteq Y : A \text{ is closed}\}; \\ C_{cb}(Y) &= \{A \subseteq Y : A \text{ is bounded and convex}\}. \end{aligned}$$

In space  $2^Y$ , the definitions of the addition operation and the number multiplication operation can be stated:

$$A + B = \{x + x' : x \in A, x' \in B\}, \quad \lambda A = \{\lambda x : x \in A\}$$

where  $A, B \in 2^Y$  and  $\lambda \in \mathbb{R}$ , the set of real numbers. Furthermore, if we have two subsets  $A, B \in C_c(Y)$ , then there has a well definition  $A \oplus B = \overline{A + B}$ . An easy observation shall show that

$$(\lambda + \mu)A \subseteq \lambda A + \mu A, \quad \lambda A + \lambda B = \lambda(A + B).$$

we can easily achieve  $(\lambda + \mu)C = \lambda C + \mu C$  for all  $C \in C_{cb}(Y)$  and  $\lambda, \mu \in \mathbb{R}^+$ .

From two any subsets  $A, B \in 2^Y$ , based on the above analysis, we give out the definition of the distance mapping  $d(\cdot, A)$  and the support mappings  $s(\cdot, A)$  in the following:

$$\begin{aligned} d(x, B) &= \inf\{\|x - y\| : y \in B\}, & x \in Y, \\ d(A, B) &= \sup_{x \in A} \inf\{\|x - y\| : y \in B\} = \sup_{x \in A} d(x, B), & A, B \in C_{cb}(Y), \\ s(x^*, B) &= \sup\{\langle x^*, x \rangle : x \in B\}, & x^* \in Y^*. \end{aligned}$$

From two any subsets  $A, B \in C_b(Y)$ , we give out the definition of the Hausdorff distance which is stated in the following

$$h(A, B) = \inf \{ \lambda > 0 : A \subseteq B + \lambda B_Y, \quad B \subseteq A + \lambda B_Y \}$$

or

$$h(A, B) = \sup \{ d(A, B), d(B, A) \}$$

whence  $B_Y$  denotes a unit closed ball in  $Y$ . These two definitions of the Hausdorff distance are equivalent under some cases. Next, some properties of the Hausdorff distance are furthermore explored by a short and simple way.

**PROPOSITION 3.** ([25], [10]) *From any subsets  $A, A', B, B' \in C_{cb}(Y)$  and  $\mu > 0$ , the following conclusions hold true*

- (1)  $h(A \oplus A', B \oplus B') \leq h(A, B) + h(A', B')$ ;
- (2)  $h(\mu A, \mu B) = \mu h(A, B)$ .

*Proof.* From the definition of  $h(A, B)$  and  $h(A', B')$ , there have

$$A \subseteq B + h(A, B)B_Y, B \subseteq A + h(A, B)B_Y$$

and

$$A' \subseteq B' + h(A', B')B_Y, B' \subseteq A' + h(A', B')B_Y.$$

Hence there has

$$A + A' \subseteq B + B' + (h(A, B) + h(A', B'))B_Y$$

and

$$B + B' \subseteq A + A' + (h(A, B) + h(A', B'))B_Y$$

and, consequently, we have

$$\overline{A + A'} \subseteq \overline{B + B'} + (h(A, B) + h(A', B'))B_Y$$

and

$$\overline{B + B'} \subseteq \overline{A + A'} + (h(A, B) + h(A', B'))B_Y$$

Therefore (1) holds true.

According to the definition of  $h(A, B)$ , there is

$$\mu A \subseteq \mu B + \mu h(A, B)B_Y \quad \text{or} \quad \mu B \subseteq \mu A + \mu h(A, B)B_Y.$$

Hence  $h(\mu A, \mu B) \leq \mu h(A, B)$ . Conversely, we have  $A \subseteq B + \frac{h(\mu A, \mu B)}{\mu} B_Y, B \subseteq A + \frac{h(\mu A, \mu B)}{\mu} B_Y$  and the converse inequality remains true. The desired results can be obtained. □

The complete metric semigroup  $(C_{cb}(Y), \oplus, h)$  has been achieved if the set  $Y$  is a Banach space (see [2]).

From the Euler-Lagrange set-valued functional inequalities

$$h(f(\alpha x + \beta y) \oplus f(\alpha x - \beta y), 2\alpha^2 f(x) \oplus 2\beta^2 f(y)) \leq \varphi(x, y)$$

with an approximate mapping  $\varphi$  and

$$h(\alpha f\left(\frac{x+y}{\alpha} + z\right), \alpha f\left(\frac{x-y}{\alpha} + z\right) \oplus f(y)) \leq \varphi(x, y, z)$$

with an approximate control mapping  $\varphi$ , respectively, we have an attempt to study that, due to introducing the fixed-point alternative theorem, the set-valued mappings

$Q$ , near  $f$ , are respectively the solution for the Euler-Lagrange set-valued functional equations

$$f(\alpha x + \beta y) \oplus f(\alpha x - \beta y) = 2\alpha^2 f(x) \oplus 2\beta^2 f(y)$$

and

$$\alpha f\left(\frac{x+y}{\alpha} + z\right) = \alpha f\left(\frac{x-y}{\alpha} + z\right) \oplus f(y).$$

For simplicity in notation, we give out the following definition.

**DEFINITION 4.** A set-valued Euler-Lagrange mapping  $f : X \rightarrow C_c(Y)$  is an arbitrary solution of the Euler-Lagrange set-valued functional equation

$$f(\alpha x + \beta y) \oplus f(\alpha x - \beta y) = 2\alpha^2 f(x) \oplus 2\beta^2 f(y)$$

for any  $x, y \in X$  with  $\alpha, \beta \in R$ , the real field.

Similarly, another definition can be presented in the following way.

**DEFINITION 5.** A set-valued Cauchy-Jensen mapping  $f : X \rightarrow C_c(Y)$  is an arbitrary function satisfying the set-valued Cauchy-Jensen functional equation

$$\alpha f\left(\frac{x+y}{\alpha} + z\right) = \alpha f\left(\frac{x-y}{\alpha} + z\right) \oplus f(y)$$

for any  $x, y, z \in X$  with  $\alpha R$ .

Throughout the context of the literature, we always assume that  $X$  is a real normed space and call  $Y$  is a real Banach space.

### 3. Stability problems for the Set-Valued inequalities

Through introducing the fixed-point alternative theorem, we prove the Hyers-Ulam-Rassias stability of set-valued Euler-Lagrange functional equation.

**THEOREM 6.** Suppose that  $\varphi : X^2 \rightarrow [0, \infty)$  is a suitable control mapping with a constant  $\alpha \in [0, 1)$  fulfilling

$$\varphi(x, y) \leq \frac{1}{\alpha} \varphi(\alpha x, \alpha y)$$

for any  $x, y \in X$ . Assume that  $f : X \rightarrow (C_{cb}(Y), h)$  is an operator with  $f(0) = \{0\}$  satisfying

$$(1) \quad h(f(\alpha x + \beta y) \oplus f(\alpha x - \beta y), 2\alpha^2 f(x) \oplus 2\beta^2 f(y)) \leq \varphi(x, y)$$

for all  $x, y \in X$ . In addition, if  $\text{diam } f(x) \leq M\|x\|^r$  for all  $x \in X$ , and for some  $r < 2$  and  $M > 0$ , then we achieve a unique determined set-valued Euler-lagrange mapping  $Q : X \rightarrow (C_{cb}(Y), h)$  fulfilling the following approximate inequality

$$h(f(x), Q(x)) \leq \frac{1}{1-\alpha} \varphi(x, x)$$

for all  $x \in X$ .

*Proof.* By setting  $y = 0$  in (1), due to convexity of  $f(x)$ , there has

$$h(2f(\alpha x), 2\alpha^2 f(x)) \leq \varphi(x, 0)$$

whence

$$h\left(\alpha^2 f\left(\frac{1}{\alpha}x\right), f(x)\right) \leq \frac{1}{2\alpha} \varphi(x, 0)$$

for all  $x \in X$ . Set

$$S := \{g : g : X \rightarrow C_{cb}(Y), g(0) = \{0\}\}$$

and therefore we can set the generalized metric on  $X$  as

$$(2) \quad d(g, f) = \inf\{a \in (0, \infty) : h(g(x), f(x)) \leq a\varphi(x, 0), \quad x \in X\},$$

whence  $\inf \phi = +\infty$  and we can achieve that  $(S, d)$  is complete generalized metric semigroup (see [6, 22]). According to the analysis, we can define mapping  $J : S \rightarrow S$  by, for every  $x \in X$

$$Jg(x) = \alpha^2 g\left(\frac{x}{\alpha}\right).$$

We claim that  $J$  is a contractive mapping on  $X$  if  $\alpha \in [0, 1)$ . From the definition, we can choose  $g, f \in S$  fulfilling  $d(g, f) = \varepsilon$  and an obvious conclusion that  $g \in Y$ . Then there has, from the definition of the generalized metric, for every  $x \in X$ ,

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, 0).$$

An easy computation shows that, for every  $x \in X$

$$\begin{aligned} h(Jg(x), Jf(x)) &= h\left(\alpha^2 g\left(\frac{x}{\alpha}\right), \alpha^2 f\left(\frac{x}{\alpha}\right)\right) \\ &= \alpha^2 h\left(g\left(\frac{x}{\alpha}\right), f\left(\frac{x}{\alpha}\right)\right) \\ &\leq \alpha\varepsilon\varphi(x, 0) \end{aligned}$$

Observing that  $d(g, f) = \varepsilon$ , we can conclude that  $d(Jg, Jf) \leq \alpha\varepsilon$  and also it can be rewritten as

$$d(Jg, Jf) \leq \alpha d(g, f)$$

for any two elements  $g, f \in S$ . Combined with the conclusion  $d(f, Jf) \leq \frac{1}{2\alpha}$  with  $0 \leq \alpha < 1$  and by introducing the fixed-point alternative theorem, there has an operator  $Q : X \rightarrow Y$  fulfilling equality :

- (1) There has a unique determined fixed point of  $J$  in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

fulfilling

$$Q(x) = \alpha^2 Q\left(\frac{x}{\alpha}\right)$$

for any  $x \in X$  and  $\alpha > 1$ . Hence the mapping  $Q$  fulfilling via some  $\mu \in (0, \infty)$  and for every  $x \in X$

$$h(f(x), Q(x)) \leq \mu\varphi(x, 0).$$

- (2) There has  $d(J^n f, Q) \rightarrow 0$  as  $n \rightarrow \infty$  and concludes that for all  $x \in X$

$$\lim_{n \rightarrow \infty} \alpha^{2n} Q\left(\frac{x}{\alpha^n}\right) = Q(x).$$

- (3) There exist  $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$  and

$$d(f, Q) \leq \frac{1}{\alpha - 1}.$$

Finally, we prove the operator  $Q$  is an Euler-Lagrange set-valued mapping for all  $x, y \in X$

$$\begin{aligned} & h(Q(\alpha x + \beta y) \oplus Q(\alpha x - \beta y), 2\alpha^2 Q(x) \oplus 2\beta^2 Q(y)) \\ &= \lim_{n \rightarrow \infty} \alpha^{2n} h\left(f\left(\frac{x+y}{\alpha^n}\right) \oplus f\left(\frac{x-y}{\alpha^n}\right), 2\alpha^2 f\left(\frac{x}{\alpha^n}\right) \oplus 2\beta^2 f\left(\frac{y}{\alpha^n}\right)\right) \\ &\leq \lim_{n \rightarrow \infty} \alpha^{2n} \varphi\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right) \\ &\leq \lim_{n \rightarrow \infty} \alpha^n \varphi(x, y) = 0. \end{aligned}$$

According to the condition  $\text{diam } f(x) \leq M\|x\|^r$  for all  $x \in X$ ,  $\text{diam } \alpha^{2n} f\left(\frac{x}{\alpha^n}\right) \leq \alpha^{2n-rn} M\|x\|^r$  for every  $x \in X$  and hence  $Q(x) = \alpha^{2n} f\left(\frac{x}{\alpha^n}\right)$  is a singleton set satisfying the following set-valued equation for all  $x, y \in X$

$$2\alpha^2 Q(x) \oplus 2\beta^2 Q(y) = Q(\alpha x + \beta y) \oplus Q(\alpha x - \beta y).$$

□

Compared with the direct method, the fixed point alternative method is more direct, simple and short. Based on it, a typical application can be stated by a short, direct and simple way. Look it.

**COROLLARY 7.** *Suppose  $p > 2$  and  $\theta > 0$  and assume an operator  $f : X \rightarrow (C_{cb}(Y), h)$  with  $f(0) = \{0\}$  fulfilling*

$$(3) \quad h(f(\alpha x + \beta y) \oplus f(\alpha x - \beta y), 2\alpha^2 f(x) \oplus 2\beta^2 f(y)) \leq \theta (\|x\|^p + \|y\|^p)$$

*In addition, if  $\text{diam } f(x) \leq M\|x\|^r$  for every  $x \in X$ , and for some  $r < 2$  and  $M > 0$ , then we achieve a unique determined set-valued Euler-Lagrange mapping  $Q : X \rightarrow Y$  fulfilling for an arbitrary element  $x \in X$*

$$h(f(x), Q(x)) \leq \frac{\alpha^p \theta}{\alpha^p - \alpha^2} \|x\|^p.$$

*Proof.* According to Theorem 6, we can easily achieve that for any two elements  $x, y \in X$

$$\varphi(x, y) := \theta (\|x\|^p + \|y\|^p).$$

Then there is the constant  $L = \alpha^{2-p}$  in the fixed-point alternative theorem. The desired result can be achieved. □

In another direction, a similar results of Theorem 6 can be described by a short way in the following.

**THEOREM 8.** *Let  $\varphi : X^2 \rightarrow [0, \infty)$  be an approximate control mapping such that there exists a constant  $\alpha > 1$  satisfying*

$$\varphi(x, y) \leq \alpha \varphi\left(\frac{1}{\alpha}x, \frac{1}{\alpha}y\right)$$

*for any arbitrary elements  $x, y \in X$ . Assume an operator  $f : X \rightarrow (C_{cb}(Y), h)$  with  $f(0) = \{0\}$  satisfying*

$$(4) \quad h(f(\alpha x + \beta y) \oplus f(\alpha x - \beta y), 2\alpha^2 f(x) \oplus 2\beta^2 f(y)) \leq \varphi(x, y)$$



for all  $x, y \in X$ . In addition, if  $\text{diam } f(x) \leq M\|x\|^r$  for all  $x \in X$ , and for some  $r < 2$  and  $M > 0$ , then we achieve a unique determined set-valued Euler-Lagrange mapping  $Q : X \rightarrow (C_{cb}(Y), h)$  fulfilling the following approximate inequality

$$h(f(x), Q(x)) \leq \frac{\alpha}{\alpha - 1} \varphi(x, 0)$$

for any  $x \in X$ .

*Proof.* By setting  $y = 0$  in (4), due to convexity of  $f(x)$ , there has

$$h(2f(\alpha x), 2\alpha^2 f(x)) \leq \varphi(x, 0)$$

whence

$$h\left(\frac{1}{\alpha^2} f(\alpha x), f(x)\right) \leq \frac{1}{2\alpha^2} \varphi(x, 0)$$

for any arbitrary element  $x \in X$ . Set

$$S := \{g : g : X \rightarrow C_{cb}(Y), g(0) = \{0\}\}$$

and we naturally defined metric on  $X$  in the following

$$(5) \quad d(g, f) = \inf\{\mu \in (0, \infty) : h(g(x), f(x)) \leq \mu\varphi(x, 0), x \in X\},$$

whence  $\inf \phi = +\infty$  and we can conclude a complete generalized metric semigroup  $(S, d)$  (see [6, 22]). According to the analysis, we can define mapping  $J : S \rightarrow S$  by, for any  $x \in X$

$$Jg(x) = \frac{1}{\alpha^2} g(\alpha x)$$

We claim that  $J$  is a contractive mapping on  $X$  if  $\alpha \in [0, 1)$ . From the definition, we can choose  $g, f \in S$  fulfilling  $d(g, f) = \varepsilon$  and an obvious conclusion that  $g \in Y$ . Then, from the definition of the generalized metric, for every  $x \in X$ ,

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, 0).$$

An easy computation shall that, for every  $x \in X$

$$\begin{aligned} h(Jg(x), Jf(x)) &= h\left(\frac{1}{\alpha^2} g(\alpha x), \frac{1}{\alpha^2} f(\alpha x)\right) \\ &= \frac{1}{\alpha^2} h(g(\alpha x), f(\alpha x)) \\ &\leq \frac{1}{\alpha} \varepsilon\varphi(x, 0). \end{aligned}$$

From  $d(g, f) = \varepsilon$ , we can get  $d(Jg, Jf) \leq \frac{1}{\alpha} \varepsilon$  and also it can be rewritten as

$$d(Jg, Jf) \leq \frac{1}{\alpha} d(g, f)$$

for any  $g, f \in S$ . There has a operator  $Q : X \rightarrow Y$  fulfilling:

(1) We achieve a unique determined fixed point  $Q$  of  $J$  in the following subset

$$M = \{g \in S : d(f, g) < \infty\}.$$

such that

$$Q(x) = \frac{1}{\alpha^2} Q(\alpha x)$$

for any  $x \in X$  via  $\alpha > 1$ . Hence there is a unique mapping  $Q$  satisfying (5) and

$$h(f(x), Q(x)) \leq \mu\varphi(x, 0)$$

for some  $\mu \in (0, \infty)$  and for any  $x \in X$ .

(2) There has  $d(J^n f, Q) \rightarrow 0$  as  $n \rightarrow \infty$  and concludes that from  $x \in X$

$$\lim_{n \rightarrow \infty} \frac{1}{\alpha^{2n}} Q(\alpha^n x) = Q(x).$$

(3) There exists  $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$  and concludes that

$$d(f, Q) \leq \frac{\alpha}{\alpha - 1}.$$

Finally, we prove the operator  $Q$  is a set-valued Euler-Lagrange mapping for all  $x, y \in X$

$$\begin{aligned} & h(Q(\alpha x + \beta y) \oplus Q(\alpha x - \beta y), 2\alpha^2 Q(x) \oplus 2\beta^2 Q(y)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\alpha^{2n}} h(f(\alpha^n x + \beta^n y) \oplus f(\alpha^n x - \beta^n y), 2\alpha^{2n} f(\alpha^n x) \oplus 2\beta^{2n} f(\alpha^n y)) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{\alpha^{2n}} \varphi\left(\frac{x}{\alpha^n}, \frac{y}{\alpha^n}\right) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{\alpha^n} \varphi(x, y) = 0. \end{aligned}$$

Since we achieve the condition  $\text{diam } f(x) \leq M\|x\|^r$  for every  $x \in X$ , then there has  $\text{diam } \frac{1}{\alpha^{2n}} f(x\alpha^n) \leq \alpha^{-2n+rn} M\|x\|^r$  for any  $x \in X$ . Thus  $Q(x) = \alpha^{2n} f\left(\frac{x}{\alpha^n}\right)$  proved to be a singleton set fulfilling the following set-valued equation for all  $x, y \in X$

$$Q(\alpha x + \beta y) \oplus Q(\alpha x - \beta y) = 2\alpha^2 Q(x) \oplus 2\beta^2 Q(y).$$

□

Note that an easy application of Theorem 8 has been depicted by a simple way. Now, we state it.

**COROLLARY 9.** Assume  $0 < p < 2$  and  $\theta > 0$  and assume an operator  $f : X \rightarrow (C_{cb}(Y), h)$  with  $f(0) = \{0\}$  and  $\alpha > 1$  fulfilling

$$(6) \quad h(f(\alpha x + \beta y) \oplus f(\alpha x - \beta y), 2\alpha^2 f(x) \oplus 2\beta^2 f(y)) \leq \theta (\|x\|^p + \|y\|^p)$$

In addition, if  $\text{diam } f(x) \leq M\|x\|^r$  for every  $x \in X$ , and for some  $r < 2$  and  $M > 0$ , then we achieve a unique determined set-valued Euler-Lagrange mapping  $Q : X \rightarrow Y$  fulfilling for an arbitrary element  $x \in X$

$$h(F(x), Q(x)) \leq \frac{\alpha^{2\theta}}{\alpha^2 - \alpha^p} \|x\|^p.$$

*Proof.* According to Theorem 8, we can easily achieve that for any two elements  $x, y \in X$

$$\varphi(x, y) := \theta (\|x\|^p + \|y\|^p).$$

Then there is the constant  $L = \alpha^{p-2}$ . The desired result can be achieved. □

As a second part of the set-valued stability problems, another set-valued functional equation has been depicted. In contrast with the results of Euler-Lagrange set-valued functional equation, the problem is more complicated.

**THEOREM 10.** Let  $\varphi : X^3 \rightarrow [0, \infty)$  be a suitable control mapping with  $0 \leq L < \frac{\alpha+1}{\alpha+2}$  such that

$$\varphi(x, y, z) \leq \frac{(\alpha + 2)L}{\alpha} \varphi\left(\frac{\alpha}{\alpha + 2}x, \frac{\alpha}{\alpha + 2}y, \frac{\alpha}{\alpha + 2}z\right)$$

for any  $x, y, z \in X$ . Assume that an operator can be given  $f : X \rightarrow (C_{cb}(Y), h)$  and

$$(7) \quad h\left(\alpha f\left(\frac{x+y}{\alpha} + z\right), \alpha f\left(\frac{x-y}{\alpha} + z\right) \oplus f(y)\right) \leq \varphi(x, y, z)$$

for any  $x, y \in X$  and  $0 \leq \alpha < 1$ . In addition, if  $\text{diam } f(x) \leq M\|x\|^r$  for all  $x \in X$ , and for some  $r < \frac{\log(\alpha+1)}{\log(\alpha+2)}$ ,  $M > 0$ , then we achieve a unique determined set-valued Cauchy-Jensen mapping  $Q : X \rightarrow (C_{cb}(Y), h)$  fulfilling the following approximate inequality

$$h(f(x), Q(x)) \leq \frac{\alpha + 1}{\alpha + 1 - (\alpha + 2)L} \varphi(x, x, x)$$

for every  $x \in X$ .

*Proof.* By setting  $x = y = z$  in (7), due to convexity of  $f(x)$ , there has

$$h\left(\alpha f\left(\frac{\alpha + 2}{\alpha}x\right), (\alpha + 1)f(x)\right) \leq \varphi(x, x, x)$$

whence

$$h\left(\frac{\alpha}{\alpha + 1}f\left(\frac{\alpha + 2}{\alpha}x\right), f(x)\right) \leq \frac{1}{\alpha + 1} \varphi(x, x, x)$$

for every  $x \in X$ . Assume the following set

$$S := \{g : g : X \rightarrow C_{cb}(Y), g(0) = \{0\}\}$$

and we naturally defined a metric on  $X$  in the following

$$(8) \quad d(g, f) = \inf\{\mu \in (0, \infty) : h(g(x), f(x)) \leq \mu\varphi(x, x, x), x \in X\},$$

whence  $\inf \phi = +\infty$  and and we can conclude a complete generalized metric semigroup  $(S, d)$  (see [6, 22]). According to the analysis, we can define the mapping  $J : S \rightarrow S$  by, for any  $x \in X$

$$Jg(x) = \frac{\alpha}{\alpha + 1}g\left(\frac{\alpha + 2}{\alpha}x\right)$$

From the definition, we can choose  $g, f \in S$  fulfilling  $d(g, f) = \varepsilon$  and an obvious conclusion that  $g \in Y$ . Then, from the definition of the generalized metric, for every  $x \in X$

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, x, x).$$

An easy computation shall that, for every  $x \in X$

$$\begin{aligned} h(Jg(x), Jf(x)) &= h\left(\frac{\alpha}{\alpha + 1}g\left(\frac{\alpha + 2}{\alpha}x\right), \frac{\alpha}{\alpha + 1}f\left(\frac{\alpha + 2}{\alpha}x\right)\right) \\ &= \frac{\alpha}{\alpha + 1}h\left(g\left(\frac{\alpha + 2}{\alpha}x\right), f\left(\frac{\alpha + 2}{\alpha}x\right)\right) \\ &\leq \frac{(\alpha + 2)L}{\alpha + 1}\varepsilon\varphi(x, x, x) \\ &= A\varepsilon\varphi(x, x, x) \end{aligned}$$

where  $A = \frac{(\alpha+2)L}{\alpha+1} < 1$ . From  $d(g, f) = \varepsilon$ , we can get  $d(Jg, Jf) \leq A\varepsilon$  and also it can be rewritten as

$$d(Jg, Jf) \leq Ad(g, f)$$

for any  $g, f \in S$ . There has an operator  $Q : X \rightarrow Y$  fulfilling:

- (1) We achieve a unique determined fixed point  $Q$  of  $J$  in the following subset

$$M = \{g \in S : d(f, g) < \infty\}.$$

such that

$$Q(x) = \frac{\alpha}{\alpha+1} Q\left(\frac{\alpha+2}{\alpha}x\right)$$

for any  $x \in X$  via  $\alpha > 1$ . Hence there is a unique mapping  $Q$  with some  $\mu \in (0, \infty)$  satisfying (8)

$$h(f(x), Q(x)) \leq \mu\varphi(x, x, x)$$

for all  $x \in X$ .

- (2) There has  $d(J^n f, Q) \rightarrow 0$  as  $n \rightarrow \infty$  and conclude that from  $x \in X$

$$\lim_{n \rightarrow \infty} \left(\frac{\alpha}{\alpha+1}\right)^n f\left(\left(\frac{\alpha+2}{\alpha}\right)^n x\right) = Q(x).$$

- (3) There exists  $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$  and conclude that

$$d(f, Q) \leq \frac{\alpha+1}{\alpha+1 - (\alpha+2)L}.$$

Finally, we prove the operator  $Q$  is a set-valued Cauchy-Jensen mapping for any  $x, y \in X$

$$\begin{aligned} & h\left(\alpha Q\left(\frac{x+y}{\alpha} + z\right), \alpha Q\left(\frac{x-y}{\alpha} + z\right) \oplus Q(y)\right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{\alpha}{\alpha+1}\right)^n h\left(f\left(\left(\frac{\alpha+2}{\alpha}\right)^n \left(\frac{x+y}{\alpha^n} + z\right)\right), f\left(\left(\frac{\alpha+2}{\alpha}\right)^n \left(\frac{x-y}{\alpha^n} + z\right)\right) \oplus f\left(\left(\frac{\alpha+2}{\alpha}\right)^n y\right)\right) \\ &\leq \lim_{n \rightarrow \infty} A^n \varphi(x, x, x) = 0. \end{aligned}$$

According to the condition  $\text{diam } f(x) \leq M\|x\|^r$  for any  $x \in X$ , thus there has

$$\text{diam} \left( \left(\frac{\alpha}{\alpha+1}\right)^n f\left(\left(\frac{\alpha+2}{\alpha}\right)^n x\right) \right) \leq \alpha^{(1-r)n} \left[\frac{(\alpha+2)^r}{\alpha+1}\right]^n M\|x\|^r$$

for every  $x \in X$  and thus we prove that

$$Q(x) = \lim_{n \rightarrow \infty} \left(\frac{\alpha}{\alpha+1}\right)^n f\left(\left(\frac{\alpha+2}{\alpha}\right)^n x\right)$$

is a singleton set satisfying the equality for any  $x, y \in X$

$$\alpha f\left(\frac{x+y}{\alpha} + z\right) = \alpha f\left(\frac{x-y}{\alpha} + z\right) \oplus f(y).$$

The desired results can be achieved. □

Nowadays, we will not repeat the similar applications for the Euler-Lagrange set-valued functional equations. In a direct and similar manner, the following results will be presented by a converse direction.

**THEOREM 11.** *Let  $\varphi : X^3 \rightarrow [0, \infty)$  be a suitable control mapping such that there exists a constant  $\alpha > 1$  satisfying*

$$\varphi(x, y, z) \leq \frac{\alpha}{\alpha + 2} \varphi\left(\frac{\alpha + 2}{\alpha}x, \frac{\alpha + 2}{\alpha}y, \frac{\alpha + 2}{\alpha}z\right)$$

for any  $x, y, z \in X$ . Suppose an operator  $f : X \rightarrow (C_{cb}(Y), h)$  with  $f(0) = \{0\}$  satisfying

$$(9) \quad h\left(\alpha f\left(\frac{x + y}{\alpha} + z\right), \alpha f\left(\frac{x - y}{\alpha} + z\right) \oplus f(y)\right) \leq \varphi(x, y, z)$$

for every  $x, y \in X$ . In addition, if  $\text{diam } f(x) \leq M\|x\|^r$  for all  $x \in X$ , and for some  $r > 1$  and  $M > 0$ , then we achieve a unique determined set-valued Cauchy-Jensen mapping  $Q : X \rightarrow (C_{cb}(Y), h)$  fulfilling the following approximate inequality

$$(10) \quad h(f(x), Q(x)) \leq (\alpha + 2)\varphi(x, x, x)$$

for any  $x \in X$ .

*Proof.* By setting  $x = y = z$  in (9), due to convexity of  $f(x)$ , there has

$$h\left(\alpha f\left(\frac{\alpha + 2}{\alpha}x\right), (\alpha + 1)f(x)\right) \leq \varphi(x, x, x)$$

whence

$$h\left(\frac{\alpha + 1}{\alpha}f\left(\frac{\alpha}{\alpha + 2}x\right), f(x)\right) \leq \frac{1}{\alpha}\varphi(x, x, x)$$

for any  $x \in X$ . The process of the certificate of the rest is similar to the certificate of Theorem 10 which will not be repeated later. The desired results can be achieved.  $\square$

Next, we give out the generalization of Euler-Lagrange functional equation. A non-linear mapping  $Q : X \rightarrow Y$  is called Euler-Lagrange quadratic functional equation [16] if it satisfies the fundamental functional equation

$$(11) \quad m_1^2 m_2 Q(a_1 x) + m_1 Q(m_2 a_2 x) = m_0^2 m_2 Q\left(\frac{m_1}{m_0} a_1 x\right) + m_0^2 m_1 Q\left(\frac{m_2}{m_0} a_2 x\right),$$

with

$$m_0 = \frac{m_1 m_2 + 1}{m_1 + m_2}$$

for every  $x \in X$ , and any fixed real numbers  $a_i$  and positive real numbers  $m_i (i = 1, 2)$  :

$$m = \frac{m_1 a_1^2 + m_2 a_2^2}{m_0} > 1.$$

and

$$(12) \quad m_1 m_2 Q(a_1 x_1 + a_2 x_2) + Q(m_2 a_2 x_1 - m_1 a_1 x_2) = (m_1 a_1^2 + m_2 a_2^2) [m_2 Q(x_1) + m_1 Q(x_2)].$$

hold for every 2-dimensional vectors  $(x_1, x_2) \in X^2$ , and any fixed reals  $a_i$  and positive reals  $m_i (i = 1, 2) : m > 1$ .

It is obvious that the mapping  $Q$  has been called quadratic since the above two equations has a common solution  $f(x) = x^2$ . Now, we need use a suitable control function  $\varphi(x, y)$  such that  $\varphi(x, x) = \varphi(x, y)$  from two arbitraries elements  $x, y$  in  $X$ .

**THEOREM 12.** ; Suppose that  $\varphi : X^2 \rightarrow [0, \infty)$  is a suitable control mapping fulfilling

$$\varphi(x, y) \leq \frac{1}{(m_1/m_0) a_1} \varphi((m_1/m_0) a_1 x, (m_2/m_0) a_2 y)$$

and

$$\varphi(x, y) \leq m\varphi\left(\frac{1}{m}x, \frac{1}{m}y\right)$$

for any  $x, y \in X$ , where the related parameters will be determined in the later. Assume that  $f : X \rightarrow (C_{cb}(Y), h)$  is an operator fulfilling  $f(0) = \{0\}$  such that the fundamental functional inequality

$$(13) \quad h\left(\bar{f}(x), \bar{f}(x)\right) \leq \frac{\varphi(x, x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)}$$

establishes for every  $x \in X$ , two arbitraries fixed real numbers  $a_1, a_2$  and two positive real numbers  $m_1, m_2$  satisfying  $m > 1$ , whence

$$\bar{f}(x) = m_0^2 \frac{m_2 f((m_1/m_0) a_1 x) \oplus m_1 f((m_2/m_0) a_2 x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)},$$

via  $m_0 = \frac{m_1 m_2 + 1}{m_1 + m_2}$  and

$$\bar{f}(x) = \frac{m_1 m_2 f(a_1 x) \oplus f(m_2 a_2 x)}{m_2 (m_1 a_1^2 + m_2 a_2^2)},$$

have been called as 2-dimensional quadratic-weighted means of first and second form, respectively, for fixed real  $m > 1$ .

Suppose further that  $f : X \rightarrow (C_{cb}(Y), h)$  is an operator for a suitable control function fulfilling the nonlinear Euler-Lagrange functional inequality

$$(14) \quad h\left(m_1 m_2 f(a_1 x_1 + a_2 x_2) \oplus f(m_2 a_2 x_1 - m_1 a_1 x_2), (m_1 a_1^2 + m_2 a_2^2) [m_2 f(x_1) \oplus m_1 f(x_2)]\right) \leq \varphi(x, y)$$

for all 2-dimensional vectors  $(x_1, x_2) \in X^2$ . In addition, if  $\text{diam } f(x) \leq M \|x\|^r$  for all  $x \in X$ , and for some  $r < 2$  and  $M > 0$ , then there have the limit of the equation

$$Q(x) = \lim_{n \rightarrow \infty} m^{-2n} f(m^n x)$$

for all  $x \in X$ , all  $n \in \mathbb{N}$  and an operator  $Q : X \rightarrow (C_{cb}(Y), h)$  is a unique 2-dimensional Euler-Lagrange quadratic set-valued mapping fulfilling functional equation (12) and mean equation (11) or equivalently Euler-Lagrange equation (12) and mean equation (11), such that

$$d(f, Q) \leq \frac{m c_1}{m - 1}, \quad m > 1$$

for every  $x \in X$  and

$$c_1 = \frac{m + m_1 m + m_1 a_1}{m_1 m_2 m (m_1 a_1^2 + m_2 a_2^2)} \varphi(x, y)$$

satisfying

$$(15) \quad m_1 m_2 Q(a_1 x_1 + a_2 x_2) \oplus Q(m_2 a_2 x_1 - m_1 a_1 x_2) = (m_1 a_1^2 + m_2 a_2^2) [m_2 Q(x_1) \oplus m_1 Q(x_2)].$$

*Proof.* By setting  $(x_1, x_2) = (x, 0)$  in (14), due to convexity of  $f(x)$ , there has

$$(16) \quad h\left(\bar{f}(x), f(x)\right) \leq \frac{\varphi(x, 0)}{m_2(m_1a_1^2 + m_2a_2^2)}$$

for all  $x \in X$ . Set  $(x_1, x_2) = (\frac{m_1a_1x}{m_0}, \frac{m_2a_2x}{m_0})$  in (14), due to convexity of  $f(x)$ , there has

$$(17) \quad h\left(\bar{f}(x), m^{-2}f(mx)\right) \leq \frac{\varphi\left(\frac{m_1a_1x}{m_0}, \frac{m_2a_2x}{m_0}\right)}{m_1m_2m^2}$$

for all  $x \in X$ . Now, we need use the restriction of the function  $\varphi(x, y)$  such that  $\varphi(x, x) = \varphi(x, y)$  from two arbitrariable elements  $x, y$  in  $X$ . Employing the equations (13), (16) and (17), we can achieve

$$(18) \quad \begin{aligned} h(f(x), m^{-2}f(mx)) &\leq h(\bar{f}(x), m^{-2}f(mx)) + h(\bar{f}(x), f(x)) + h(\bar{f}(x), \bar{f}(x)) \\ &\leq \frac{m\varphi(x, 0) + m_1m\varphi(x, 0) + \varphi(m_1a_1x, m_2a_2x)}{m_1m_2m(m_1a_1^2 + m_2a_2^2)} \\ &= \frac{m + m_1m + m_1a_1}{m_1m_2m(m_1a_1^2 + m_2a_2^2)}\varphi(x, y) = c_1 \end{aligned}$$

for all  $x \in X$ .

$$S := \{g : g : X \rightarrow C_b(Y), g(0) = \{0\}\}$$

and therefore we can set the generalized metric on  $X$  as

$$(19) \quad d(g, f) = \inf\{a \in (0, \infty) : h(g(x), f(x)) \leq a\varphi(x, y), \text{ for every } x, y \in X\},$$

whence  $\inf \phi = +\infty$  and we can achieve that  $(S, d)$  is complete generalized metric semigroup (see [6, 22]). According to the analysis, we can define mapping  $J : S \rightarrow S$  by, for every  $x \in X$

$$Jg(x) = m^{-2}g(mx).$$

We claim that  $J$  is a contractive mapping on  $X$  if  $\alpha \in [0, 1)$ . From the definition, we can choose  $g, f \in S$  fulfilling  $d(g, f) = \varepsilon$  and an obvious conclusion that  $g \in Y$ . Then there has, from the definition of the generalized metric, for every  $x \in X$ ,

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, 0).$$

An easy computation shall that, for every  $x \in X$

$$\begin{aligned} h(Jg(x), Jf(x)) &= m^{-2}h(g(mx), f(my)) \\ &\leq m^{-1}\varepsilon\varphi(x, y) \end{aligned}$$

Observing that  $d(g, f) = \varepsilon$ , we can conclude that  $d(Jg, Jf) \leq m^{-1}\varepsilon$  and also it can be rewritten as

$$d(Jg, Jf) \leq m^{-1}d(g, f)$$

for any two elements  $g, f \in S$ . Combined with the equation (18) with  $m > 1$  and by introducing fixed-point alternative theorem, there has an operator  $Q : X \rightarrow Y$  fulfilling:

- (1) There has a unique determined fixed point of  $J$  in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

fulfilling

$$Q(x) = m^{-2}Q(mx)$$

for any  $x \in X$  and  $\alpha > 1$ . Hence the mapping  $Q$  fulfilling via some  $\mu \in (0, \infty)$  and for every  $x \in X$

$$h(f(x), Q(x)) \leq \mu\varphi(x, 0).$$

(2) There has  $d(J^n f, Q) \rightarrow 0$  as  $n \rightarrow \infty$  and concludes that for all  $x \in X$

$$\lim_{n \rightarrow \infty} m^{-2n} Q\left(\frac{x}{m^n}\right) = Q(x).$$

(3) There exist  $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$  and

$$d(f, Q) \leq \frac{m c_1}{m - 1}.$$

Finally, due to the condition  $m > 1$  and , we prove the operator  $Q$  is the nonlinear Euler-Lagrange set-valued mapping for all  $x, y \in X$

$$\begin{aligned} & h\left(m_1 m_2 Q(a_1 x_1 + a_2 x_2) \oplus Q(m_2 a_2 x_1 - m_1 a_1 x_2), (m_1 a_1^2 + m_2 a_2^2) [m_2 Q(x_1) \oplus m_1 Q(x_2)]\right) \\ &= \lim_{n \rightarrow \infty} m^{-2n} h\left(m_1 m_2 f(a_1 m^n x_1 + a_2 m^n x_2) \oplus f(m_2 a_2 m^n x_1 - m_1 a_1 m^n x_2), \right. \\ &\quad \left. (m_1 a_1^2 + m_2 a_2^2) [m_2 f(m^n x_1) \oplus m_1 f(m^n x_2)]\right) \\ &\leq \lim_{n \rightarrow \infty} m^{-2n} \varphi(m^n x, m^n y) \\ &\leq \lim_{n \rightarrow \infty} m^{-n} \varphi(x, y) = 0. \end{aligned}$$

According to the condition  $\text{diam } f(x) \leq M\|x\|^r$  for all  $x \in X$ ,  $\text{diam } m^{-2n} f(m^n x) \leq m^{r n - 2n} M\|x\|^r$  for every  $x \in X$  and hence  $Q(x) = \lim_{n \rightarrow \infty} m^{-2n} f(m^n x)$  is a singleton set satisfying the equality for all  $x, y \in X$

$$m_1 m_2 Q(a_1 x_1 + a_2 x_2) \oplus Q(m_2 a_2 x_1 - m_1 a_1 x_2) = (m_1 a_1^2 + m_2 a_2^2) [m_2 Q(x_1) \oplus m_1 Q(x_2)].$$

This completes the desired assertion. □

For some particular parameters  $m_1 = 1, m_2 > 0$ , there exist  $m_0 = 1$  and  $m_1 = a_1^2 + m_2 a_2^2 > 1$ . For the particular case, we do not need the fundamental functional equation (13) where the results of this case can be stated in the following.

**COROLLARY 13.** *Suppose that  $\varphi : X^2 \rightarrow [0, \infty)$  is a suitable control mapping fulfilling*

$$\varphi(x, y) \leq m\varphi\left(\frac{1}{m}x, \frac{1}{m}y\right)$$

for any  $x, y \in X$ , where the related parameters will be determined in the later.

Suppose further that  $f : X \rightarrow (C_{cb}(Y), h)$  is an operator fulfilling the nonlinear Euler-Lagrange functional inequality

$$(20) \quad h\left(m_1 m_2 f(a_1 x_1 + a_2 x_2) \oplus f(m_2 a_2 x_1 - m_1 a_1 x_2), (m_1 a_1^2 + m_2 a_2^2) [m_2 f(x_1) \oplus m_1 f(x_2)]\right) \leq \varphi(x, y)$$

for all 2-dimensional vectors  $(x_1, x_2) \in X^2$  and any fixed reals  $a_1, a_2$  and positive reals  $m_2$  :

$$m = \frac{m_1 a_1^2 + m_2 a_2^2}{m_0} = \frac{m_1 + m_2}{m_1 m_2 + 1} (m_1 a_1^2 + m_2 a_2^2) > 1.$$



Moreover, if  $\text{diam } f(x) \leq M\|x\|^r$  for all  $x \in X$ , and for some  $r < 2$  and  $M > 0$ . Then there have the limit of the equation

$$Q(x) = \lim_{n \rightarrow \infty} m^{-2n} f(m^n x)$$

for all  $x \in X$ , all  $n \in N$ , and an arbitrary positive numbers  $m$  with  $m > 1$  and an operator  $Q : X \rightarrow (C_{cb}(Y), h)$  is a unique 2-dimensional Euler-Lagrange quadratic set-valued mapping fulfilling

$$d(f, Q) \leq \frac{mc_1}{m-1}, \quad m > 1$$

for every  $x \in X$  and

$$c_1 = \frac{1 + m_1}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)} \varphi(x, y)$$

satisfying

$$(21) \quad m_1 m_2 Q(a_1 x_1 + a_2 x_2) \oplus Q(m_2 a_2 x_1 - m_1 a_1 x_2) = (m_1 a_1^2 + m_2 a_2^2) [m_2 Q(x_1) \oplus m_1 Q(x_2)].$$

In fact, we can achieve the Euler-Lagrange functional equation from (11) if  $m_1 = m_2 = 1$ . Next, we give another similar results in a converse direction by using the similar method. However, it is not like the above dealing way of two functional equations. A non-linearly analogous mapping  $Q : X \rightarrow Y$  is called Euler-Lagrange quadratic functional equation [16] if it is satisfies (12) and the fundamental functional equation

$$(22) \quad m_1^2 m_2 Q\left(\frac{a_1 x}{m}\right) + m_1 Q\left(\frac{m_2 a_2}{m} x\right) = m_0^2 m_2 Q\left(\frac{m_1}{m m_0} a_1 x\right) + m_0^2 m_1 Q\left(\frac{m_2}{m m_0} a_2 x\right).$$

**THEOREM 14.** Suppose that  $\varphi : X^2 \rightarrow [0, \infty)$  is a suitable control mapping fulfilling

$$\varphi(x, y) \leq \frac{1}{(m_1/m_0) a_1} \varphi((m_1/m_0) a_1 x, (m_2/m_0) a_2 y)$$

and

$$\varphi(x, y) \leq \frac{1}{m} \varphi(mx, my)$$

for any  $x, y \in X$ , where the related parameters will be determined in the later. Assume that  $f : X \rightarrow (C_{cb}(Y), h)$  is an operator fulfilling  $f(0) = \{0\}$  such that the fundamental functional inequality

$$(23) \quad h(\bar{f}(x), \bar{f}(x)) \leq \frac{(m_1 a_1^2 + m_2 a_2^2) (m_1 + m_2)^2 \varphi(x, x)}{m_1 m_2 (m_1 m_2 + 1)^2}$$

holds for every  $x \in X$ , two arbitraries fixed real numbers  $a_1, a_2$  and two positive real numbers  $m_1, m_2$  satisfying  $m > 1$ , whence

$$(24) \quad \bar{f}(x) = m_0^2 m^2 \frac{m_2 f((m_1/m m_0) a_1 x) \oplus m_1 f((m_2/m m_0) a_2 x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)},$$

via  $m_0 = \frac{m_1 m_2 + 1}{m_1 + m_2}$  and

$$(25) \quad \bar{f}(x) = m^2 \frac{m_1 m_2 f\left(\frac{m_1 a_1}{m} x\right) \oplus f\left(\frac{m_2 a_2}{m} x\right)}{m_2 (m_1 a_1^2 + m_2 a_2^2)},$$

have been called as 2-dimensional quadratic-weighted means of first and second form, respectively, for fixed real  $0 < m < 1$ .

Suppose further that  $f : X \rightarrow (C_{cb}(Y), h)$  is an operator fulfilling the nonlinear Euler-Lagrange functional inequality

$$(26) \quad h \left( m_1 m_2 f(a_1 x_1 + a_2 x_2) \oplus f(m_2 a_2 x_1 - m_1 a_1 x_2), \right. \\ \left. (m_1 a_1^2 + m_2 a_2^2) [m_2 f(x_1) \oplus m_1 f(x_2)] \right) \leq \varphi(x, y)$$

for all 2-dimensional vectors  $(x_1, x_2) \in X^2$ .

In addition, if  $\text{diam } f(x) \leq M \|x\|^r$  for all  $x \in X$ , and for some  $r < 2$  and  $M > 0$ , then there have the limit of the equation

$$Q(x) = \lim_{n \rightarrow \infty} m^{2n} f(m^{-n} x)$$

for all  $x \in X$ , all  $n \in \mathbb{N}$  and an operator  $Q : X \rightarrow (C_{cb}(Y), h)$  is a unique 2-dimensional Euler-Lagrange quadratic set-valued mapping fulfilling functional equation (25) and mean equation (22) or equivalently Euler-Lagrange equation (25) and mean equation (22), such that

$$d(f, Q) \leq \frac{m c_2}{1 - m}, \quad 0 < m < 1$$

for every  $x \in X$  and

$$c_2 = \frac{m + m_1 m_2 + m_1 a_1}{m_1 m^2 m_0 m} \varphi(x, y)$$

satisfying

$$(27) \quad m_1 m_2 Q(a_1 x_1 + a_2 x_2) \oplus Q(m_2 a_2 x_1 - m_1 a_1 x_2) = \\ (m_1 a_1^2 + m_2 a_2^2) [m_2 Q(x_1) \oplus m_1 Q(x_2)].$$

*Proof.* By setting  $(x_1, x_2) = (\frac{x}{m}, 0)$  in (26), due to convexity of  $f(x)$ , there has

$$(28) \quad h \left( \bar{f}(x), m^2 f(m^{-1} x) \right) \leq \frac{m \varphi(x, 0)}{(m_0 m m_2)}$$

for all  $x \in X$ . Set  $(x_1, x_2) = (\frac{m_1 a_1 x}{m m_0}, \frac{m_2 a_2 x}{m m_0})$  in (26), due to convexity of  $f(x)$ , there has

$$(29) \quad h \left( \bar{f}(x), f(x) \right) \leq \frac{\varphi \left( \frac{m_1 a_1 x}{m m_0}, \frac{m_2 a_2 x}{m m_0} \right)}{m_1 m_2}$$

for all  $x \in X$ . Now, we need to use the restriction of the function  $\varphi(x, y)$  such that  $\varphi(x, x) = \varphi(x, y)$  from two arbitrary elements  $x, y$  in  $X$ . Employing the equations (26), (28) and (29), we can achieve

$$(30) \quad h(f(x), m^{-2} f(mx)) \leq h(\bar{f}(x), f(x)) + h(\bar{f}(x), m^2 f(m^{-1} x)) + h(\bar{f}(x), \bar{f}(x)) \\ \leq \frac{m + m_1 m^2 + m_1 a_1}{m_1 m_2 m_0 m} \varphi(x, y) = c_2$$

for all  $x \in X$ .

$$S := \{g : g : X \rightarrow C_{cb}(Y), g(0) = \{0\}\}$$

and therefore we can set the generalized metric on  $X$  as

$$(31) \quad d(g, f) = \inf\{a \in (0, \infty) : h(g(x), f(x)) \leq a \varphi(x, y), \text{ for every } x, y \in X\},$$

whence  $\inf \phi = +\infty$  and we can achieve that  $(S, d)$  is complete generalized metric semigroup (see [6, 22]). According to the analysis, we can define mapping  $J : S \rightarrow S$  by, for every  $x \in X$

$$Jg(x) = m^2g(m^{-1}x).$$

We claim that  $J$  is a contractive mapping on  $X$  if  $m \in (0, 1)$ . From the definition, we can choose  $g, f \in S$  fulfilling  $d(g, f) = \varepsilon$  and an obvious conclusion that  $g \in Y$ . Then there has, from the definition of the generalized metric, for every  $x \in X$ ,

$$h(g(x), f(x)) \leq \varepsilon\varphi(x, y).$$

An easy computation shall that, for every  $x \in X$

$$\begin{aligned} h(Jg(x), Jf(x)) &= m^2h(g(m^{-1}x), f(m^{-1}y)) \\ &\leq m\varepsilon\varphi(x, y) \end{aligned}$$

Observing that  $d(g, f) = \varepsilon$ , we can conclude that  $d(Jg, Jf) \leq m\varepsilon$  and also it can be rewritten as

$$d(Jg, Jf) \leq md(g, f)$$

for any two elements  $g, f \in S$ . The rest of the process of this proof are similar to the corresponding the process of the proof of Theorem 12 which we omit it here. This completes the desired assertion.  $\square$

For some particular parameters  $m_2 = 1, m_1 > 0$ , there exist  $m_0 = 1$  and  $m = m_1a_1^2 + a_2^2 < 1$ . For the particular case, we do not need the fundamental functional equation (23) where the results of this case can be stated in the following.

**COROLLARY 15.** *Suppose that  $\varphi : X^2 \rightarrow [0, \infty)$  is a suitable control mapping fulfilling*

$$\varphi(x, y) \leq m\varphi\left(\frac{1}{m}x, \frac{1}{m}y\right)$$

for any  $x, y \in X$ , where the related parameters will be determined in the later.

Suppose further that  $f : X \rightarrow (C_{cb}(Y), h)$  is an operator fulfilling the nonlinear Euler-Lagrange functional inequality

$$(32) \quad h\left(m_1m_2f(a_1x_1 + a_2x_2) \oplus f(m_2a_2x_1 - m_1a_1x_2), (m_1a_1^2 + m_2a_2^2)[m_2f(x_1) + m_1 \oplus f(x_2)]\right) \leq \varphi(x, y)$$

holds for all 2-dimensional vectors  $(x_1, x_2) \in X^2$  and any fixed reals  $a_1, a_2$ .

In addition, if  $\text{diam } f(x) \leq M\|x\|^r$  for all  $x \in X$ , and for some  $r < 2$  and  $M > 0$ , then there have the limit of the equation

$$Q(x) = \lim_{n \rightarrow \infty} m^{-2n}f(m^n x)$$

for all  $x \in X$ , all  $n \in N$ , and an arbitrary positive numbers  $m$  with  $0 < m < 1$  and an operator  $Q : X \rightarrow (C_{cb}(Y), h)$  is a unique 2-dimensional Euler-Lagrange quadratic set-valued mapping fulfilling

$$d(f, Q) \leq \frac{mc_1}{m-1}, \quad 0 < m < 1$$

for every  $x \in X$  and

$$c_1 = \frac{1 + m_1}{m_1m_2(m_1a_1^2 + m_2a_2^2)}\varphi(x, y)$$

In particular, there has the identical equation

$$Q(x) = m^{-2n}Q(m^n x)$$

for every  $x \in X$ , and any  $n \in N$ , and real numbers  $a_1, a_2$  and positive real numbers  $m_1, m_2$  satisfying

$$(33) \quad m_1 m_2 Q(a_1 x_1 + a_2 x_2) \oplus Q(m_2 a_2 x_1 - m_1 a_1 x_2) = (m_1 a_1^2 + m_2 a_2^2) [m_2 Q(x_1) + m_1 \oplus Q(x_2)].$$

For further studying in various normed space, we will study the stability problem in the sense of random normed space, quasi- $(2, \beta)$ -Banach space and even more general settings in the future. It particularly interested that we can also study the hyper-stability results for radical-type functional equations in normed spaces of different types.

## References

- [1] A. Batool, S. Nawaz, O. Ege, de la Sen, M. *Hyers-Ulam stability of functional inequalities: A fixed point approach*, Journal of Inequalities and Applications, **251** (2020), 1–18.
- [2] C. Castaing, M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics Springer, Berlin 1977, 580.
- [3] C. Baak, *Cauchy-Rassias stability of Cauchy-Jensen additive mappings in Banach spaces*, Acta Math. Sin. **22** (2006), 1789–1796.
- [4] D.H. Hyers, On the stability of the linear functional equation. Proc. Nat. Acad. Sci. (1941, 27, 222–224.
- [5] D. Zhang , Q. Liu, J.M. Rassias, Y., Li, *The stability of functional equations with a new direct method*, Mathematics. **7** (2022), 1188.
- [6] D. Mihet, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
- [7] D. Marinescu, M. Monea, M. Opincariu, M. Stroe, Some equivalent characterizations of inner product spaces and their consequences. Filomat. (2015, 29(7), 1587–1599.
- [8] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis.Milano **53** (1983), 113–129.
- [9] G.L. Forti, E. Shulman, *A comparison among methods for proving stability*, Aequationes Math. **94** (2020), 547–574.
- [10] G. Debreu, *Integration of correspondences. In: Proceedings of Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Journal of Fixed Point Theory and Applications Vol. II, Part I (1966), 351–372.
- [11] G. Isac, T.M. Rassias, *Stability of  $\Psi$ -additive mappings: applications to nonlinear analysis*, Int. J. Math. Math. Sci. **19** (1996), 219–228.
- [12] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge Univ. Press 1999.
- [13] J.R. Lee, C. Park, D.Y. Shin, S. Yun, *Set-Valued Quadratic Functional Equations*, Results Math. **17** (2017), 1422–6383.
- [14] J.A. Baker, *The stability of certain functional equations*, Bull. Am. Math. Soc. **112** (1991), 729–732.
- [15] J.B. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Am. Math. Soc. **74** (1968), 305–309.
- [16] J.M. Rassias, *Solution of the Ulam Stability Problem for Euler-Lagrange Quadratic Mappings*, J. Math. Anal. Appl. **220** (1998), 613–639.
- [17] K. Karthikeyan, G.S. Murugapandian, O. Ege, *On the solutions of fractional integro-differential equations involving Ulam-Hyers-Rassias stability results via  $\psi$ -fractional derivative with boundary value conditions*, Turkish Journal of Mathematics **46** (6) (2022), 2500–2512.
- [18] K.W. Jun, H. M. Kim, *On the Hyers-Ulam stability of a generalized quadratic and additive functional equation*, Bulletin of the Korean Mathematical Society **42** (1) (2005), 5133–148.

- [19] L. Čadariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
- [20] L. Čadariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **15** (2008), Art. ID 749392.
- [21] M. Mirzavaziri, M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
- [22] M.E. Gordji, C. Park, M.B. Savadkouhi, *The stability of a quartic type functional equation with the fixed point alternative*, Fixed Point Theory **11** (2010), 265–272.
- [23] O. Ege, S. Ayadi, C. Park, *Ulam-Hyers stabilities of a differential equation and a weakly singular Volterra integral equation*, Journal of Inequalities and Applications **19** (2021), 1–12.
- [24] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
- [25] P. Kaskasem, C. Klin-eam, Y.J. Cho, *On the stability of the generalized Cauchy-Jensen set-valued functional equations*, J. Fixed Point Theory Appl. **10** (2018), 1007.
- [26] R. Ger, *On alienation of two functional equations of quadratic type*, Aequat. Math. **19** (2021), 1169–1180.
- [27] S.M. Ulam, *Problems in Modern Mathematics, Chapter IV, Science Editions*, Wiley, New York 1960.
- [28] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.
- [29] S.M. Jung, *On the Hyers–Ulam–Rassias Stability of a Quadratic Functional Equation*, J. Math. Anal. Appl. **68** (1999), 384–393.
- [30] S.S. Kim, J.M. Rassias, N. Hussain, Y.J. Cho, *Generalized Hyers-Ulam stability of general cubic functional equation in random normed spaces*, Filomat. **1** (2016), 89–98.
- [31] T.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.
- [32] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
- [33] W.A.J. Luxemburg, *On the convergence of successive approximations in the theory of ordinary differential equations II. In: Proceedings of the Koninklijke Nederlandse Akademie Van Wetenschappen*, Amsterdam, Series A (5) Indag. Math. **61** (1958), 540–546.
- [34] Y.J. Cho, C. Park, T.M. Rassias, R. Saadati, *Stability of Functional Equations in Banach Algebras*, Springer International Publishing. Basel 2015.
- [35] Y.J. Cho, T.M. Rassias, R. Saadati, *Stability of Functional Equations in Random Normed Spaces*, Springer, New York 2013.
- [36] Y.J. Cho, R. Saadati, J. Vahidi, *Approximation of homomorphisms and derivations on non-Archimedean Lie  $C^*$ -algebras via fixed point method*, Discrete Dyn. Nat. Soc. 2012.
- [37] Y. Lee, *Stability of a generalized quadratic functional equation with jensen type*, Bulletin of the Korean Mathematical Society **42** (1) (2005), 57–73.
- [38] Z.X. Gao, H.X. Cao, W.T. Zheng, L. Xu, *Generalized Hyers–Ulam–Rassias stability of functional inequalities and functional equations*, J. Math. Inequal. **3** (2009), 63–67.

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