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THE MEANING OF THE CONCEPT OF LACUNARY STATISTICAL CONVERGENCE IN G-METRIC SPACES

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Abstract. In this study, the concept of lacunary statistical convergence is studied in G-metric spaces. The G-metric function is based on the concept of distance between three points. Considering this new concept of distance, we examined the relationships between GS, $GS_\theta, G\sigma_1$ and GN_θ sequence spaces.

1. Introduction

In this section, some basic definitions and results related to statistical convergence, lacunary statistical convergence and $G-$metric spaces are presented and discussed.

Zygmund first mentioned the concept of statistical convergence in her monograph in 1935 in Warsaw and it was formally introduced by Fast [6] and Steinhaus [28], independently. Later on, Schoenberg gave some basic properties of statistical convergence and studied as a summability method [27]. After the 1950s, studies on the concept of statistical convergence made rapid progress and many studies were conducted on this subject. The most well-known of these areas are number theory by Erdos and Tenenbaum [5], measure theory by Miller [19], trigonometric series by Zygmund [31] and summability theory by Freedman and Sember [7]. Also Fridy has an important study in which he studied the properties of statistical convergence [8] and Maio studied statistical convergence in topological spaces [18]. This concept was also studied with ideals, weak convergence, modulus functions and $p-$Cesáro convergence (see [3], [14], [15], [26]). Statistical convergence is based on the definition of natural density of the set $A \subseteq \mathbb{N}$ such as $d(A) = \lim\limits_{n\to\infty} \frac{A_n}{n}$ where $A_n = \{k \in A : k \leq n\}$ and $|A_n|$ gives the cardinality of A_n.

Definition 1.1. [6] A number sequence (x_k) is statistically convergent to L provided that for every $\varepsilon > 0$, $d(\{k \leq n : |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $st - \lim x_k = L$ and usually the set of statistically convergent sequences is denoted by S.

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Considering the definition of natural density, this definition can also be expressed as for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |k \leq n : |x_k - L| \geq \varepsilon| = 0.$$

Lacunary statistical convergence was defined by Fridy and Orhan in [9]. Before giving this definition, let's remind the definition of a lacunary sequence.

Definition 1.2. A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals $I_r = (k_{r-1}, k_r]$ are determined by θ and the ratio is determined $q_r = \frac{k_r}{k_{r-1}}$.

Example 1.3. $\theta = (r^2)$ is a lacunary sequence because $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

Example 1.4. $\theta = (r)$ is not a lacunary sequence because $k_0 = 0$ but $h_r = k_r - k_{r-1} = 1$ for all $r = 0, 1, ...$

Definition 1.5. [9] Let $\theta = (k_r)$ be a lacunary sequence. The number sequence $x = (x_k)$ is lacunary statistically convergent (or $S_\theta-$convergent) to L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |k \in I_r : |x_k - L| \geq \varepsilon| = 0.$$

In this case we write $S_\theta - \lim x_k = L$ and usually the set of lacunary statistically convergent sequences is denoted by S_θ.

Another concept closely related to statistical convergence is strong Cesáro summability:

$$|\sigma_1| := \left\{ x : \text{for some } L,\ \lim_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k - L| \right) = 0 \right\}.$$

Similarly, there is a close relationship between strong Cesáro summability and N_θ space:

$$N_\theta := \left\{ x : \text{for some } L,\ \lim_r \left(\frac{1}{h_r} \sum_{k\in I_r} |x_k - L| \right) = 0 \right\}.$$

As it is known, metric spaces are based on the concept of distance, and distance function is an important concept in mathematics and many other fields. Today, due to very large and complex data sets, the definition of the distance function needs to be generalized. For this purpose, many studies have already been carried out (see [2], [4], [13], [16], [17], [10], [11]).On the other hand, Mishra and his friends have similar studies in other metric spaces (see [20], [21], [22], [25], [29],([30]). Among them, we will be particularly interested in $G-$metric spaces, which allows us to establish many topological properties.

During the sixties, Gahler introduced $2-$metric spaces with a $2-$metric $d : X \times X \times X \to \mathbb{R}^+$ where X is a nonempty set and $\mathbb{R}$ is real numbers (see [12], [13]) .

Definition 1.6. [12] Let X be a nonempty set. A function $d : X \times X \times X \to \mathbb{R}^+$ satisfying the following axioms:

$d1)$ For every distinct points $x, y \in X$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

$d2)$ If at least two of three points x, y, z are the same then $d(x, y, z) = 0$.

$d3)$ $d(x, y, z) = d(x, z, y) = d(y, z, x) = ...$ (symmetry)

$d4)$ $d(x, y, z) \leq d(x, y, t) + d(y, z, t) + d(z, x, t)$ for all $x, y, z, t \in X$ (rectangle inequality)

is called a $2-$metric on X and (X, d) is called a $2-$metric space.

Gahler claimed that a $2-$metric is a generalization of the usual notion of a metric, but different authors proved that there is no relation between these two functions. Further, there is no easy relationship between results obtained in the two metrics.

These considerations led Dhage to define a new generalized metric space called $D-$metric space [4]. A function $D : X \times X \times X \to \mathbb{R}^+$ is a $D-$metric with four properties. In a $D-$metric, the two properties in the $2-$metric remain the same, but the other two properties are replaced by another property and an additional property is added. In a subsequent series of papers Dhage attempted to develop topological structures in such spaces. Subsequently, these works have been the basis for over 40 papers by Dhage and other authors. However, several errors for fundamental topological properties in a D-metric space were found in [23] and [24].

Definition 1.7. [4] Let X be a nonempty set. A function $D : X \times X \times X \to \mathbb{R}^+$ satisfying the following axioms:

$D1)$ $D(x, y, z) \geq 0$ for all $x, y, z \in X$.

$D2)$ $D(x, y, z) \geq 0$ if and only if $x = y = z$.

$D3)$ $D(x, y, z) = D(x, z, y) = D(y, z, x) = ...$ (symmetry in all three variables)

$D4)$ $D(x, y, z) \leq D(x, y, t) + D(x, t, z) + D(t, y, z)$ for all $x, y, z, t \in X$ (rectangle inequality)

is called a $D-$metric on X and (X, D) is called a $D-$metric space.

2. Statistical Convergence in $G-$metric spaces

All these developments led Mustafa and Sims to the idea of defining a more appropriate generalized metric space and they defined $G-$metric spaces with five properties. These properties are satisfied when $G(x, y, z)$ is the perimeter of a triangle with vertices at x, y and z in $\mathbb{R}^2$, further taking a in the interior of the triangle shows that $(G5)$ is best possible. $G-$metric function is a distance function that generalizes the concept of distance between 3 points [23].

Definition 2.1. [23] Let X be a nonempty set. The $G : X \times X \times X \to \mathbb{R}^+$ function that provides the following properties is called generalized metric or briefly $G-$metric on X.

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$

$(G2)$ $0 < G(x, y, z)$, for all $x, y \in X$ with $x \neq y$

$(G3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables)

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality)

The pair (X, G) is called by a $G-$metric space.

Example 2.2. Let $d(x,y,z)$ be the perimeter of the triangle with vertices at $x,y,z \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a $G-$metric space.

Example 2.3. Let $X = \{x,y\}$ and let $G(x,x,x) = G(y,y,y) = 0$, $G(x,x,y) = 1$, $G(x,y,y) = 2$ and extend G to all of $X \times X \times X$ by symmetry in the variables. Then G is a $G-$metric which is not symmetric.

Example 2.4. Let (X,d) be a metric space. The function

$$\psi(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$$

is a $G-$metric where $\psi : X \times X \times X \to \mathbb{R}^+$.

After all this work, Abazari defined statistical convergence in $g-$metric spaces and studied some basic properties [1]. In a $g-$metric space, the distance function defined between $n+1$ points. In this section, we redefine some important definitions that Abazari studied in $g-$metric spaces for $G-$metric spaces.

Definition 2.5. Let $A \in \mathbb{N}^2$ and $A(n) = \{i_1, i_2 \leq n : (i_1, i_2) \in A\}$ then, $\rho_1(A) := \lim_{n\to\infty} \frac{2}{n^2} |A(n)|$ is called 2-dimensional asymptotic (or natural) density of the set A.

Definition 2.6. Let (x_i) be a sequence in a $G-$metric space (X,G). For every $\varepsilon > 0$, if

$$\lim_{n\to\infty} \frac{2}{n^2} |\{(i_1,i_2) \in A : i_1, i_2 \leq n,\ G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}| = 0$$

then, (x_i) statistically converges to x in X. This situation is denoted by $GS-\lim x_i = x$ or $x_i \overset{GS}{\to} x$. The set of all statistically convergent sequences in a $G-$metric space is denoted by GS.

The following theorems, which occupy an important place in $G-$metric spaces, are proved for $g-$metric spaces in Abazari's article [1]. Therefore, we do not give the proof of these theorems in this section.

Theorem 2.7. *In $G-$metric spaces, every convergent sequence is statistically convergent.*

Theorem 2.8. *Statistical limit of a sequence in a $G-$metric space is unique.*

Theorem 2.9. *In $G-$metric spaces, every statistically convergent sequence has a convergent subsequence.*

3. Main Results

In this section the main definitions and results are introduced and discussed. First of all, we consider the definition of lacunary statistical convergence in $G-$metric spaces.

Definition 3.1. Let (X,G) be a $G-$metric space, (x_i) be a sequence in this space and θ be a lacunary sequence. The sequence (x_i) is said to be lacunary statistically convergent to x in X provided that for all $\varepsilon > 0$,

$$\lim_r \frac{2}{h_r^2} |\{i_1, i_2 \in I_r : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}| = 0.$$

This situation is denoted by $GS_\theta - \lim x_i = x$ or $x_i \overset{GS_\theta}{\to} x$. The set of all lacunary statistically convergent sequences in X is denoted by GS_θ.

DEFINITION 3.2. Let (X,G) be a $G-$metric space and (x_i) be a sequence in this space. The sequence (x_i) is said to be $G\sigma_1-$summable to x provided that

$$\lim_n \frac{2}{n^2}\sum_{i_1,i_2=1}^{n} G(x,x_{i_1},x_{i_2}) = 0.$$

This situation is denoted by $G\sigma_1-\lim x_i = x$ or $x_i \stackrel{G\sigma_1}{\to} x$. The set of all $G\sigma_1-$summable sequences in X is denoted by $G\sigma_1$.

DEFINITION 3.3. Let (X,G) be a $G-$metric space, (x_i) be a sequence in this space and θ be a lacunary sequence. The sequence (x_i) is said to be $GN_\theta-$summable to x provided that

$$\lim_r \frac{2}{h_r^2}\sum_{i_1,i_2\in I_r} G(x,x_{i_1},x_{i_2}) = 0.$$

This situation is denoted by $GN_\theta-\lim x_i = x$ or $x_i \stackrel{GN_\theta}{\to} x$. The set of all $GN_\theta-$summable sequences in X is denoted by GN_θ.

First, let's prove two theorems that give the relationship between $G\sigma_1$ and GS.

THEOREM 3.4. *Let (X,G) be a $G-$metric space and (x_i) be a sequence in this space.*

$$\textit{If } x_i \stackrel{G\sigma_1}{\to} x \ \textit{ then } x_i \stackrel{GS}{\to} x.$$

Proof. Suppose that $x_i \stackrel{G\sigma_1}{\to} x$ and $\varepsilon > 0$ be given. Then,

$$\begin{aligned} \tfrac{2}{n^2}\textstyle\sum_{i_1,i_2=1}^{n} G(x,x_{i_1},x_{i_2}) &\geq \tfrac{2}{n^2}\displaystyle\sum_{\substack{i_1,i_2=1\\ G(x,x_{i_1},x_{i_2})\geq\varepsilon}}^{n} G(x,x_{i_1},x_{i_2}) \\ &\geq \varepsilon\tfrac{2}{n^2}\left|\{i_1,i_2\leq n : G(x,x_{i_1},x_{i_2})\geq\varepsilon\}\right| \end{aligned}$$

If we do the necessary mathematical operations and take the limit of both sides $x_i \stackrel{GS}{\to} x$ is obtained. □

THEOREM 3.5. *Let (X,G) be a $G-$metric space and G be a bounded function in X.*

$$\textit{If } x_i \stackrel{GS}{\to} x \ \textit{ then } x_i \stackrel{G\sigma_1}{\to} x.$$

Proof. This time suppose that $x_i \stackrel{GS}{\to} x$ and $\varepsilon > 0$ be given. From the boundedness of G there is a positive M such that $G(x,x_{i_1},x_{i_2}) \leq M$ for all $x, x_{i_1}, x_{i_2} \in X$. Then,

$$\begin{aligned} \tfrac{2}{n^2}\textstyle\sum_{i_1,i_2=1}^{n} G(x,x_{i_1},x_{i_2}) &= \tfrac{2}{n^2}\displaystyle\sum_{\substack{i_1,i_2=1\\ G(x,x_{i_1},x_{i_2})\geq\varepsilon}}^{n} G(x,x_{i_1},x_{i_2}) + \tfrac{2}{n^2}\sum_{\substack{i_1,i_2=1\\ G(x,x_{i_1},x_{i_2})<\varepsilon}}^{n} G(x,x_{i_1},x_{i_2}) \\ &\leq M\tfrac{2}{n^2}\left|\{i_1,i_2\leq n : G(x,x_{i_1},x_{i_2})\geq\varepsilon\}\right| + \varepsilon. \end{aligned}$$

Considering the limits of both side we have $x_i \stackrel{G\sigma_1}{\to} x$. □

In the following two theorems, we explain the relationship between GS_θ and GN_θ and the role of boundedness in this relationship.

THEOREM 3.6. *Let (X,G) be a $G-$metric space, (x_i) be a sequence in this space and θ be a lacunary sequence.*

$$\textit{If } x_i \stackrel{GN_\theta}{\to} x \ \textit{ then } x_i \stackrel{GS_\theta}{\to} x.$$

Proof. If $\varepsilon > 0$ and $x_i \overset{GN_\theta}{\to} x$ we have,

$$\lim_r \frac{2}{h_r^2} \sum_{i_1,i_2 \in I_r} G(x, x_{i_1}, x_{i_2}) = 0.$$

With this information we can write that

$$\begin{aligned} \frac{2}{h_r^2} \textstyle\sum_{i_1,i_2 \in I_r} G(x, x_{i_1}, x_{i_2}) &\geq \frac{2}{h_r^2} \textstyle\sum_{\substack{i_1,i_2 \in I_r \\ G(x,x_{i_1},x_{i_2}) \geq \varepsilon}} G(x, x_{i_1}, x_{i_2}) \\ &\geq \varepsilon \frac{2}{h_r^2} \left|\{i_1, i_2 \in I_r : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}\right| \end{aligned}$$

which yields the result. □

THEOREM 3.7. *Let (X, G) be a $G-$metric space, θ be a lacunary sequence and G be a bounded function in X.*

$$\text{If } x_i \overset{GS_\theta}{\to} x \text{ then } x_i \overset{GN_\theta}{\to} x.$$

Proof. Let $\varepsilon > 0$, G be a bounded function and $x_i \overset{GS_\theta}{\to} x$. From the boundedness of G there is a positive M such that $G(x, x_{i_1}, x_{i_2}) \leq M$ for all $x, x_{i_1}, x_{i_2} \in X$. Hence,

$$\begin{aligned} \frac{2}{h_r^2} \textstyle\sum_{i_1,i_2 \in I_r} G(x, x_{i_1}, x_{i_2}) &= \frac{2}{h_r^2} \textstyle\sum_{\substack{i_1,i_2 \in I_r \\ G(x,x_{i_1},x_{i_2}) \geq \varepsilon}} G(x, x_{i_1}, x_{i_2}) \\ &+ \frac{2}{h_r^2} \textstyle\sum_{\substack{i_1,i_2 \in I_r \\ G(x,x_{i_1},x_{i_2}) < \varepsilon}} G(x, x_{i_1}, x_{i_2}) \\ &\leq M \frac{2}{h_r^2} \left|\{i_1, i_2 \in I_r : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}\right| + \varepsilon \end{aligned}$$

Considering that $x_i \overset{GS_\theta}{\to} x$, we have the result. □

In the following theorem, we explain the relationship between GS and GS_θ.

THEOREM 3.8. *For any lacunary sequence θ in (X, G) with $\liminf_r q_r > 1$, $GS - \lim x_i = x$ implies $GS_\theta - \lim x_i = x$.*

Proof. Assume that $\liminf_r q_r > 1$. Then, there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r and therefore $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$.

For every $\varepsilon > 0$, we know that the limit of the set $\frac{2}{k_r^2} \left|\{i_1, i_2 \leq k_r : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}\right|$ is 0. Therefore,

$$\begin{aligned} \frac{2}{k_r^2} \left|\{i_1, i_2 \leq k_r : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}\right| &\geq \frac{2}{k_r^2} \left|\{i_1, i_2 \in I_r : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}\right| \\ &= \left(\frac{h_r}{k_r}\right)^2 \frac{2}{h_r^2} \left|\{i_1, i_2 \in I_r : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}\right| \\ &\geq \left(\frac{\delta}{1+\delta}\right)^2 \frac{2}{h_r^2} \left|\{i_1, i_2 \in I_r : G(x, x_{i_1}, x_{i_2}) \geq \varepsilon\}\right|. \end{aligned}$$

Considering that $GS - \lim x_i = x$ then, we have $GS_\theta - \lim x_i = x$. □

Finally, in the last theorem, we explain the relationship between $G\sigma_1$ and GN_θ.

THEOREM 3.9. *For any lacunary sequence θ in (X, G) with $\liminf_r q_r > 1$, $G\sigma_1 - \lim x_i = x$ implies $GN_\theta - \lim x_i = x$.*

Proof. Assume that $\liminf_r q_r > 1$. Then, there exists a $\delta > 0$ such that $q_r \geq 1+\delta$ for sufficiently large r. Since $h_r = k_r - k_{r-1}$ we have $\frac{k_r}{k_{r-1}} \geq 1+\delta$ for sufficiently large r which implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$.

$$\begin{aligned}
\frac{2}{k_r^2}\sum_{i_1,i_2=1}^{k_r} G(x,x_{i_1},x_{i_2}) &\geq \frac{2}{k_r^2}\sum_{i_1,i_2\in I_r} G(x,x_{i_1},x_{i_2}) \\
&= \left(\frac{h_r}{k_r}\right)^2 \frac{2}{h_r^2}\sum_{i_1,i_2\in I_r} G(x,x_{i_1},x_{i_2}) \\
&\geq \left(\frac{\delta}{1+\delta}\right)^2 \frac{2}{h_r^2}\sum_{i_1,i_2\in I_r} G(x,x_{i_1},x_{i_2})
\end{aligned}$$

If we take the limit of both sides we have the proof. □

4. Conclusions

As we know, the concept of distance is very important in functional analysis and many new concepts are defined on the concept of distance. The metric function is a way of generalizing distance, and different metrics have been studied in different spaces. Some of them are $2-$metrics, $G-$metrics and $g-$metrics. Therefore, in this study, it was interesting to see the the meaning of lacunary statistical convergence in $G-$metric spaces.

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