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BERNSTEIN-WALSH TYPE INEQUALITIES FOR DERIVATIVES OF ALGEBRAIC POLYNOMIALS

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Abstract. In this work, we study Bernstein-Walsh-type estimations for the derivative of an arbitrary algebraic polynomial in regions with piecewise smooth boundary without cusps of the complex plane. Also, estimates are given on the whole complex plane.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$ such that $0 \in G$; $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$; $\Delta(w_0, R) := \{w : |w - w_0| > R\}$, $\Delta := \Delta(0, 1)$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto Δ such that $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$; $\Psi := \Phi^{-1}$.

Let $\{z_j\}_{j=1}^{l}$ be the fixed system of distinct points on the curve L. We consider generalized Jacobi weight function $h(z)$ which is defined as follows:

$$h(z) := \prod_{j=1}^{l} |z - z_j|^{\gamma_j}, \quad z \in \mathbb{C}, \tag{1}$$

where $\gamma_j > -2$ for all $j = 1, 2, \ldots, l$.

Let $\wp_n$ denote the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $p > 0$ and σ be the two-dimensional Lebesgue measure. For the Jordan region G, we introduce:

$$\begin{aligned} \|P_n\|_p &:= \|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \, |P_n(z)|^p \, d\sigma_z \right)^{1/p}, \ 0 < p < \infty, \\ \|P_n\|_\infty &:= \|P_n\|_{A_\infty(1,G)} := \max_{z \in \overline{G}} |P_n(z)|, \ p = \infty, \end{aligned} \tag{2}$$

and $A_p(1, G) \equiv A_p(G)$.

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When L is rectifiable, for any $p>0$, let

$$\begin{aligned} \|P_n\|_{\mathcal{L}_p(h,G)} &:= \left(\int_L h(z)\,|P_n(z)|^p\,|dz|\right)^{1/p} < \infty,\ 0<p<\infty, \\ \|P_n\|_{\mathcal{L}_\infty(1,G)} &:= \max_{z\in L}|P_n(z)|\,,\ p=\infty, \end{aligned} \tag{3}$$

and $\mathcal{L}_p(1,L)\equiv\mathcal{L}_p(L)$.

For $R>1$, let us set $L_R:=\{z:|\Phi(z)|=R\}\,,G_R:=intL_R$, $\Omega_R:=extL_R$. Then, well known Bernstein-Walsh Lemma [24] says that:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n\,\|P_n\|_{C(\overline{G})}\,. \tag{4}$$

Hence, setting $R=1+\frac{1}{n}$, we see that the C-norm of polynomials $P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ is identical, i.e., the norm $\|P_n\|_{C(\overline{G})}$ increases up to multiplication by a constant in $\overline{G}_R$.

Also, in [24] some similar estimates were given for various norms on the right-hand side of (3). Analogous estimation with respect to the quasinorm (4) for $p>0$ was obtained in [19] for $h(z)\equiv 1$ (i.e., $\gamma_j=0$ for all $j=1,2,\ldots,l$) as following:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}}\,\|P_n\|_{\mathcal{L}_p(L)}\,,\quad p>0. \tag{5}$$

Moreover, in [7, Lemma 2.4] this estimate has been generalized for $h(z)\neq 1$, defined as in (1) and the following was proved:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}}\,\|P_n\|_{\mathcal{L}_p(h,L)}\,,\ \gamma^*=\max\{0;\gamma_j:1\leq j\leq\ l\}\,. \tag{6}$$

To give a similar estimation to (6) for the $A_p(h,G)$-norm, first of all, we give the following definition.

Definition ([20, p. 97], [22])**.** The Jordan arc (or curve) L is called K-*quasiconformal* ($K\geq 1$) if there is a K-quasiconformal mapping f of the region $D\supset L$ such that $f(L)$ is a line segment (or a circle).

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of the region $D\supset L$ such that $f(L)$ is a line segment (or a circle) and let

$$K_L:=\inf\{K(f):f\in F(L)\}\,,$$

where $K(f)$ is the maximal dilatation of f. Then, L is a quasiconformal curve if $K_L<\infty$ and L is a K-quasiconformal curve if $K_L\leq K$.

A curve L is called a *quasiconformal* if it is a K-*quasiconformal* for some $K>1$.

In [3] (also, see [1]), The Bernstein-Walsh type estimates for the norm (2), the regions with quasiconformal boundary, weight function $h(z)$ defined in (1) with $\gamma_j>-2$ and all $p>0$ are as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1\ {R^*}^{n+\frac{1}{p}}\,\|P_n\|_{A_p(h,G)}\,, \tag{7}$$

where $R^* := 1 + c_2(R-1)$, $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent from n and R. It is well known that quasicircles can be non-rectifiable (see, for example, [16], [20, p. 104]).

In [6, Theorem1.1], analogous estimate was studied for $A_p(1, G)$-norm, $p > 0$, for arbitrary Jordan region and the following estimate was obtained: for any $P_n \in \wp_n$, $R_1 = 1 + \frac{1}{n}$ and arbitrary R, $R > R_1$,

$$\|P_n\|_{A_p(G_R)} \le cR^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}$$

holds, where $c = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right]$, $n \to \infty$. Note that the constant c is asymptotically sharp as $n \to \infty$.

N. Stylianopoulos in [23] replaced the norm $\|P_n\|_{C(\overline{G})}$ with the norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (4) and found a new version of the Bernstein-Walsh Lemma: *Assume that L is quasiconformal and rectifiable. Then there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \le c\frac{\sqrt{n}}{d(z, L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega, \tag{8}$$

where $d(z, L) := \inf\{|\zeta - z| : \zeta \in L\}$, holds for every $P_n \in \wp_n$.

On the other hand, using the mean value theorem, for an arbitrary Jordan region G, $P_n \in \wp_n$ and any $p > 0$, we can find:

$$|P_n(z)| \le \left(\frac{1}{\sqrt{\pi} d(z, L)}\right)^{\frac{2}{p}} \|P_n\|_{A_p(G)}, \quad z \in G. \tag{9}$$

Hence, from (8) and (9), we obtain an estimation on the growth of $|P_n(z)|$ for any $P_n \in \wp_n$ on the whole complex plane as following:

$$|P_n(z)| \le \frac{c_3}{d(z, L)} \|P_n\|_{A_2(G)} \begin{cases} 1, & z \in G, \\ \sqrt{n}\, |\Phi(z)|^{n+1}, & z \in \Omega. \end{cases} \tag{10}$$

In this work, we study the estimation in bounded region $\overline{G}$ with piecewise smooth boundary which according to (4), is also true for the $\overline{G}_{1+\varepsilon_0 n^{-1}}$. Additionally, we study pointwise estimation in unbounded region $\Omega_{1+\varepsilon_0 n^{-1}} = \overline{\mathbb{C}} \backslash \overline{G}_{1+\varepsilon_0 n^{-1}}$ for sufficiently small $\varepsilon_0 > 0$, for the derivative $|P_n'(z)|$ in the following type:

$$|P_n'(z)| \le c_4 \|P_n\|_p \begin{cases} \nu_n(G, h, p) & z \in \overline{G}, \\ \eta_n(G, h, p, d(z, L))\ |\Phi(z)|^{n+1} & z \in \Omega_{1+\varepsilon_0 n^{-1}}, \end{cases} \tag{11}$$

where $c_4 = c_4(G, p) > 0$ is a constant independent of n, h, P_n and $\nu_n(G, h, p) \to \infty$, $\eta_n(G, h, p, d(z, L)) \to \infty$ depending on the properties of the G, h as $n \to \infty$.

Analogous results of (11)-type for $|P_n(z)|$, a different weight funtion h, unbounded region G and some norms were obtained in [17, pp. 418–428], [4, 5, 7–11, 14, 21] and the others.

2. Definitions and main results

Throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations) which in general, depends on G and parameters that inessential for the argument. Otherwise, the dependence will be explicitly stated. For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k+1, \ldots, m$.

To formulate our problem, first of all, let us give the necessary definitions and notations.

Let $z = z(s)$, $s \in [0, mesL]$ denote the natural representation of L.

Definition. We say that $L \in C_\theta$ if L has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. Then, we write $G \in C_\theta \Leftrightarrow \partial G \in C_\theta$.

According to [22], we have the following:

Corollary 2.1. *If $L \in C_\theta$, then $L = \partial G$ is $(1+\varepsilon)$-quasiconformal for all $\varepsilon > 0$.*

Now we give the definitions of regions with a piecewise smooth curve which we present our main result and some notation that will be used later in the text.

Definition. We say that a Jordan region $G \in C_\theta(\lambda_1, \ldots, \lambda_l)$, $0 < \lambda_j < 2$, $j = 1, 2, \ldots, l$ if $L = \partial G$ consists of the union of finite smooth arcs $\{L_j\}_{j=1}^{l}$ such that they have exterior (with respect to $\overline{G}$) angles $\lambda_j \pi$, $0 < \lambda_j < 2$ at the corner points $\{z_j\}_{j=1}^{l} \in L$, where two arcs meet.

If $l = 1$, we put $G \in C_\theta(\lambda)$, $0 < \lambda < 2$.

According to the "three-point" criterion [13, p. 100], every piecewise smooth curve (without cusps) is quasiconformal.

For $0 < \delta_j < \delta_0 := \frac{1}{4} \min\{|z_i - z_j| : i, j = 1, 2, \ldots, l, \ i \neq j\}$, let

$$\Omega(z_j, \ \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}; \ \delta := \min_{1 \leq j \leq l} \delta_j,$$

$$\Omega(\delta) := \bigcup_{j=1}^{l} \Omega(z_j, \ \delta), \ \widehat{\Omega} := \Omega \setminus \Omega(\delta). \tag{12}$$

For simplicity, here we consider the case when there is only one singular point on the curve L, i.e., $l = 1$, and we assume that $\lambda_1 = \lambda$, $\gamma_1 =: \gamma$. For $l > 1$ the reasoning is similar.

Theorem A. *Let $p > 1$; $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, and $h(z)$ be defined as in (1) for $l = 1$. Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$ and $R_1 = 1 + \frac{1}{n}$, we have:*

$$|P_n(z)| \leq c \frac{G_n(p, \lambda, \varepsilon)}{d(z, L_{R_1})} \|P_n\|_p \ |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}, \tag{13}$$

where $c = c(G, p, \varepsilon) > 0$,

(14)

$$G_n(p,\lambda,\varepsilon) := \begin{cases} n^{\frac{1}{p}+\varepsilon_p}, & \begin{array}{l} \text{if } p \geq 2,\ 0 < \lambda < 2,\ -2 < \gamma < \frac{1}{\lambda} + (p-2), \\ \text{or } p < 2,\ 1 \leq \lambda < 2,\ -2 < \gamma < \frac{1}{\lambda} - (2-p), \\ \text{or } p < 2,\ 0 < \lambda < 1,\ -2 < \gamma < \frac{p-1}{\lambda}, \end{array} \\ n^{\frac{\gamma\lambda}{p}+(\frac{2}{p}-1)\lambda+\varepsilon}, & \begin{array}{l} \text{if } p \geq 2,\ 0 < \lambda < 2,\ \gamma \geq \frac{1}{\lambda} + (p-2), \\ \text{or } p < 2,\ 1 \leq \lambda < 2,\ \gamma \geq \frac{1}{\lambda} - (2-p), \end{array} \\ n^{\frac{\gamma\lambda}{p}+(\frac{2}{p}-1)+\varepsilon}, & \text{if } p < 2, 0 < \lambda < 1,\ \gamma \geq \frac{p-1}{\lambda}, \end{cases}$$

and $\varepsilon_p = \varepsilon$ *if* $p \neq 2$ *and* $\varepsilon_p = 0$ *if* $p = 2$.

In particular, in case of $p = 2$, we obtain:

$$|P_n(z)| \leq c \frac{G_n(2,\lambda,\varepsilon)}{d(z, L_{R_1})} \|P_n\|_2\ |\Phi(z)|^{n+1}, \quad z \in \Omega_{R_1}, \tag{15}$$

where $c = c(G, \varepsilon) > 0$ and $G_n(2, \lambda, \varepsilon)$ defined as in (14) for $p = 2$.

In this work, we study similar problems for $|P_n'(z)|$ in regions with piecewise smooth boundary (without cusps) and generalized Jacobi weight function $h(z)$ as defined in (1) in $A_p(h, G)$, $p > 1$.

Now, we start to formulate the new results.

Theorem 2.2. *Let* $p > 1$; $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, *and* $h(z)$ *be defined as in* (1) *for* $l = 1$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$ *and* $\forall \varepsilon > 0$

$$|P_n'(z)| \leq c_1 \frac{\|P_n\|_p}{d(z, L_{R_1})} \left[|\Phi(z)|^{n+1} D_n(p,\lambda,\varepsilon) + |\Phi(z)|^{2n+2} n^{\widehat{\lambda}} G_n(p,\lambda,\varepsilon) \right] \tag{16}$$

holds, where $c_1 = c_1(G, p, \varepsilon) > 0$,

$$D_n(p,\lambda,\varepsilon) := \begin{cases} n^{\frac{\widetilde{\gamma}+2}{p}\ \widehat{\lambda}}, & \gamma \geq -2 + \frac{1}{\widehat{\lambda}}, \\ 1, & -2 < \gamma < -2 + \frac{1}{\widehat{\lambda}}, \end{cases}$$

$$\widehat{\lambda} := \begin{cases} \max\{1, \lambda\} + \varepsilon, & z \in \Omega(\delta) \cap \Omega_{R_1}, \\ 1 + \varepsilon & z \in \widehat{\Omega}(\delta) \cap \Omega_{R_1}, \end{cases}$$

$$\widetilde{\gamma} := \max\{0, \gamma\},$$

and $G_n(p, \lambda, \varepsilon)$ *defined as in* (14).

Corollary 2.3. *Let* $p > 1$; $G \in C_\theta$ *and* $h(z)$ *be defined as in* (1) *for* $l = 1$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$ *and* $\forall \varepsilon > 0$

$$|P_n'(z)| \leq c_1 \frac{\|P_n\|_p}{d(z, L_{R_1})} \left[|\Phi(z)|^{n+1} D_n(p,1,\varepsilon) + |\Phi(z)|^{2n+2} n^{1+\varepsilon} G_n(p,1,\varepsilon) \right] \tag{17}$$

holds, where $c_1 = c_1(G, p, \varepsilon) > 0$; $D_n(p, 1, \varepsilon)$ *and* $G_n(p, 1, \varepsilon)$ *are defined from* (14) *and* (16) *for* $\lambda = 1$*, respectively, as following*:

$$D_n(p,1,\varepsilon) = \begin{cases} n^{\frac{\widetilde{\gamma}+2}{p}}, & \gamma \geq -1, \\ 1, & -2 < \gamma < -1, \end{cases} \tag{18}$$
$$G_n(p,1,\varepsilon) = \begin{cases} n^{\frac{1}{p}+\varepsilon_p}, & \text{if } -2 < \gamma < p-1, \\ n^{\frac{\gamma+2}{p}-1+\varepsilon}, & \text{if } \gamma \geq p-1, \end{cases}$$

where $\varepsilon_p = \varepsilon$ *if* $p \neq 2$ *and* $\varepsilon_p = 0$ *if* $p = 2$.

Theorem 2.4. *Let* $p > 0$; $G \in C_\theta(\lambda)$ *for* $0 < \lambda < 2$; $h(z)$ *be defined as in* (1) *for* $l = 1$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$ *and arbitrary small* $\varepsilon > 0$*, we have*:

$$\left\|P_n'\right\|_\infty \leq c_2 \mu_n \left\|P_n\right\|_p, \tag{19}$$

where $c_2 = c_2(G, \gamma, \lambda, p, \varepsilon) > 0$ *is the constant independent of* z *and* n*, and*

$$\mu_n := \begin{cases} n^{\left(\frac{2+\widetilde{\gamma}}{p}+1\right)\widetilde{\lambda}}, & \text{if } \gamma \geq -2 + \frac{1}{\widetilde{\lambda}}, \\ n^{\frac{1}{p}+\widetilde{\lambda}}, & \text{if } \gamma < -2 + \frac{1}{\widetilde{\lambda}}, \end{cases}$$
$$\widetilde{\gamma} := \max\{0, \gamma\}, \quad \widetilde{\lambda} := \max\{1, \lambda\} + \varepsilon.$$

According to (4) (applied to $|P_n'(z)|$), the estimation (19) is true again for the $z \in \overline{G}_R$, $R = 1 + \frac{1}{n}$ with a different constant.

Now we can estimate $|P_n'(z)|$ on the whole complex plane.

Combining Theorems 2.2 (for the $z \in \overline{G}_{R_1}$) and 2.4, we obtain the following estimation for the growth of $|P_n'(z)|$ on the whole complex plane:

Theorem 2.5. *Let* $p > 1$; $G \in C_\theta(\lambda)$ *for some* $0 < \lambda < 2$*, and* $h(z)$ *be defined as in* (1) *for* $l = 1$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$ *and* $\forall \varepsilon > 0$

$$|P_n'(z)| \leq c_3 \|P_n\|_p \begin{cases} \mu_n, \quad z \in \overline{G}_{R_1}, \\ \frac{1}{d(z, L_{R_1})} \left[\begin{array}{c} |\Phi(z)|^{n+1} D_n(p, \lambda, \varepsilon) + \\ |\Phi(z)|^{2n+2} n^{\widehat{\lambda}} G_n(p, \lambda, \varepsilon) \end{array} \right], \quad z \in \Omega_{R_1}, \end{cases}$$

where $c_3 = c_3(G, \gamma, \lambda, p, \varepsilon) > 0$ *is the constant independent of* z *and* n, $D_n(p, \lambda, \varepsilon)$ *and* $G_n(p, \lambda, \varepsilon)$ *are defined as in* (18).

Analogously, combining estimation (19) (for $\lambda = 1$ and $z \in \overline{G}_{R_1}$) with Corollary 2.3, we obtain the following theorem.

Theorem 2.6. *Let* $p > 1$; $\partial G \in C_\theta$ *and* $h(z)$ *be defined as in* (1) *for* $l = 1$. *Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$ *and* $\forall \varepsilon > 0$

$$|P_n'(z)| \leq c_4 \|P_n\|_p \begin{cases} \mu_n, z \in \overline{G}_{R_1}, \\ \frac{1}{d(z, L_{R_1})} \left[\begin{array}{c} |\Phi(z)|^{n+1} D_n(p, 1, \varepsilon) + \\ |\Phi(z)|^{2n+2} n^{1+\varepsilon} G_n(p, 1, \varepsilon) \end{array} \right], \quad z \in \Omega_{R_1}, \end{cases}$$

where $c_4 = c_4(G, \gamma, \lambda, p, \varepsilon) > 0$ *is the constant independent of* z *and* n, $D_n(p, 1, \varepsilon)$ *and* $G_n(p, 1, \varepsilon)$ *are defined as in* (18).

Particularly, in case of $p = 2$, from Theorem 2.2, we obtain the following two more useful results:

Corollary 2.7. *Let $p = 2$; $G \in C_\theta(\lambda)$, $0 < \lambda < 2$, and $h(z)$ be defined in* (1) *for $l = 1$. Then, for any $P_n \in \wp_n, n \in \mathbb{N}$ and $R_1 = 1 + \frac{1}{n}$, we have*:

$$|P_n'(z)| \leq c_5 \frac{\|P_n\|_2}{d(z, L_{R_1})} \left[|\Phi(z)|^{n+1} D_n(2, \lambda, \varepsilon) + |\Phi(z)|^{2n+2} n^{\widehat{\lambda}} G_n(2, \lambda, \varepsilon) \right],$$

where $c_5 = c_5(G, \gamma, \lambda, \varepsilon) > 0$, $D_n(2, \lambda, \varepsilon)$ and $G_n(2, \lambda, \varepsilon)$ are defined as

$$D_n(2, \lambda, \varepsilon) = \begin{cases} n^{\frac{\widetilde{\gamma}+2}{2}\ \widehat{\lambda}}, & \gamma \geq -2 + \frac{1}{\widehat{\lambda}}, \\ 1, & -2 < \gamma < -2 + \frac{1}{\widehat{\lambda}}, \end{cases}$$

$$G_n(2, \lambda, \varepsilon) = \begin{cases} n^{\frac{1}{2}}, & -2 < \gamma < \frac{1}{\lambda}, \quad 0 < \lambda < 2, \\ n^{\frac{\gamma\lambda}{2} + \varepsilon}, & \gamma \geq \frac{1}{\lambda}, \quad 0 < \lambda < 2. \end{cases}$$

Corollary 2.8. *Under the conditions of Corollary 2.7, for $z \in \Omega(\delta) \cap \Omega_{R_1}$, we have*:

$$|P_n'(z)| \prec \frac{\|P_n\|_2}{d(z, L_{R_1})} \begin{cases} n^{\frac{1}{2} + \ \lambda + \varepsilon}, & -2 < \gamma < \frac{1}{\lambda}, \\ n^{\frac{\gamma\lambda}{2} + \ \lambda + \varepsilon}, & \gamma \geq \frac{1}{\lambda}. \end{cases}$$

In here and throughout this paper, for $a > 0$ and $b > 0$, we use the expression "$a \prec b$" (order inequality), if $a \leq cb$. The expression "$a \asymp b$" means that "$a \prec b$" and "$b \prec a$" simultaneously.

2.1. Sharpness of the inequalities

Theorem 2.9. *The statement* (19) *from Theorem 2.4 is sharp.*

3. Some auxiliary results

Lemma 3.1 ([2])**.** *Let L be a K-quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{R_0})\}$; $w_j = \Phi(z_j)$, $(z_2, z_3 \in G \cap \{z : |z - z_1| \prec d(z_1, L_{R_0})\}$; $w_j = \varphi(z_j))$, $j = 1, 2, 3$. Then,*

a) *The statements $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.*

b) *If $|z_1 - z_2| \prec |z_1 - z_3|$, then*

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^2} \prec \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \prec \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^{-2}},$$

where $\varepsilon < 1$, $c > 1$, $R_0 > 1$ are constants depending on G.

Corollary 3.2. *Under the assumptions of Lemma 3.1, if $z_3 \in L_{R_0}$, then*

$$|w_1 - w_2|^{K^2} \prec |z_1 - z_2| \prec |w_1 - w_2|^{K^{-2}}.$$

Corollary 3.3. *If $L \in C_\theta$, then*

$$|w_1 - w_2|^{1+\varepsilon} \prec |z_1 - z_2| \prec |w_1 - w_2|^{1-\varepsilon}$$

for all $\varepsilon > 0$.

Let $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. Additionally to the notations (12) for $\eta_j = \min\limits_{t \in \partial\Phi(\Omega(z_j,\delta_j))} |t - w_j| > 0$ and $\eta := \min\left\{\eta_j, j = \overline{1,l}\right\}$ let us set: $\Delta_j(\eta_j) := \{t : |t - w_j| \leq \eta_j\} \subset \Phi(\Omega(z_j,\, \delta_j))$, $\Delta(\eta) := \bigcup_{j=1}^{l} \Delta_j(\eta)$, $\widehat{\Delta}_j = \Delta \backslash \Delta(\eta_j)$; $\widehat{\Delta}(\eta) := \Delta \backslash \Delta(\eta)$; $\Delta_1^{'} := \Delta_1^{'}(1)$, $\Delta_1^{'}(\rho) := \left\{t = R \cdot e^{i\theta} : R \geq \rho > 1,\ \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\}$, $\Delta_j^{'} := \Delta_j^{'}(1)$, $\Delta_j^{'}(\rho) := \left\{t = R \cdot e^{i\theta} : R \geq \rho > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_0}{2}\right\}$, $j = 2, 3, \ldots, l$, where $\varphi_0 = 2\pi - \varphi_l$; $\Omega_j := \Psi(\Delta_j^{'})$, $L_{R_1}^{j} := L_{R_1} \cap \Omega_j$. Clearly, $\Omega = \bigcup_{j=1}^{l} \Omega_j$.

The following lemma is a consequence of the results given in [25] and [18].

Lemma 3.4. *Let $G \in C_\theta(\lambda, \ldots, \lambda_l)$, $0 < \lambda_j < 2$, $j = 1, 2, \ldots, l$. Then, for arbitrary small $\varepsilon > 0$*

i) *for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j + \varepsilon} \prec |\Psi(w) - \Psi(w_j)| \prec |w - w_j|^{\lambda_j - \varepsilon}$, $|w - w_j|^{\lambda_j - 1 + \varepsilon} \prec |\Psi^{'}(w)| \prec |w - w_j|^{\lambda_j - 1 - \varepsilon}$,*

ii) *for any $w \in \overline{\Delta} \backslash \Delta_j$, $(|w| - 1)^{1+\varepsilon} \prec d(\Psi(w), L)| \prec (|w| - 1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \prec |\Psi^{'}(w)| \prec (|w| - 1)^{-\varepsilon}$.*

Let $\{z_j\}_{j=1}^{l}$ be a fixed system of distinct points on curve L ordered in the positive direction and the weight function $h(z)$ be defined as in (1).

Lemma 3.5 ([11])**.** *Let L be a K-quasiconformal curve $R = 1 + \frac{c}{n}$. Then, for any fixed $\varepsilon \in (0, 1)$, there exists a level curve $L_{1+\varepsilon(R-1)}$ such that the following holds for any polynomial $P_n(z) \in \wp_n$, $n \in \mathbb{N}$:*

$$\|P_n\|_{\mathcal{L}_p\left(\frac{h}{|\Phi'|},\ L_{1+\varepsilon(R-1)}\right)} \prec n^{\frac{1}{p}} \|P_n\|_p\,,\ p > 0. \tag{20}$$

Lemma 3.6 ([11])**.** *Let L be a K-quasiconformal curve; $h(z)$ be defined as in (1). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n = 1, 2, \ldots$, we have*

$$\|P_n\|_{A_p(h,G_R)} \prec \widetilde{R}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}\,,\ p > 0, \tag{21}$$

where $\widetilde{R} = 1 + c(R - 1)$ and c is independent from n and R.

4. Proof of Theorems

4.1. Proof of Theorem 2.2

Proof. Suppose that $G \in C_\theta(\lambda)$ for some $0 < \lambda < 2$ and $h(z)$ be defined as in (1) for $l = 1$. For $z \in \Omega$ we define:

$$T_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}. \tag{22}$$

Then

$$T_n'(z) = \frac{P_n'(z)}{\Phi^{n+1}(z)} + P_n(z)\left(\frac{1}{\Phi^{n+1}(z)}\right)', \ z \in \Omega,$$

and, so

$$P_n'(z) = \Phi^{n+1}(z)\left[T_n'(z) - P_n(z)\left(\frac{1}{\Phi^{n+1}(z)}\right)'\right].$$

For any $R > 1$ and $R_1 := 1 + \frac{R-1}{2}$, Cauchy integral representation for the region Ω_{R_1} gives

$$\begin{aligned} T_n'(z) &= -\frac{1}{2\pi i}\int_{L_{R_1}} T_n(\zeta)\frac{d\zeta}{(\zeta - z)^2} \\ &= -\frac{1}{2\pi i}\int_{L_{R_1}} \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta - z)^2}, \ z \in \Omega_{R_1}, \end{aligned}$$

and

$$\left(\frac{1}{\Phi^{n+1}(z)}\right)' = -\frac{1}{2\pi i}\int_{L_{R_1}} \frac{1}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta - z)^2}, \ z \in \Omega_{R_1}.$$

Then

$$P_n'(z) = \Phi^{n+1}(z)\left[-\frac{1}{2\pi i}\int_{L_{R_1}} \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta - z)^2} + \frac{P_n(z)}{2\pi i}\int_{L_{R_1}} \frac{1}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta - z)^2}\right].$$

Therefore,

$$|P_n'(z)| \le \frac{\left|\Phi^{n+1}(z)\right|}{2\pi}\left[\int_{L_{R_1}} \left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right| \frac{|d\zeta|}{|\zeta - z|^2} + |P_n(z)|\int_{L_{R_1}} \left|\frac{1}{\Phi^{n+1}(\zeta)}\right| \frac{|d\zeta|}{|\zeta - z|^2}\right].$$

Since $|\ \Phi(\zeta)| > 1$ for $\zeta \in L_{R_1}$, then we have

$$\begin{aligned} |P_n'(z)| &< \frac{|\Phi(z)|^{n+1}}{2\pi}\left[\int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^2} + |P_n(z)|\int_{L_{R_1}} \frac{|d\zeta|}{|\zeta - z|^2}\right] \\ &\le \frac{|\Phi(z)|^{n+1}}{2\pi}\left[\frac{1}{d(z, L_{R_1})}\int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} + |P_n(z)|\int_{L_{R_1}} \frac{|d\zeta|}{|\zeta - z|^2}\right]. \end{aligned} \tag{23}$$

Denoted by

$$A_n(z) := \int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|}; \ B_n(z) := \int_{L_{R_1}} \frac{|d\zeta|}{|\zeta - z|^2}, \tag{24}$$

and we estimate these integrals separately.

To estimate A_n, first of all, replacing the variable $\tau = \Phi(\zeta)$ and multiplying the numerator and denominator of the integrant by $|\Psi(\tau) - \Psi(w_1)|^{\frac{\gamma}{p}}\ |\Psi'(\tau)|^{\frac{2}{p}}$, and then, applying the Hölder inequality, we obtain:

$$A_n(z) = \int_{L_{R_1}} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|}$$

$$= \sum_{i=1}^{2}\int_{F_{R_1}^{i}} \frac{\left|\Psi(\tau)-\Psi(w_1)\right|^{\frac{\gamma_j}{p}}\left|P_n\left(\Psi(\tau)\right)\left(\Psi'(\tau)\right)^{\frac{2}{p}}\right|\left|\Psi'(\tau)\right|^{1-\frac{2}{p}}}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\frac{\gamma}{p}}\left|\Psi(\tau)-\Psi(w)\right|}\left|d\tau\right|$$
$$\leq \sum_{i=1}^{2}\left(\int_{F_{R_1}^{i}}\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma}\left|P_n\left(\Psi(\tau)\right)\right|^{p}\left|\Psi'(\tau)\right|^{2}\left|d\tau\right|\right)^{\frac{1}{p}}$$
$$\times\left(\int_{F_{R_1}^{i}}\left(\frac{\left|\Psi'(\tau)\right|^{1-\frac{2}{p}}}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\frac{\gamma}{p}}\left|\Psi(\tau)-\Psi(w)\right|}\right)^{q}\left|d\tau\right|\right)^{\frac{1}{q}}$$
$$=: \sum_{i=1}^{2} A_n^{i},$$

where $F_{R_1}^1 := \Phi(L_{R_1}^1) = \Delta_1^{'} \cap \{\tau : |\tau| = R_1\}$, $F_{R_1}^2 := \Phi(L_{R_1}^1)\backslash F_{R_1}^1$ and

$$A_n^i(z) := \left(\int_{F_{R_1}^{i}}\left|f_{n,p}(\tau)\right|^{p}\left|d\tau\right|\right)^{\frac{1}{p}}\left(\int_{F_{R_1}^{i}}\frac{\left|\Psi'(\tau)\right|^{2-q}}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma(q-1)}\left|\Psi(\tau)-\Psi(w)\right|^{q}}\left|d\tau\right|\right)^{\frac{1}{q}}$$
$$=: J_{n,1}^{i}\cdot J_{n,2}^{i}(z),$$
$$f_{n,p}(\tau) := \left(\Psi(\tau)-\Psi(w_1)\right)^{\frac{\gamma}{p}} P_n\left(\Psi(\tau)\right)\left(\Psi'(\tau)\right)^{\frac{2}{p}},\ |\tau| = R_1.$$

Applying to Lemma 3.5, we get:

$$J_{n,1}^{i} \prec n^{\frac{1}{p}}\left\|P_n\right\|_p,\ i = 1,2. \tag{25}$$

For the estimation of the integral $J_{n,2}^{i}$ for $i = 1,2$, we set:

$$E_{R_1}^{11} := \left\{\tau : \tau \in F_{R_1}^1,\ |\tau - w_1| < c_1(R_1-1)\right\},$$
$$E_{R_1}^{12} := \left\{\tau : \tau \in F_{R_1}^1,\ c_1(R_1-1)\ \leq |\tau - w_1| < \eta\right\},$$
$$E_{R_1}^{13} := \left\{\tau : \tau \in \Phi(L_{R_1}^1),\ |\tau - w_1| \geq \eta\right\},$$

where $0 < c_1 < \eta$ is chosen so that $\{\tau : |\tau - w_1| < c_1(R_1-1)\} \cap \Delta \neq \emptyset$ and $\Phi(L_{R_1}^1) = \bigcup_{k=1}^{3} E_{R_1}^{1k}$. Taking into consideration these notations, (25) can be written as:

$$\begin{aligned} J_{n,2}^{1}(z) + J_{n,2}^{2}(z) &=: J_2(z) = J_2(E_{R_1}^{11}) + J_2(E_{R_1}^{12}) + J_2(E_{R_1}^{13}) \\ &=: J_2^1(z) + J_2^2(z) + J_2^3(z), \end{aligned} \tag{26}$$

and consequently,

$$\begin{aligned} A_n(z) = A_n^1(z) + A_n^2(z) &\prec n^{\frac{1}{p}}\left\|P_n\right\|_p \cdot \left(J_2^1(z) + J_2^2(z) + J_2^3(z)\right) \\ &=: A_{n,1}(z) + A_{n,2}(z) + A_{n,3}(z), \end{aligned} \tag{27}$$

where

$$A_{n,k}(z):=n^{\frac{1}{p}}\left\|P_n\right\|_p\cdot J_2^k(z),\ k=1,2,3.$$

$$(28)\qquad \left(J_2^k(z)\right)^q:=\int_{E_{R_1}^{1k}}\frac{\left|\Psi'(\tau)\right|^{2-q}|d\tau|}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma(q-1)}\left|\Psi(\tau)-\Psi(w)\right|^q},\ k=1,2,3.$$

For any $k=1,2$, denote by

$$E_{R_1,1}^{1k}:=\left\{\tau\in E_{R_1}^{1k}:\left|\Psi(\tau)-\Psi(w_1)\right|>\left|\Psi(\tau)-\Psi(w)\right|\right\},\ E_{R_1,2}^{1k}:=E_{R_1}^{1k}\backslash E_{R_1,1}^{1k},$$

$$(29)\qquad \left(I(E_{R_1,1}^{1k})\right)^q:=\begin{cases}\displaystyle\int_{E_{R_1,1}^{1k}}\frac{\left|\Psi'(\tau)\right|^{2-q}|d\tau|}{\left|\Psi(\tau)-\Psi(w)\right|^{\gamma(q-1)+q}}, & \text{if } \gamma\geq 0,\\ \displaystyle\int_{E_{R_1,1}^{1k}}\frac{\left|\Psi(\tau)-\Psi(w_1)\right|^{(-\gamma)(q-1)}\left|\Psi'(\tau)\right|^{2-q}|d\tau|}{\left|\Psi(\tau)-\Psi(w)\right|^q}, & \text{if } \gamma<0,\end{cases}$$

$$\left(I(E_{R_1,2}^{1k})\right)^q:=\int_{E_{R_1,2}^{1k}}\frac{\left|\Psi'(\tau)\right|^{2-q}|d\tau|}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma(q-1)+q}}\ ,\ k=1,2,$$

and we estimate the last integrals. Note that without loss of generality we will consider only the points $w\in\Delta(w_1,\eta)$ for which Lemma 3.4 is used. Otherwise, for $w\in\Delta(w_1,\eta)$, the inequality $|\Psi(\tau)-\Psi(w)|\succ 1$ or $|\Psi(\tau)-\Psi(w)|\succ(R-1)^{1+\varepsilon}$ holds which shows a decrease in the degree of growth of the integrals that we are estimating.

Given the possible values q ($q>2$ and $q<2$), λ ($0<\lambda<1$ and $1<\lambda<2$), and γ ($-2<\gamma<0$ and $\gamma\geq 0$), we will consider the cases separately.

Case 1. Let $1<q\leq 2$ ($p\geq 2$). Then,

$$\left(I(E_{R_1,1}^{1k})\right)^q=\int_{E_{R_1,1}^{1k}}\frac{\left|\Psi'(\tau)\right|^{2-q}|d\tau|}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma(q-1)}\left|\Psi(\tau)-\Psi(w)\right|^q}.$$

1.1. Let $1\leq\lambda<2$.

1.1.1. If $\gamma\geq 0$, applying Lemma 3.4 to (29), we get:

$$(30)\qquad \begin{aligned}\left(I(E_{R_1,1}^{11})\right)^q&<\int_{E_{R_1,1}^{11}}\frac{\left|\Psi'(\tau)\right|^{2-q}|d\tau|}{\left|\Psi(\tau)-\Psi(w)\right|^{\gamma(q-1)+q}}\\ &\prec\int_{E_{R_1,1}^{11}}\frac{\left|\tau-w_1\right|^{(\lambda-1-\varepsilon)(2-q)}}{\left|\tau-w\right|^{[\gamma(q-1)+q](\lambda+\varepsilon)}}|d\tau|\\ &\prec\left(\frac{1}{n}\right)^{(\lambda-1-\varepsilon)(2-q)}\int_{E_{R_1}^{11}}\frac{|d\tau|}{(|\tau|-1)^{[\gamma(q-1)+q](\lambda+\varepsilon)}}\\ &\prec n^{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1+\varepsilon};\\ I(E_{R_1,1}^{11})&\prec n^{\frac{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1}{q}+\varepsilon}=n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ \forall\varepsilon>0.\end{aligned}$$

$$(31)\qquad \left(I(E_{R_1,2}^{11})\right)^q=\int_{E_{R_1,2}^{11}}\frac{\left|\Psi'(\tau)\right|^{2-q}|d\tau|}{\left|\Psi(\tau)-\Psi(w_1)\right|^{\gamma(q-1)+q}}$$

$$\begin{aligned}
&\prec \int_{E^{11}_{R_1,1}} \frac{|\tau - w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau - w_1|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} |d\tau| \\
&\prec \left(\frac{1}{n}\right)^{(\lambda-1-\varepsilon)(2-q)} \int_{E^{11}_{R_1}} \frac{|d\tau|}{(|\tau|-1)^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\
&\prec n^{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1+\varepsilon}; \\
I(E^{11}_{R_1,2}) &\prec n^{\frac{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1}{q}+\varepsilon} = n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ \forall\varepsilon>0.
\end{aligned}$$

$$\begin{aligned}
(32) \qquad \left(I(E^{12}_{R_1,1})\right)^q &\prec \int_{E^{12}_{R_1,1}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+q}} \\
&\prec \int_{E^{12}_{R_1,1}} \frac{|\tau - w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau - w|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} |d\tau| \\
&\prec \left(\frac{1}{n}\right)^{(\lambda-1-\varepsilon)(2-q)} \int_{E^{12}_{R_1}} \frac{|d\tau|}{(|\tau|-1)^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\
&\prec n^{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1+\varepsilon}, \ [\gamma(q-1)+q]\,\lambda \geq 1;
\end{aligned}$$

$$I(E^{12}_{R_1,1}) \prec n^{\frac{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1}{q}+\varepsilon} = n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ \gamma \geq \frac{p-1}{\lambda} - p, \ \forall\varepsilon>0.$$

$$\begin{aligned}
(33) \qquad \left(I(E^{12}_{R_1,2})\right)^q &< \int_{E^{12}_{R_1,2}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}} \\
&\prec \int_{E^{12}_{R_1,2}} \frac{|\tau - w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau - w_1|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} |d\tau| \\
&\prec \left(\frac{1}{n}\right)^{(\lambda-1-\varepsilon)(2-q)} \int_{E^{12}_{R_1}} \frac{|d\tau|}{(|\tau|-1)^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\
&\prec n^{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1+\varepsilon}, \ [\gamma(q-1)+q]\,\lambda \geq 1; \\
I(E^{12}_{R_1,2}) &\prec n^{\frac{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1}{q}+\varepsilon} = n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ \forall\varepsilon>0.
\end{aligned}$$

For $\tau \in E^{13}_{R_1}$ and $w \in \{\tau : |\tau - w_1| < c_1(R_1-1)\} \cap \Delta(w_1,\eta)$ we see that $\eta < |\tau - w_1| < 2\pi\dot{R}_1$, $|\tau - w| \geq \eta - c_1$. Therefore, from Lemma 3.1, $|\Psi(\tau)-\Psi(w_1)| \succ 1$ and from Corollary 3.3, $|w-w_1| \geq \eta$ $|\Psi(\tau)-\Psi(w)| \succ |\tau - w|^{1+\varepsilon}$. Then, applying Lemma 3.4, we get:

$$\begin{aligned}
(34) \qquad &\left(J^3_2(z)\right)^q \prec \int_{E^{13}_{R_1}} \frac{|\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau)-\Psi(w)|^q} \prec \int_{E^{13}_{R_1}} \frac{|\tau - w|^{-\varepsilon} |d\tau|}{|\tau - w|^{q+\varepsilon}} \prec n^{q-1+\varepsilon}; \\
&\left(J^3_2(z)\right)^q \prec n^{\frac{1}{p}+\varepsilon}.
\end{aligned}$$

Combining (30)-(34), for $p \geq 2, \gamma \geq 0$ and $1 \leq \lambda < 2$, we get:

$$\sum_{k=1}^{3} J_2^k(z) \prec \begin{cases} n^{\frac{\gamma+2}{p}\ \lambda-\frac{1}{p}+\varepsilon}, & z \in \Omega(\delta), \\ n^{\frac{1}{p}+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases} \quad \forall\varepsilon > 0,$$

and, from (27) we obtain:

$$A_n(z) \prec \|P_n\|_p \begin{cases} n^{\frac{\gamma+2}{p}\ \lambda+\varepsilon}, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases} \quad \forall\varepsilon > 0. \tag{35}$$

1.1.2. If $\gamma < 0$, analogously we have:

$$\begin{aligned} \left(I(E^{11}_{R_1,1})\right)^q &:= \int_{E^{11}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau)-\Psi(w)|^q} \\ &\prec \int_{E^{11}_{R_1,1}} \frac{|\tau-w_1|^{-\gamma(q-1)(\lambda-\varepsilon)} |\tau-w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau-w|^{q(\lambda+\varepsilon)}} |d\tau| \\ &\prec \left(\frac{1}{n}\right)^{-\gamma(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)} \int_{E^{11}_{R_1}} \frac{|d\tau|}{(|\tau|-1)^{q(\lambda+\varepsilon)}} \\ &\prec n^{q(\lambda+\varepsilon)-(-\gamma)(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)} \int_{E^{11}_{R_1}} |d\tau| \\ &\prec n^{q(\lambda+\varepsilon)+\gamma(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)-1}; \\ I(E^{11}_{R_1,1}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ \forall\varepsilon > 0. \end{aligned} \tag{36}$$

$$\begin{aligned} \left(I(E^{11}_{R_1,2})\right)^q &:= \int_{E^{11}_{R_1,2}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau)-\Psi(w_1)|^q} \\ &\prec \int_{E^{11}_{R_1,1}} \frac{|\tau-w_1|^{-\gamma(q-1)(\lambda-\varepsilon)} |\tau-w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau-w_1|^{q(\lambda+\varepsilon)}} |d\tau| \\ &\prec \left(\frac{1}{n}\right)^{-\gamma(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)} \int_{E^{11}_{R_1}} \frac{|d\tau|}{(|\tau|-1)^{q(\lambda+\varepsilon)}} \\ &\prec n^{q(\lambda+\varepsilon)+\gamma(q-1)(\lambda-\varepsilon)-(\lambda-1-\varepsilon)(2-q)-1}; \\ I(E^{11}_{R_1,2}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ \forall\varepsilon > 0. \end{aligned} \tag{37}$$

For $\tau \in E^{12}_{R_1}$ and $w \in \{\tau : |\tau-w_1| < c_1(R_1-1)\} \cap \Delta(w_1,\eta)$ we see that $|\tau-w_1| < \eta$, and from Lemma 3.1, $|\Psi(\tau)-\Psi(w_1)| \prec 1$. Then, applying Lemma 3.4, we get:

$$\left(I(E^{12}_{R_1,1})\right)^q := \int_{E^{12}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau)-\Psi(w)|^q} \tag{38}$$

$$\prec \int_{E^{12}_{R_1,1}} \frac{|\tau - w_1|^{(\lambda-1-\varepsilon)(2-q)} |d\tau|}{|\tau - w|^{q(\lambda+\varepsilon)}}$$
$$\prec n^{q\lambda-1+(\lambda-1-\varepsilon)(2-q)+\varepsilon}, \ q\lambda > 1;$$
$$I(E^{12}_{R_1,1}) \prec n^{\frac{2}{p}\lambda - \frac{1}{p}+\varepsilon}, \ \lambda > 1 - \frac{1}{p}.$$

(39) $$\left(I(E^{12}_{R_1,2})\right)^q := \int_{E^{12}_{R_1,2}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w_1)|^q}$$
$$\prec n^{\varepsilon} \int_{E^{12}_{R_1,1}} \frac{|\tau - w_1|^{(\lambda-1-\varepsilon)(2-q)} |d\tau|}{|\tau - w_1|^{q(\lambda+\varepsilon)}}$$
$$\prec n^{q\lambda-1+(\lambda-1-\varepsilon)(2-q)+\varepsilon}, \ q\lambda > 1;$$
$$I(E^{12}_{R_1,2}) \prec n^{\frac{2}{p}\lambda - \frac{1}{p}+\varepsilon}, \ \lambda > 1 - \frac{1}{p}.$$

For $\tau \in E^{13}_{R_1}$ and $w \in \{\tau : |\tau - w_1| < c_1(R_1 - 1)\} \cap (w_1, \eta)$ we see that $\eta < |\tau - w_1| < 2\pi \dot{R}_1$, $|\tau - w| \geq \eta - c_1$. Therefore, from Lemma 3.1, $|\Psi(\tau) - \Psi(w)| \succ 1$ and from Corollary 3.3, $|w - w_1| \geq \eta$, $|\Psi(\tau) - \Psi(w)| \succ |\tau - w|^{1+\varepsilon}$. Then, applying Lemma 3.4, we get:

(40) $$\begin{aligned} \left(I(E^{13}_{R_1})\right)^q &:= \int_{E^{13}_{R_1,1}} \frac{|\Psi(\tau) - \Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau) - \Psi(w)|^q} \\ &\prec \int_{E^{13}_{R_1}} \frac{|\tau - w|^{-\varepsilon} |d\tau|}{|\tau - w|^{q-\varepsilon}} \prec n^{q-1+\varepsilon}; \\ I(E^{13}_{R_1}) &\prec n^{\frac{1}{p}+\varepsilon}. \end{aligned}$$

Then, for the $\gamma < 0$, from (36)-(40), we obtain:

$$\sum_{k=1}^{3} J_2^k(z) \prec \begin{cases} n^{\frac{\gamma+2}{p}\lambda - \frac{1}{p}+\varepsilon}, & z \in \Omega(\delta), \\ n^{\frac{1}{p}+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases} \quad \forall \varepsilon > 0,$$

and consequently, in this case, from (27), we have:

(41) $$A_n(z) \prec \|P_n\|_p \cdot \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases} \quad \forall \varepsilon > 0.$$

Therefore, combining (35) and (41), for any $\gamma > -2$, $1 \leq \lambda < 2$, $p \geq 2$, we obtain:

(42) $$A_n(z) \prec \|P_n\|_p \cdot \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases} \quad \forall \varepsilon > 0.$$

1.2. Let now $0 < \lambda < 1$.

1.2.1. If $\gamma \geq 0$, applying Lemma 3.4 to (29), we get:

$$
\begin{aligned}
\left(I(E^{11}_{R_1,1})\right)^q &< \int_{E^{11}_{R_1,1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+q}} \\
&\prec \int_{E^{11}_{R_1,1}} \frac{|\tau-w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau-w|^{[\gamma(q-1)+q](\lambda+\varepsilon)}}|d\tau| \\
&\prec \int_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{[\gamma(q-1)+q](\lambda+\varepsilon)-(\lambda-1-\varepsilon)(2-q)}} \\
&\prec n^{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1+\varepsilon}; \\
I(E^{11}_{R_1,1}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ (\gamma+2)\,\lambda\geq 1,\ \forall\varepsilon>0.
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\left(I(E^{11}_{R_1,2})\right)^q &= \int_{E^{11}_{R_1,2}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}} \\
&\prec \int_{E^{11}_{R_1,1}} \frac{|\tau-w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau-w_1|^{[\gamma(q-1)+q](\lambda+\varepsilon)}}|d\tau| \\
&\prec \left(\frac{1}{n}\right)^{(\lambda-1-\varepsilon)(2-q)} \int_{E^{11}_{R_1}} \frac{|d\tau|}{(|\tau|-1)^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\
&\prec n^{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1+\varepsilon}; \\
I(E^{11}_{R_1,2}) &\prec n^{\frac{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1}{q}+\varepsilon} \\
&= n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ [\gamma+2]\,\lambda>1,\ \forall\varepsilon>0.
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
\left(I(E^{12}_{R_1,1})\right)^q &< \int_{E^{12}_{R_1,1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+q}} \\
&\prec \int_{E^{12}_{R_1,1}} \frac{|\tau-w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau-w|^{[\gamma(q-1)+q](\lambda+\varepsilon)}}|d\tau| \\
&\prec \left(\frac{1}{n}\right)^{(\lambda-1-\varepsilon)(2-q)} \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau-w|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\
&\prec n^{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1+\varepsilon}, \\
&\quad [\gamma(q-1)+q]\,\lambda\geq 1,\ \forall\varepsilon>0; \\
I(E^{12}_{R_1,1}) &\prec n^{\frac{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1}{q}+\varepsilon} \\
&= n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ \gamma\geq\frac{p-1}{\lambda}-p,\ \forall\varepsilon>0.
\end{aligned} \tag{45}
$$

$$(46)\qquad \left(I(E^{12}_{R_1,2})\right)^q < \int_{E^{12}_{R_1,2}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}}$$
$$\prec \int_{E^{12}_{R_1,2}} \frac{|\tau-w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau-w_1|^{[\gamma(q-1)+q](\lambda+\varepsilon)}}|d\tau|$$
$$\prec \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{[\gamma(q-1)+q](\lambda+\varepsilon)-(\lambda-1-\varepsilon)(2-q)}}$$
$$\prec n^{[\gamma(q-1)+q]\lambda-(\lambda-1)(2-q)-1+\varepsilon};$$
$$I(E^{12}_{R_1,2}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\quad (\gamma+2)\,\lambda \geq 1,\ \forall\varepsilon>0.$$

For $\tau \in E^{13}_{R_1}$ and $w \in \{\tau : |\tau-w_1| < c_1(R_1-1)\} \cap (w_1,\eta)$ we see that $\eta < |\tau-w_1| < 2\pi R_1$, $|\tau-w| \geq \eta - c_1$. Therefore, from Lemma 3.1, $|\Psi(\tau)-\Psi(w)| \succ 1$ and from Corollary 3.3, $|w-w_1| \geq \eta$ $|\Psi(\tau)-\Psi(w)| \succ |\tau-w|^{1+\varepsilon}$. Then, applying Lemma 3.4, we get:

$$(47)\qquad \begin{aligned} &\left(J^3_2(z)\right)^q \prec \int_{E^{13}_{R_1,1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^q} \prec \int_{E^{13}_{R_1}} \frac{|\tau-w|^{-\varepsilon}|d\tau|}{|\tau-w|^{q-\varepsilon}} \prec n^{q-1+\varepsilon};\\ &J^3_2(z) \prec n^{\frac{1}{p}+\varepsilon}. \end{aligned}$$

Combining (43)-(47), we get:

$$\sum_{k=1}^{3} J^k_2(z) \prec \begin{cases} n^{\frac{\gamma^*+2}{p}\lambda-\frac{1}{p}+\varepsilon}, & z\in\Omega(\delta),\\ n^{\frac{1}{p}+\varepsilon}, & z\in\widehat{\Omega}(\delta), \end{cases} \quad (\gamma+2)\,\lambda\geq 1,\ \forall\varepsilon>0.$$

In this case, from (27) and (29) for A_n, we obtain:

$$(48)\quad A_n(z) \prec \|P_n\|_p \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & z\in\Omega(\delta),\\ n^{\frac{2}{p}+\varepsilon}, & z\in\widehat{\Omega}(\delta), \end{cases} \quad \gamma\geq 0,\ (\gamma+2)\,\lambda\geq 1,\ \forall\varepsilon>0.$$

1.2.2. If $\gamma < 0$, analogously, we have:

$$(49)\qquad \left(I(E^{11}_{R_1,1})\right)^q := \int_{E^{11}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w)|^q}$$
$$\prec \int_{E^{11}_{R_1,1}} \frac{|\tau-w_1|^{-\gamma(q-1)(\lambda-\varepsilon)}|\tau-w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau-w|^{q(\lambda+\varepsilon)}}|d\tau|$$
$$\prec \left(\frac{1}{n}\right)^{-\gamma(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)} \int_{E^{11}_{R_1}} \frac{|d\tau|}{(|\tau|-1)^{q(\lambda+\varepsilon)}}$$
$$\prec n^{q(\lambda+\varepsilon)-(-\gamma)(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)} \int_{E^{11}_{R_1}} |d\tau|$$
$$\prec n^{q(\lambda+\varepsilon)-(-\gamma)(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)} \cdot mesE^{11}_{R_1}$$
$$\prec n^{q(\lambda+\varepsilon)+\gamma(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)-1};$$

$$I(E_{R_1,1}^{11}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ [\gamma+2]\,\lambda \geq 1\ \forall\varepsilon > 0.$$

$$\begin{aligned}
(50)\qquad \left(I(E_{R_1,2}^{11})\right)^q &:= \int_{E_{R_1,2}^{11}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|\Psi'(\tau)|^{2-q}\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^q} \\
&\prec \int_{E_{R_1,1}^{11}} \frac{|\tau-w_1|^{-\gamma(q-1)(\lambda-\varepsilon)}\,|\tau-w_1|^{(\lambda-1-\varepsilon)(2-q)}}{|\tau-w_1|^{q(\lambda+\varepsilon)}}\,|d\tau| \\
&\prec \left(\frac{1}{n}\right)^{-\gamma(q-1)(\lambda-\varepsilon)+(\lambda-1-\varepsilon)(2-q)} \int_{E_{R_1}^{11}} \frac{|d\tau|}{(|\tau|-1)^{q(\lambda+\varepsilon)}} \\
&\prec n^{q(\lambda+\varepsilon)+\gamma(q-1)(\lambda-\varepsilon)-(\lambda-1-\varepsilon)(2-q)} \int_{E_{R_1}^{11}} |d\tau|\,;
\end{aligned}$$

$$I(E_{R_1,2}^{11}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ [\gamma+2]\,\lambda \geq 1,\ \forall\varepsilon > 0.$$

For $\tau \in E_{R_1}^{12}$ and $w \in \{\tau : |\tau - w_1| < c_1(R_1 - 1)\} \cap \Delta(0, R)$ we see that $|\tau - w_1| < \eta$, and from Lemma 3.1, $|\Psi(\tau) - \Psi(w_1)| \prec 1$. Then, applying Lemma 3.4, we get:

$$\begin{aligned}
(51)\qquad \left(I(E_{R_1,1}^{12})\right)^q &:= \int_{E_{R_1,1}^{12}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|\Psi'(\tau)|^{2-q}\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^q} \\
&\prec \int_{E_{R_1,1}^{12}} \frac{|\tau-w_1|^{-\gamma(q-1)+(2-q)(\lambda-1+\varepsilon)}\,|d\tau|}{|\tau-w|^{q(\lambda+\varepsilon)}} \\
&\prec \int_{E_{R_1,1}^{12}} \frac{|d\tau|}{|\tau-w|^{q(\lambda+\varepsilon)-(2-q)(\lambda-1+\varepsilon)+(-\gamma)(q-1)}} \\
&\prec n^{q(\lambda+\varepsilon)-(2-q)(\lambda-1+\varepsilon)+(-\gamma)(q-1)-1},\ (\gamma+2)\lambda \geq 1;
\end{aligned}$$

$$I(E_{R_1,1}^{12}) \prec n^{\frac{\gamma+2}{p}\lambda-\frac{1}{p}+\varepsilon},\ (\gamma+2)\lambda \geq 1.$$

$$\begin{aligned}
(52)\qquad \left(I(E_{R_1,2}^{12})\right)^q &:= \int_{E_{R_1,2}^{12}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|\Psi'(\tau)|^{2-q}\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^q} \\
&\prec \int_{E_{R_1,2}^{12}} \frac{|\tau-w_1|^{-\gamma(q-1)+(2-q)(\lambda-1+\varepsilon)}\,|d\tau|}{|\tau-w_1|^{q(\lambda+\varepsilon)}} \\
&\prec \int_{E_{R_1,2}^{12}} \frac{|d\tau|}{|\tau-w_1|^{q(\lambda+\varepsilon)-(2-q)(\lambda-1+\varepsilon)+(-\gamma)(q-1)}} \\
&\prec n^{q(\lambda+\varepsilon)-(2-q)(\lambda-1+\varepsilon)+(-\gamma)(q-1)-1};
\end{aligned}$$

$$I(E_{R_1,2}^{12}) \prec n^{\frac{2}{p}\lambda-\frac{1}{p}+\varepsilon},\ (\gamma+2)\lambda \geq 1.$$

For $\tau \in E_{R_1}^{13}$ and $w \in \{\tau : |\tau - w_1| < c_1(R_1 - 1)\} \cap \Delta(0, R)$ we see that $\eta < |\tau - w_1| < 2\pi\dot{R}_1$, $|\tau - w| \geq \eta - c_1$. Therefore, from Lemma 3.1, we have

$|\Psi(\tau)-\Psi(w)| \succ |\tau-w|^{1+\varepsilon}$, $|\Psi(\tau)-\Psi(w_1)| \succ 1$. Then, applying Lemma 3.4, we get:

$$\begin{aligned}\left(J_2^3(z)\right)^q &:= \int_{E^{13}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)} |\Psi'(\tau)|^{2-q} |d\tau|}{|\Psi(\tau)-\Psi(w)|^q} \\ &\prec \int_{E^{13}_{R_1,1}} \frac{|\tau-w_1|^{-\varepsilon} |d\tau|}{|\Psi(\tau)-\Psi(w)|^q} \prec n^{\varepsilon} \int_{E^{13}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{q+\varepsilon}} \prec n^{q-1+\varepsilon}; \\ J_2^3(z) &\prec n^{\frac{1}{p}+\varepsilon}.\end{aligned} \tag{53}$$

In this case, from (49)-(53) we get:

$$\sum_{k=1}^{3} J_2^k(z) \prec \begin{cases} n^{\frac{\gamma^*+2}{p}\lambda-\frac{1}{p}+\varepsilon}, & z\in\Omega(\delta), \\ n^{\frac{1}{p}+\varepsilon}, & z\in\widehat{\Omega}(\delta),\end{cases} \quad \gamma<0,\ (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0,$$

and, from (27) for A_n, we obtain:

$$A_n(z) \prec \|P_n\|_p \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & z\in\Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & z\in\widehat{\Omega}(\delta),\end{cases} \quad \gamma<0,\ (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0. \tag{54}$$

Case 2. Let $q>2$ $(p<2)$. Then, $2-q<0$ and, so

$$\begin{aligned}\left(I(E^{1k}_{R_1,1})\right)^q &:= \begin{cases} \int_{E^{1k}_{R_1,1}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{\gamma(q-1)+q}}, & \text{if } \gamma\geq 0, \\ \int_{E^{1k}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w)|^{q}}, & \text{if } \gamma<0,\end{cases} \\ \left(I(E^{1k}_{R_1,2})\right)^q &:= \int_{E^{1k}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}},\ k=1,2, \\ \left(J_2^3(z)\right)^q &:= \int_{E^{13}_{R_1}} \frac{|\Psi'(\tau)|^{2-q}|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}|\Psi(\tau)-\Psi(w)|^{q}}.\end{aligned} \tag{55}$$

2.1. Let $1\leq\lambda<2$.

2.1.1. If $\gamma\geq 0$, applying Lemma 3.4 to (55), we obtain:

$$\begin{aligned}\left(I(E^{11}_{R_1,1})\right)^q &\prec \int_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)}|\tau-w|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)} mes E^{11}_{R_1,1} \\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1}; \\ I(E^{11}_{R_1,1}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ \forall\varepsilon>0.\end{aligned} \tag{56}$$

$$\begin{aligned}\left(I(E^{11}_{R_1,2})\right)^q &\prec \int_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)}|\tau-w_1|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)} mes E^{11}_{R_1,1}\end{aligned} \tag{57}$$

$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};$$

$$I(E^{11}_{R_1,2}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ \forall\varepsilon>0.$$

$$\begin{aligned}(58)\qquad \left(I(E^{12}_{R_1,1})\right)^q &\prec \int_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)}\,|\tau-w|^{[\gamma(q-1)+q](\lambda+\varepsilon)}}\\ &\prec \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau-w|^{[\gamma(q-1)+q](\lambda+\varepsilon)+(\lambda-1+\varepsilon)(2-q)}}\\ &\prec n^{[\gamma(q-1)+q]\lambda+(\lambda-1)(2-q)-1+\varepsilon};\\ I(E^{12}_{R_1,1}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0.\end{aligned}$$

$$\begin{aligned}(59)\qquad \left(I(E^{12}_{R_1,2})\right)^q &< \int_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}}\\ &\prec \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)}\,|\tau-w_1|^{[\gamma(q-1)+q](\lambda+\varepsilon)}}\\ &\prec \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)}}\\ &\prec n^{[\gamma(q-1)+q]\lambda+(\lambda-1)(2-q)-1+\varepsilon};\\ I(E^{12}_{R_1,2}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0.\end{aligned}$$

For $\tau\in E^{13}_{R_1}$ and $w\in\{\tau:|\tau-w_1|<c_1(R_1-1)\}\cap\Delta(w_1,\eta)$ we see that $\eta<|\tau-w_1|<2\pi\dot{R}_1$, $|\tau-w|\geq\eta-c_1$. Therefore, from Lemma 3.1, we have $|\Psi(\tau)-\Psi(w_1)|\succ 1$ or for $w\in\widehat{\Delta}(w_1,\eta)$ $|\Psi(\tau)-\Psi(w)|\succ|\tau-w|^{1+\varepsilon}$. Then, applying Lemma 3.4, we get:

$$\begin{aligned}\left(J^3_2(z)\right)^q &:= \int_{E^{13}_{R_1}} \frac{|\Psi'(\tau)|^{2-q}\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)}\,|\Psi(\tau)-\Psi(w)|^q}\\ (60)\qquad &\prec n^{\varepsilon}\int_{E^{13}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{q+\varepsilon}} \prec n^{q-1+\varepsilon};\\ J^3_2(z) &\prec n^{\frac{1}{p}+\varepsilon}.\end{aligned}$$

From (56)-(60) and (27), for $\gamma\geq 0$, we have:

$$(61)\qquad A_n(z)\prec\|P_n\|_p\begin{cases} n^{\frac{\gamma+2}{p}\ \lambda+\varepsilon}, & z\in\Omega(\delta),\\ n^{\frac{2}{p}+\varepsilon}, & z\in\widehat{\Omega}(\delta),\end{cases}\quad\forall\varepsilon>0.$$

2.1.2. If $\gamma<0$, analogously, we have:

$$\left(I(E^{11}_{R_1,1})\right)^q \prec \int_{E^{11}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^q}$$

$$\prec \int_{E^{11}_{R_1,1}} \frac{|\tau-w_1|^{-\gamma(\lambda-\varepsilon)(q-1)}}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)}\,|\tau-w|^{(\lambda+\varepsilon)q}}\,|d\tau|$$
$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)} mes E^{11}_{R_1,1}$$
$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};$$
$$I(E^{11}_{R_1,1}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ \forall\varepsilon>0.$$

$$\left(I(E^{11}_{R_1,2})\right)^q \prec \int_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}}$$
$$\prec \int_{E^{11}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)}}$$
$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)} mes E^{11}_{R_1,1}$$
$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};$$
$$I(E^{11}_{R_1,2}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ \forall\varepsilon>0.$$

$$\left(I(E^{12}_{R_1,1})\right)^q \prec \int_{E^{12}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^{q}}$$
$$\prec \int_{E^{12}_{R_1,1}} \frac{|\tau-w_1|^{-\gamma(\lambda-\varepsilon)(q-1)}}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)}\,|\tau-w|^{(\lambda+\varepsilon)q}}\,|d\tau|$$
$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};$$
$$I(E^{12}_{R_1,1}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ \forall\varepsilon>0.$$

$$\left(I(E^{12}_{R_1,2})\right)^q \prec \int_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}}$$
$$\prec \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)}}$$
$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};$$
$$I(E^{12}_{R_1,2}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ \forall\varepsilon>0.$$

$$\left(J^3_2(z)\right)^q \prec \int_{E^{13}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^{q}} \prec \int_{E^{13}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\varepsilon}\,|\tau-w|^{(1+\varepsilon)q}}$$
$$\prec n^{\varepsilon}\int_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{q-1+\varepsilon}} \prec n^{q-1+\varepsilon};$$
$$J^3_2(z) \prec n^{\frac{1}{p}+\varepsilon}.$$

So, for $\gamma < 0$, from (27), we have:

$$A_n(z) \prec \|P_n\|_p \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & z \in \Omega(\delta), \\ n^{\frac{2}{p}+\varepsilon}, & z \in \widehat{\Omega}(\delta), \end{cases} \quad \forall \varepsilon > 0. \tag{62}$$

2.2. Let $0 < \lambda < 1$.

2.2.1. If $\gamma \geq 0$, applying Lemma 3.4 to (55), we obtain:

$$\begin{aligned} \left(I(E^{11}_{R_1,1})\right)^q &\prec \int_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau - w_1|^{(\lambda-1+\varepsilon)(q-2)} |\tau - w|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)} mes E^{11}_{R_1,1} \\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1}; \\ I(E^{11}_{R_1,1}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ (\gamma+2)\lambda \geq 1, \ \forall \varepsilon > 0. \end{aligned} \tag{63}$$

$$\begin{aligned} \left(I(E^{11}_{R_1,2})\right)^q &\prec \int_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\tau - w_1|^{(\lambda-1+\varepsilon)(q-2)} |\tau - w_1|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)} mes E^{11}_{R_1,1} \\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1}; \\ I(E^{11}_{R_1,2}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ (\gamma+2)\lambda \geq 1, \ \forall \varepsilon > 0. \end{aligned} \tag{64}$$

$$\begin{aligned} \left(I(E^{12}_{R_1,1})\right)^q &\prec \int_{E^{12}_{R_1,1}} \frac{|d\tau|}{|\tau - w_1|^{(\lambda-1+\varepsilon)(q-2)} |\tau - w|^{[\gamma(q-1)+q](\lambda+\varepsilon)}} \\ &\prec \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau - w|^{[\gamma(q-1)+q](\lambda+\varepsilon)+(\lambda-1+\varepsilon)(2-q)}} \\ &\prec n^{[\gamma(q-1)+q]\lambda+(\lambda-1)(2-q)-1+\varepsilon}; \\ I(E^{12}_{R_1,1}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ (\gamma+2)\lambda \geq 1, \ \forall \varepsilon > 0. \end{aligned} \tag{65}$$

(66)

$$\begin{aligned} \left(I(E^{12}_{R_1,2})\right)^q &< \int_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2} |\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)+q}} \\ &\prec \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau - w_1|^{(\lambda-1+\varepsilon)(q-2)} |\tau - w_1|^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)}} \\ &\prec n^{[\gamma(q-1)+q]\lambda+(\lambda-1)(2-q)-1+\varepsilon}, \ [\gamma(q-1)+q]\,\lambda \geq 1; \\ I(E^{12}_{R_1,2}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon}, \ (\gamma+2)\lambda \geq 1, \ \forall \varepsilon > 0. \end{aligned}$$

For $\tau \in E^{13}_{R_1}$ and $w \in \{\tau : |\tau - w_1| < c_1(R_1-1)\} \cap \Delta(w_1, \eta)$ we see that $\eta < |\tau - w_1| < 2\pi \dot{R}_1$, $|\tau - w| \geq \eta - c_1$. Therefore, from Lemma 3.1, we have

$|\Psi(\tau)-\Psi(w_1)|\succ 1$ or $|\Psi(\tau)-\Psi(w)|\succ |\tau-w|^{1+\varepsilon}$ for $w\in\widehat{\Delta}(w_1,\eta)$. Then, applying Lemma 3.4, we get:

$$\begin{aligned}\left(J_2^3(z)\right)^q &\prec \int_{E^{13}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^q}\\ &\prec \int_{E^{13}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\varepsilon}\,|\tau-w|^{(1+\varepsilon)q}}\\ &\prec n^{\varepsilon}\int_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{q-1+\varepsilon}} \prec n^{q-1+\varepsilon};\\ J_2^3(z) &\prec n^{\frac{1}{p}+\varepsilon}.\end{aligned} \tag{67}$$

For $\gamma\geq 0$, from (63)-(67) and (27), we have:

$$A_n(z)\prec \|P_n\|_p \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & z\in\Omega(\delta),\\ n^{\frac{2}{p}+\varepsilon}, & z\in\widehat{\Omega}(\delta),\end{cases} \quad (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0. \tag{68}$$

2.2.2. If $\gamma<0$, analogously, we have:

$$\begin{aligned}\left(I(E^{11}_{R_1,1})\right)^q &\prec \int_{E^{11}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^q}\\ &\prec \int_{E^{11}_{R_1,1}} \frac{|\tau-w_1|^{-\gamma(\lambda-\varepsilon)(q-1)}}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)}\,|\tau-w|^{(\lambda+\varepsilon)q}}\,|d\tau|\\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)} mes E^{11}_{R_1,1}\\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};\\ I(E^{11}_{R_1,1}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0.\end{aligned}$$

$$\begin{aligned}\left(I(E^{11}_{R_1,2})\right)^q &\prec \int_{E^{11}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}}\\ &\prec \int_{E^{11}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)}}\\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)} mes E^{11}_{R_1,1}\\ &\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};\\ I(E^{11}_{R_1,2}) &\prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0.\end{aligned}$$

$$\begin{aligned}\left(I(E^{12}_{R_1,1})\right)^q &\prec \int_{E^{12}_{R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^q}\\ &\prec \int_{E^{12}_{R_1,1}} \frac{|\tau-w_1|^{-\gamma(\lambda-\varepsilon)(q-1)}}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)}\,|\tau-w|^{(\lambda+\varepsilon)q}}\,|d\tau|\end{aligned}$$

$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};$$

$$I(E^{12}_{R_1,1}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ (\gamma+2)\lambda \geq 1,\ \forall\varepsilon>0;$$

$$\left(I(E^{12}_{R_1,2})\right)^q \prec \int_{E^{12}_{R_1,2}} \frac{|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w_1)|^{\gamma(q-1)+q}}$$
$$\prec \int_{E^{12}_{R_1}} \frac{|d\tau|}{|\tau-w_1|^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)}}$$
$$\prec n^{(\lambda-1+\varepsilon)(q-2)+[\gamma(q-1)+q](\lambda+\varepsilon)-1};$$

$$I(E^{12}_{R_1,2}) \prec n^{\frac{[\gamma+2]\lambda-1}{p}+\varepsilon},\ (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0.$$

$$\left(J_2^3(z)\right)^q \prec \int_{E^{13}_{R_1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{-\gamma(q-1)}\,|d\tau|}{|\Psi'(\tau)|^{q-2}\,|\Psi(\tau)-\Psi(w)|^q} \prec \int_{E^{13}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{\varepsilon}\,|\tau-w|^{(1+\varepsilon)q}}$$
$$\prec n^{\varepsilon}\int_{E^{11}_{R_1,1}} \frac{|d\tau|}{|\tau-w|^{q-1+\varepsilon}} \prec n^{q-1+\varepsilon};$$
$$J_2^3(z) \prec n^{\frac{1}{p}+\varepsilon}.$$

So, for $\gamma<0$, we have:

$$A_n(z) \prec \|P_n\|_p \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & z\in\Omega(\delta),\\ n^{\frac{2}{p}+\varepsilon}, & z\in\widehat{\Omega}(\delta),\end{cases} \quad (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0. \tag{69}$$

Therefore, for any $\gamma\geq -2,\ 0<\lambda<1$, $p<2$, from (68) and (69) we get:

$$A_n(z) \prec \|P_n\|_p \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & z\in\Omega(\delta),\\ n^{\frac{2}{p}+\varepsilon}, & z\in\widehat{\Omega}(\delta),\end{cases} \quad (\gamma+2)\lambda\geq 1,\ \forall\varepsilon>0. \tag{70}$$

Combining (35), (41), (48), (54), (61), (62), (68) and (70), for any $p>1$, $\gamma>-2$, and for all sufficiently small $\varepsilon>0$, we obtain:

$$\begin{aligned} A_n(z) &= \sum_{k=1}^{3} A^1_{n,k} \\ &\prec \|P_n\|_p \cdot \begin{cases} n^{\frac{\gamma+2}{p}\lambda+\varepsilon}, & \frac{1}{\gamma+2}\leq\lambda<2, \quad z\in\Omega(\delta),\\ n^{\frac{2}{p}+\varepsilon}, & \frac{1}{\gamma+2}\leq\lambda<2, \quad z\in\widehat{\Omega}(\delta),\\ 1, & \text{otherwise.}\end{cases}\end{aligned} \tag{71}$$

Now, let us estimate $B_n(z)$. For this, first of all, replacing the variable $\tau=\Phi(\zeta)$ and according to Lemma 3.4, we obtain:

$$\begin{aligned} B_n(z) &= \int_{L_{R_1}} \frac{|d\zeta|}{|\zeta-z|^2} = \int_{|\tau|=R_1} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^2} \\ &= \int_{\{|\tau|=R_1\}\cap\Delta_1} \frac{|\tau-w_1|^{\lambda-1-\varepsilon}\,|d\tau|}{|\tau-w|^{2(\lambda+\varepsilon)}} \end{aligned} \tag{72}$$

$$+\int_{\{|\tau|=R_1\}\cap\,\widehat{\Delta}_1}\frac{(|\tau|-1)^{-\varepsilon}\,|d\tau|}{|\tau-w|^{2(1+\varepsilon)}}$$
$$=: B_n^1(z)+B_n^2(z).$$

Let us set:

$$F_1 := \{\{\tau : |\tau| = R_1\}\cap\Delta_1 : |\tau-w_1| \geq |\tau-w|\},$$
$$F_2 := (\{\tau : |\tau| = R_1\}\cap\Delta_1)\backslash F_1.$$

Under this notations we have:

$$B_n^1(z) = \int_{F_1}\frac{|\tau-w_1|^{\lambda-1-\varepsilon}\,|d\tau|}{|\tau-w|^{2(\lambda+\varepsilon)}}+\int_{F_2}\frac{|\tau-w_1|^{\lambda-1-\varepsilon}\,|d\tau|}{|\tau-w|^{2(\lambda+\varepsilon)}}$$
$$\prec \begin{cases} \left(\frac{1}{n}\right)^{\lambda-1-\varepsilon}\int_{F_1}\frac{|d\tau|}{|\tau-w|^{2(\lambda-\varepsilon)}}+\int_{F_2}\frac{|d\tau|}{|\tau-w|^{\lambda+1+\varepsilon}}, & \text{if } \lambda\geq 1,\\ \int_{F_1}\frac{|d\tau|}{|\tau-w|^{2(\lambda+\varepsilon)-\lambda+1-\varepsilon}}+\left(\dfrac{1}{n}\right)^{\lambda-1-\varepsilon}\int_{F_2}\frac{|d\tau|}{|\tau-w|^{2(\lambda+\varepsilon)}}, & \text{if } \lambda<1,\end{cases}$$
$$\prec \begin{cases} n^{\lambda+\varepsilon}, & \text{if } \lambda\geq 1,\\ n^{\lambda+\varepsilon}, & \text{if } \lambda<1,\end{cases}\quad \forall\varepsilon>0;$$
$$B_n^2(z) = \int_{\{|\tau|=R_1\}\cap\,\widehat{\Delta}_1}\frac{(|\tau|-1)^{-\varepsilon}\,|d\tau|}{|\tau-w|^{2(1-\varepsilon)}}\prec n^{1+\varepsilon},\ \forall\varepsilon>0.$$

So, from (72), we have:

$$B_n(z)\prec\begin{cases} n^{\lambda+\varepsilon}, & z\in\Omega(\delta),\\ n^{1+\varepsilon}, & z\in\widehat{\Omega}(\delta),\end{cases}\quad\forall\varepsilon>0. \tag{73}$$

Now, combining (23), (24), (71), (73), we obtain:

$$|P_n'(z)|\prec|\Phi(z)|^{n+1}\left[\frac{1}{d(z,L_{R_1})}\int_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|}+|P_n(z)|\int_{L_{R_1}}\frac{|d\zeta|}{|\zeta-z|^2}\right]$$
$$\prec\frac{1}{d(z,L_{R_1})}\|P_n\|_p\begin{cases}|\Phi(z)|^{n+1}\begin{cases} n^{\frac{\gamma+2}{p}\ \lambda+\varepsilon}, & \frac{1}{\gamma+2}\leq\lambda<2, & z\in\Omega(\delta),\\ n^{\frac{2}{p}+\varepsilon}, & \frac{1}{\gamma+2}\leq\lambda<2, & z\in\widehat{\Omega}(\delta),\\ 1, & \text{otherwise},\end{cases}\\ +|\Phi(z)|^{2(n+1)}G_n(p,\lambda,\varepsilon)\begin{cases} n^{\lambda+\varepsilon}, & z\in\Omega(\delta),\\ n^{1+\varepsilon}, & z\in\widehat{\Omega}(\delta),\end{cases}\end{cases}$$

where $G_n(p,\lambda,\varepsilon)$ defined as in (14). Therefore, we complete the proof of Theorem 2.2. □

4.2. Proof of Theorem 2.4

Proof. Suppose that $G\in C_\theta(\lambda)$ for some $0<\lambda<2$ and $h(z)$ be defined as in (1) for $l=1$. For an arbitrary point $z\in L$ and $R=1+cn^{-1}$, by Cauchy integral formula, we have:

$$P_n'(z)=\frac{1}{2\pi i}\int_{L_R}P_n(\zeta)\frac{d\zeta}{(\zeta-z)^2}.$$

Then,

$$\left|P_n'(z)\right| \leq \frac{1}{2\pi}\int_{L_R}\frac{\left|P_n(\zeta)\right|\left|d\zeta\right|}{\left|\zeta-z\right|^2} \leq \frac{1}{2\pi}\max_{\zeta\in\overline{G_R}}\left|P_n(\zeta)\right|\int_{L_R}\frac{\left|d\zeta\right|}{\left|\zeta-z\right|^2}.$$

According to (4) and [15, Theorem 1.1], we get:

$$\left|P_n'(z)\right| \prec \left\|P_n\right\|_{C(\overline{G})}\cdot\int_{L_R}\frac{\left|d\zeta\right|}{\left|\zeta-z\right|^2} \prec \mu_{n,1}\cdot\left\|P_n\right\|_p\cdot S_n(z)\ , \tag{74}$$

where

$$S_n(z) := \int_{L_R}\frac{\left|d\zeta\right|}{\left|\zeta-z\right|^2},\ z\in L,$$

and

$$\mu_{n,1} := \begin{cases} n^{\frac{(2+\widetilde{\gamma})\cdot\widetilde{\lambda}}{p}}, & \text{if } (2+\gamma)\cdot\widetilde{\lambda} > 1, \\ (n\ln n)^{\frac{1}{p}}, & \text{if } (2+\gamma)\cdot\widetilde{\lambda} = 1, \\ n^{\frac{1}{p}}, & \text{if } (2+\gamma)\cdot\widetilde{\lambda} < 1, \end{cases}$$

$$\widetilde{\gamma} := \max\{0,\gamma\}\ ,\ \widetilde{\lambda} := \max\{1,\lambda\}+\varepsilon.$$

It remains for us to estimate of $S_n(z)$. For this, first of all, replacing the variable $\tau = \Phi(\zeta)$, and according to Lemma 3.4, we obtain:

$$\begin{aligned} S_n(z) &= \int_{L_R}\frac{\left|d\zeta\right|}{\left|\zeta-z\right|^2} = \int_{|\tau|=R}\frac{\left|\Psi'(\tau)\right|\left|d\tau\right|}{\left|\Psi(\tau)-\Psi(w)\right|^2} \\ &= \int_{M_1}\frac{\left|\tau-w_1\right|^{\lambda-1-\varepsilon}\left|d\tau\right|}{|\tau-w|^{2(\lambda+\varepsilon)}} + \int_{M_2}\frac{(|\tau|-1)^{-\varepsilon}\left|d\tau\right|}{|\tau-w|^{2(1+\varepsilon)}} \\ &=: S_n^1(z)+S_n^2(z), \end{aligned} \tag{75}$$

where we set: $M_1 := \{t : |t| = R\}\cap\Delta_1$; $M_2 := \{t : |t| = R\}\cap\widehat{\Delta}_1$. Under this notations, for all $z\in L$, we have:

$$S_n^1(z) = \int_{M_1}\frac{\left|\tau-w_1\right|^{\lambda-1-\varepsilon}\left|d\tau\right|}{|\tau-w|^{2(\lambda+\varepsilon)}} \prec \begin{cases} \left(\frac{1}{n}\right)^{\lambda-1-\varepsilon}\int_{M_1}\frac{|d\tau|}{|\tau-w|^{2(\lambda-\varepsilon)}}, \\ \int_{M_1}\frac{|d\tau|}{|\tau-w|^{2(\lambda+\varepsilon)-\lambda+1-\varepsilon}}, \end{cases}$$

$$\prec \begin{cases} n^{\lambda+\varepsilon}, & \text{if } \lambda\geq 1, \\ n^{\lambda+\varepsilon}, & \text{if } \lambda < 1, \end{cases}\quad \forall\varepsilon>0;$$

$$S_n^1(z) \prec n^{\lambda+\varepsilon},\ 0<\lambda<2,\ \forall\varepsilon>0. \tag{76}$$

$$S_n^2(z) = \int_{M_2}\frac{(|\tau|-1)^{-\varepsilon}\left|d\tau\right|}{|\tau-w|^{2(1-\varepsilon)}} \prec n^{1+\varepsilon},\ \forall\varepsilon>0. \tag{77}$$

So, from (75)-(77), for all $z\in L$ we have:

$$S_n(z) \prec n^{\widetilde{\lambda}},\ \forall\varepsilon>0. \tag{78}$$

Combining (74) with (78) we complete the proof. □

4.3. Proof of Theorem 2.9

Proof. The sharpness of Theorem 2.4 is obtained as follows:

a) Let $h^*(z)\equiv 1$. If $Q_n(z):=\sum_{j=0}^{n}(j+1)z^j$, $G_1^*=B(0,1)=\{z:|z|<1\}$ and $p=2$, then $Q_n'(z):=\sum_{j=0}^{n}j(j+1)z^{j-1}$, and in this case, we have:

$$\|Q_n'\|_\infty=\frac{n(n+1)(2n+1)}{6};\quad \|Q_n\|_{A_2(B)}=\sqrt{\frac{\pi(n+1)(n+2)}{2}}.$$

Then,

$$\|Q_n'\|_\infty\geq\frac{1}{3\sqrt{2\pi}}n^2\,\|Q_n\|_{A_2(B)}\,.$$

b) Let $h^*(z)\neq 1$. In this case, the sharpness is seen from the following two facts:

b1) For an arbitrary polynomial $P_n\in\wp_n$, the following well known exact Bernstein inequalities holds:

$$\|P_n'\|_\infty\leq n\,\|P_n\|_\infty\,.$$

b2) For all polynomial $P_n\in\wp_n$ and $\gamma\geq-1$, the exact estimate holds ([12, Theorem 2.1; Remark 2.2]):

$$\|P_n\|_\infty\leq c_8n^2\,\|P_n\|_2\,. \qquad \square$$

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