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PAIRED HAYMAN CONJECTURE AND UNIQUENESS OF COMPLEX DELAY-DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, the paired Hayman conjecture of different types are considered, namely, the zeros distribution of $f(z)^n L(g) - a(z)$ and $g(z)^n L(f) - a(z)$, where $L(h)$ takes the derivatives $h^{(k)}(z)$ or the shift $h(z+c)$ or the difference $h(z+c)-h(z)$ or the delay-differential $h^{(k)}(z+c)$, where k is a positive integer, c is a non-zero constant and $a(z)$ is a non-zero small function with respect to $f(z)$ and $g(z)$. The related uniqueness problems of complex delay-differential polynomials are also considered.

1. Introduction and main results

We assume that the reader is familiar with the basic notations and fundamental results of Nevanlinna theory [8, 21], such as the proximity function $m(r, f)$, the counting function $N(r, f)$, the characteristic function $T(r, f)$, the order $\rho(f)$, the hyper-order $\rho_2(f)$, and so on. A small function $a(z)$ with respect to $f(z)$ means $T(r, a(z)) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. We say that two meromorphic functions $f(z)$ and $g(z)$ share a small function $a(z)$ CM, if $f(z) - a(z)$ and $g(z) - a(z)$ admit the same zeros with the same multiplicities, the abbreviation CM for counting multiplicities.

In 1959, Hayman published one of his significant papers [7], where the zero distribution of complex differential polynomials was considered. For example, [7, Theorem 10] can be stated as follows.

Theorem A. *If $f(z)$ is a transcendental entire function and $n \geq 2$ is a positive integer, then $f(z)^n f'(z) - a$ has infinitely many zeros, where a is a non-zero constant.*

Recall that Clunie [4] proved that Theorem A is also true for the case $n = 1$. The well-known Hayman conjecture is also presented in [7]:

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Hayman conjecture. *If $f(z)$ is a transcendental meromorphic function and n is a positive integer, then $f(z)^n f'(z) - a$ has infinitely many zeros, where a is a non-zero constant.*

Hayman conjecture has been proved completely. The case of $n \geq 3$ is proved by Hayman [7, Corollary to Theorem 9], by Mues [17] for $n = 2$ and by Bergweiler and Eremenko [2, Theorem 2], Chen and Fang [3] and Zalcman [22] for $n = 1$, respectively. Hence, Theorem A can be seen as the start to consider the zero distribution of complex differential polynomials. In 2007, Laine and Yang [11, Theorem 2] obtained the zero distribution of complex difference polynomials as follows, which can be viewed as the difference version of Hayman conjecture.

Theorem B. *If $f(z)$ is a transcendental entire function of finite order, c is a non-zero constant and $n \geq 2$, then $f(z)^n f(z+c) - a$ has infinitely many zeros, where a is a non-zero constant.*

Now, some improvements on Theorem B have been obtained, such as the constant a can be replaced by a non-zero small function $a(z)$ with respect to $f(z)$, $f(z)^n$ can be improved to certain polynomials, see [16, Theorem 1]. Liu and Yang [15, Theorem 1.4] also considered the zeros of $f(z)^n[f(z+c) - f(z)] - p(z)$, where $p(z)$ is a non-zero polynomial.

For the case that $f(z)$ is a transcendental meromorphic function of hyper-order less than one in Theorem B, Liu, Liu and Cao [13] proved Theorem B is also true for $n \geq 6$, Wang and Ye [19] proved Theorem B is true for the case $n \geq 4$, and counter-examples of Theorem B of $n \leq 3$ can be found in [13], more details and improvements of difference versions of Hayman conjecture can be found in Liu, Laine and Yang [12, Chapter 2].

The delay-differential version of Hayman conjecture was first considered by Liu, Liu and Zhou [14], using the ideas of common zeros and common poles in Wang and Ye [19], the improvement below and more results can be found in [12, Chapter 2]. The latest results related to delay-differential versions of Hayman conjecture can also be found in Laine and Latreuch [10].

Theorem C. *Let $f(z)$ be a transcendental meromorphic function of hyper-order $\rho_2(f) < 1$ and $a(z)$ be a non-zero small function with respect to $f(z)$. If $n \geq k + 4$, resp. if $n \geq 3$ and $f(z)$ is transcendental entire, then $f(z)^n f^{(k)}(z + c) - a(z)$ has infinitely many zeros.*

In this paper, we will consider the paired Hayman conjecture of complex delay-differential polynomials of different types. In fact, we consider the zeros distribution of $f(z)^n L(g) - a(z)$ and $g(z)^n L(f) - a(z)$, where $L(h)$ takes the derivatives $h^{(k)}(z)(k \geq 1)$ or the shift $h(z + c)$ or the difference $h(z + c) - h(z)$ (when considering this case, we assume that $h(z)$ is not a periodic function with period c in the paper) or the delay-differential $h^{(k)}(z + c)(k \geq 1)$, where c is a non-zero constant and $a(z)$ is a non-zero small function with respect to

$f(z)$ and $g(z)$. Denote $\mathcal{M}$ for the function class of transcendental meromorphic functions and $\mathcal{M}'$ for transcendental meromorphic functions of hyper-order less than one. Denote $\mathcal{E}$ for the function class of transcendental entire functions and $\mathcal{E}'$ for transcendental entire functions of hyper-order less than one.

Theorem 1.1. *If one of the following conditions is satisfied*:

(i) $L(h) = h^{(k)}(z)$, $n \geq k+4$ *and* $h \in \mathcal{M}$ *or* $n \geq 3$ *and* $h \in \mathcal{E}$;
(ii) $L(h) = h(z+c)$, $n \geq 4$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$;
(iii) $L(h) = h(z+c) - h(z)$, $n \geq 5$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$;
(iv) $L(h) = h^{(k)}(z+c)$, $n \geq k+4$ *and* $h \in \mathcal{M}'$ *or* $n \geq 3$ *and* $h \in \mathcal{E}'$,

then at least one of $f(z)^n L(g) - a(z)$ *and* $g(z)^n L(f) - a(z)$ *has infinitely many zeros, where* $a(z)$ *is a non-zero small function with respect to* $f(z)$ *and* $g(z)$.

Obviously, if $f \equiv g$, then Theorem 1.1 may reduce to Hayman conjecture of different types. We proceed to give some observations on the case of transcendental entire functions $f(z), g(z)$ and $a(z) \equiv 0$. If $f(z)^n g^{(k)}(z)$ and $g(z)^n f^{(k)}(z)$ have finitely many zeros, then the functions $f(z), f^{(k)}(z), g(z), g^{(k)}(z)$ must have finitely many zeros. If $k \geq 2$, then $f(z) = P_1(z)e^{Q_1(z)}$ and $g(z) = P_2(z)e^{Q_2(z)}$, where $P_i(z), Q_i(z)$, $(i = 1, 2)$ are polynomials by [8, Theorem 3.8]. In particular, if $f(z)^n g^{(k)}(z)$ and $g(z)^n f^{(k)}(z)$ $(k \geq 2)$ have no zeros, then $f(z) = e^{A_1 z + B_1}$ and $g(z) = e^{A_2 z + B_2}$, where A_1, A_2 are non-zero constants, see [8, Theorem 3.8]. If $k = 1$, we can take $f(z) = e^{h_1(z)}$ and $g(z) = e^{h_2(z)}$ where $h_i'(z)(i = 1, 2)$ are entire functions without zeros. The same discussions can be applied to the case that $f(z)^n g^{(k)}(z+c)$ and $g(z)^n f^{(k)}(z+c)$ have finitely many zeros. If $f(z)^n g(z+c)$ and $g(z)^n f(z+c)$ have finitely many zeros, then $f(z)$ and $g(z)$ must have finitely many zeros. If $f(z)^n(g(z+c) - g(z))$ and $g(z)^n(f(z+c) - f(z))$ have finitely many zeros, then the functions $f(z)$, $f(z+c) - f(z)$, $g(z)$ and $g(z+c) - g(z)$ must have finitely many zeros. Furthermore, assume that $f, g \in \mathcal{E}'$, we can obtain $f(z) = e^{A_1 z + B_1}$ and $g(z) = e^{A_2 z + B_2}$, where A_1, A_2 are non-zero constants. From Hadamard factorization theorem, we can assume that $f(z) = P(z)e^{H(z)}$, where $P(z)$ is a polynomial and $H(z)$ is an entire function with $\rho(H) < 1$, we then assume $f(z+c) - f(z) = T(z)e^{S(z)}$, where $T(z)$ is a polynomial and $S(z)$ is an entire function with $\rho(S) < 1$. Thus,

$$P(z+c)e^{H(z+c)} - P(z)e^{H(z)} = T(z)e^{S(z)}.$$

Rewrite the above equation as follows:

$$\frac{P(z+c)}{T(z)}e^{H(z+c)-S(z)} - \frac{P(z)}{T(z)}e^{H(z)-S(z)} = 1.$$

Using the second main theorem of Nevanlinna, we see $H(z+c) - S(z)$ must be a constant, and we also have $H(z) - S(z)$ must be a constant. Thus, $H(z+c) - H(z)$ is also a constant, and it implies that $\rho(H) \geq 1$ if $H(z)$ is transcendental, which is a contradiction. Thus $H(z)$ is a linear polynomial. It is also an interesting problem to obtain the forms of f, g such that $f(z)^n L(g) - a(z)$ and $g(z)^n L(f) - a(z)$ have both finitely many zeros, where $a(z)$ is a non-zero small function with respect to $f(z), g(z)$ and n does not satisfy the conditions in Theorem 1.1.

Remark 1.2. If $n = 1$, then Theorem 1.1 is not true. For example, take $f(z) = e^z$, $g(z) = e^{-z}$ and $a(z)(\not\equiv 0, 1, -1)$ is a polynomial in Case (i), take $f(z) = 1 + e^z$ and $g(z) = -1 - e^z$ and $e^c = -1$ and $a(z) = -1$ in Case (ii), take $f(z) = z + e^z$, $g(z) = -z + e^z$, $e^c = 1$ and $a(z) = -zc$ in Case (iii), take $f(z) = e^z$, $g(z) = e^{-z}$, $e^c = -1$ and $a(z)(\not\equiv 0, 1, -1)$ is a polynomial in Case (iv).

Remark 1.3. The condition of hyper-order less than one cannot be deleted in Cases (ii), (iii), (iv). For example, take $f(z) = e^{e^z}$ and $g(z) = e^{-e^z}$, where $e^c = n$ and $a(z)$ is a non-constant polynomial in Case (ii), $a(z) \equiv 1$ in (iii), $a(z)$ is a polynomial of e^z for different k in (iv).

In the following, we will consider the uniqueness problem on complex delay-differential polynomials sharing common small functions or common values. Yang and Hua [20] have considered the uniqueness problems on $f(z)^n f'(z)$ and $g(z)^n g'(z)$ share one non-zero constant a CM as follows.

Theorem D. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic (entire) functions, $n \geq 11 (n \geq 7)$ be an integer. If $f(z)^n f'(z)$ and $g(z)^n g'(z)$ share the value a CM, then either $f(z) = tg(z)$, where $t^{n+1} = 1$ or $f(z) = c_1 e^{-cz}$ and $g(z) = c_2 e^{cz}$, where c, c_1 and c_2 are constants and $(c_1 c_2)^{n+1} c^2 = -a^2$.*

Combining Theorem 1.1 and Theorem D, it evokes us to consider the uniqueness problems provided that $f(z)^n L(g)$ and $g(z)^n L(f)$ share a non-zero small function $a(z)$ CM, where $a(z)$ is a small function with respect to $f(z)$ and $g(z)$. For the case of meromorphic functions $f(z), g(z)$, we obtain:

Theorem 1.4. *Let f and g be transcendental meromorphic functions. If $f(z)^n L(g)$ and $g(z)^n L(f)$ share a non-zero small function $a(z)$ CM, and one of the following conditions is satisfied*

(i) $L(h) = h^{(k)}(z)$, $n \geq 3k + 16$ *and* $f, g \in \mathcal{M}$;
(ii) $L(h) = h(z + c)$, $n \geq 16$ *and* $f, g \in \mathcal{M}'$;
(iii) $L(h) = h(z + c) - h(z)$, $n \geq 19$ *and* $f, g \in \mathcal{M}'$;
(iv) $L(h) = h^{(k)}(z + c)$, $n \geq 3k + 16$ *and* $f, g \in \mathcal{M}'$,

then $f(z)^n L(g) = g(z)^n L(f)$ *or* $f(z)^n L(g) g(z)^n L(f) = a(z)^2$.

For the case of entire functions $f(z)$ and $g(z)$, we obtain:

Theorem 1.5. *Let f and g be transcendental entire functions. If $f(z)^n L(g)$ and $g(z)^n L(f)$ share a non-zero small function $a(z)$ CM, and one of the following conditions is satisfied*

(i) $L(h) = h^{(k)}(z)$, $n \geq 8$ *and* $f, g \in \mathcal{E}$;
(ii) $L(h) = h(z + c)$, $n \geq 8$ *and* $f, g \in \mathcal{E}'$;
(iii) $L(h) = h(z + c) - h(z)$, $n \geq 8$ *and* $f, g \in \mathcal{E}'$;

(iv) $L(h) = h^{(k)}(z+c)$, $n \geq 8$ *and* $f, g \in \mathcal{E}'$,

then $f(z)^n L(g) = g(z)^n L(f)$ *or* $f(z)^n L(g) g(z)^n L(f) = a(z)^2$.

Corollary 1.6. *Let* f *and* g *be transcendental entire functions and* $n \geq 8$. *If* $f(z)^n g'(z)$ *and* $g(z)^n f'(z)$ *share a non-zero constant* a *CM, then* $f(z) = tg(z)$ *and* $t^{n-1} = 1$ *or* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$ *where* c, c_1 *and* c_2 *are constants and* $(c_1 c_2)^{n+1} c^2 = a^2$.

Remark 1.7. If $n = 1$, then Corollary 1.6 is not true. For example $f(z) = e^z + e^{-z}$ and $g(z) = 2e^z + 2e^{-z}$, then $f(z)g'(z)$ and $g(z)f'(z)$ share any non-zero value a CM. If $n = 2$, then Corollary 1.6 is also not true. For example $f(z) = e^z + 1$ and $g(z) = e^{-z} - 1$, then $f(z)^2 g'(z)$ and $g(z)^2 f'(z)$ share the value 2 CM.

Corollary 1.8. *Let* f *and* g *be transcendental entire functions with hyper order less than one and* $n \geq 8$. *If* $f(z)^n g(z+c)$ *and* $g(z)^n f(z+c)$ *share one non-zero constant* a *CM, then* $f(z) = c_1 g(z)$ *where* $c_1^{n-1} = 1$ *or* $f(z)g(z) = c_2$ *where* $c_2^{n+1} = a^2$.

Remark 1.9. If $n = 1$, then Corollary 1.8 is not true. For example $f(z) = 1 + e^z$ and $g(z) = e^{-z} - 1$, $e^c = -1$, then $f(z)g(z+c)$ and $g(z)f(z+c)$ share -2 CM.

For further studying, we raise the following question.

Question 1. Can we reduce $n \geq 3$ to $n \geq 2$ in Theorem 1.1 for entire functions f, g in $\mathcal{E}$ or $\mathcal{E}'$? And what is the sharp value n in Theorem 1.1 for meromorphic functions f, g in $\mathcal{M}$ or $\mathcal{M}'$?

Question 2. How to describe the relationship between f and g from the equations $f(z)^n L(g) = g(z)^n L(f)$ and $f(z)^n L(g) g(z)^n L(f) = a(z)^2$, where $L(h)$ is defined in Theorem 1.4 and $a(z)$ is a small function with respect to f and g?

2. Lemmas

Related to the estimate on the zeros of derivatives of meromorphic functions, one basic result is stated as follows, see [21, Theorem 1.14].

Lemma 2.1. *Let* $f(z)$ *be a non-constant meromoprhic function and* k *be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}(z)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + k\overline{N}(r, f(z)) + S(r, f(z)). \tag{2.1}$$

The characteristic function of $L(h)$ is important to estimate the value n in our results, when $L(h)$ takes the derivatives $h^{(k)}(z)$ or the shift $h(z+c)$ or the difference $h(z+c) - h(z)$ or the delay-differential $h^{(k)}(z+c)$, we summarize the results in the following lemma.

Lemma 2.2. (1) $T\left(r, \frac{1}{h^{(k)}(z)}\right) \leq (k+1)T(r,h(z)) + S(r,h(z))$, *where* $h \in \mathcal{M}$, *and* $T\left(r, \frac{1}{h^{(k)}(z)}\right) \leq T(r,h(z)) + S(r,h(z))$, *where* $h \in \mathcal{E}$.

(2) $T\left(r, \frac{1}{h(z+c)}\right) \leq T(r,h(z)) + S(r,h(z))$, *where* $h \in \mathcal{M}'$.

(3) $T\left(r, \frac{1}{h(z+c)-h(z)}\right) \leq 2T(r,h(z)) + S(r,h(z))$, *where* $h \in \mathcal{M}'$, *and*

$$T\left(r, \frac{1}{h(z+c)-h(z)}\right) \leq T(r,h(z)) + S(r,h(z)),$$

where $h \in \mathcal{E}'$.

(4) $T\left(r, \frac{1}{h^{(k)}(z+c)}\right) \leq (k+1)T(r,h(z)) + S(r,h(z))$, *where* $h \in \mathcal{M}'$, *and*

$$T\left(r, \frac{1}{h^{(k)}(z+c)}\right) \leq T(r,h(z)) + S(r,h(z)),$$

where $h \in \mathcal{E}'$.

Proof. The Case (1) is obvious. The Cases (2) and (3) are obtained by [6, Lemma 8.3] and the first main theorem of Nevanlinna. The Case (4) also can be obtained by Case (1) and [6, Lemma 8.3]. □

Lemma 2.3. *If* $f, g \in \mathcal{M}$, *then*

$$\begin{aligned} nT(r,f) - (k+1)T(r,g) &\leq T(r, f^n g^{(k)}) + S(r,g) \\ &\leq nT(r,f) + (k+1)T(r,g). \end{aligned} \tag{2.2}$$

If $f, g \in \mathcal{E}$, *then*

$$nT(r,f) - T(r,g) \leq T(r, f^n g^{(k)}) + S(r,g) \leq nT(r,f) + T(r,g). \tag{2.3}$$

Proof. Since f and g are transcendental meromorphic functions, from Valiron-Mohon'ko theorem, see [9, Theorem 2.2.5], and Lemma 2.2(1), then

$$\begin{aligned} nT(r,f) = T(r,f^n) &= T\left(r, f^n g^{(k)} \frac{1}{g^{(k)}}\right) \leq T(r, f^n g^{(k)}) + T\left(r, \frac{1}{g^{(k)}}\right) \\ &\leq T(r, f^n g^{(k)}) + (k+1)T(r,g) + S(r,g). \end{aligned}$$

Thus, the left-hand side of (2.2) is proved. The right hand-side of (2.2) is trivial. We also can get (2.3) by considering f, g are entire functions. □

Using the similar method to the above, we can also obtain the following three lemmas.

Lemma 2.4. *If* $f, g \in \mathcal{M}'$ *or* $\mathcal{E}'$, *then*

$$nT(r,f) - T(r,g) \leq T(r, f(z)^n g(z+c)) + S(r,g) \leq nT(r,f) + T(r,g). \tag{2.4}$$

Lemma 2.5. *If* $f, g \in \mathcal{M}'$ *and* $g(z+c) - g(z) \not\equiv 0$, *then*

$$\begin{aligned} nT(r,f) - 2T(r,g) &\leq T(r, f(z)^n (g(z+c) - g(z))) + S(r,g) \\ &\leq nT(r,f) + 2T(r,g). \end{aligned} \tag{2.5}$$

If $f, g \in \mathcal{E}'$ and $g(z+c) - g(z) \not\equiv 0$, then

$$nT(r,f) - T(r,g) \leq T(r, f(z)^n(g(z+c) - g(z))) + S(r,g) \leq nT(r,f) + T(r,g). \tag{2.6}$$

Lemma 2.6. *If $f, g \in \mathcal{M}'$, then*

$$nT(r,f) - (k+1)T(r,g) \leq T(r, f(z)^n g^{(k)}(z+c)) + S(r,g) \leq nT(r,f) + (k+1)T(r,g). \tag{2.7}$$

If $f, g \in \mathcal{E}'$, then

$$nT(r,f) - T(r,g) \leq T(r, f(z)^n g^{(k)}(z+c)) + S(r,g) \leq nT(r,f) + T(r,g). \tag{2.8}$$

Remark 2.7. The left-hand sides of Lemma 2.3 to Lemma 2.6 are important to get the estimate of n in Theorem 1.1, and the right-hand sides of Lemma 2.3 to Lemma 2.6 are useful when we get the relations among small functions, such as $S(r, f^n g^{(k)}) = S(r,f) + S(r,g)$ from (2.2).

Let p be a positive integer and $a \in \mathbb{C}$. We denote by $N_p(r, \frac{1}{f-a})$ the counting function of the zeros of $f - a$ where an m-fold zero is counted m times if $m \leq p$ and p times if $m > p$. Similarly, $N_p(r, f)$ denotes the counting function of the poles of f where an m-fold pole is counted m times if $m \leq p$ and p times if $m > p$. The following lemma is given by An and Phuong [1], which is important for the proof of Theorem 1.4.

Lemma 2.8 ([1, Lemma 2])**.** *Let f and g be non-constant meromorphic functions, and let α be a non-zero small function with respect to f and g. If f and g share α CM, then precisely one of the following statements hold*:

(i) $T(r,f) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$, *and* $T(r,g) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + N_2(r,f) + N_2(r,g) + S(r,f) + S(r,g)$,

(ii) $f \equiv g$,

(iii) $f \cdot g \equiv \alpha^2$.

Remark 2.9. Considering the meromorphic functions f and g share a non-zero small function α CM, the readers should also be very carefully since $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ may not share the value 1 CM. For example, $f = \frac{1}{z} + e^z$ and $g(z) = \frac{1}{z} + \frac{e^z}{z}$ share $\alpha(z) = \frac{1}{z}$ CM, but $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ do not share the value 1. The observation is first given by Schweizer [18]. See more details in [18] and [1].

3. Proofs of Theorems

Proof of Theorem 1.1. Let $\Psi(z) := f(z)^n L(g) - a(z)$. Then

$$nm(r, f(z)) = m(r, f(z)^n) = m\left(\frac{\Psi(z) + a(z)}{L(g)}\right) \tag{3.1}$$

$$\leq m(r,\Psi+a)+m\left(\frac{1}{L(g)}\right)+O(1),$$

and

$$nN(r,f(z))=N(r,f(z)^n)=N\left(\frac{\Psi(z)+a(z)}{L(g)}\right)$$
$$\leq N(r,\Psi+a)+N\left(\frac{1}{L(g)}\right)-\overline{N}_0(r)-\overline{N}_1(r), \tag{3.2}$$

where $\overline{N}_0(r)$, resp. $\overline{N}_1(r)$ are reducing counting functions, stands for the common zeros, resp. poles, of $\Psi+a$ and $L(g)$. Combining (3.1), (3.2) and the first main theorem of Nevanlinna, we obtain

$$nT(r,f(z))\leq T(r,\Psi+a)+T\left(r,\frac{1}{L(g)}\right)-\overline{N}_0(r)-\overline{N}_1(r)+O(1). \tag{3.3}$$

From the expression of Ψ, we have

$$\overline{N}(r,\Psi+a)\leq \overline{N}(r,f(z))+\overline{N}_1(r)\leq T(r,f)+\overline{N}_1(r), \tag{3.4}$$

and

$$\overline{N}\left(r,\frac{1}{\Psi+a}\right)\leq \overline{N}\left(r,\frac{1}{f(z)}\right)+\overline{N}_0(r)\leq T(r,f)+\overline{N}_0(r). \tag{3.5}$$

Using the second main theorem for three small functions, see [8, Theorem 2.5], (3.4) and (3.5), we obtain

$$T(r,\Psi+a)\leq \overline{N}(r,\Psi+a)+\overline{N}\left(r,\frac{1}{\Psi+a}\right)+\overline{N}\left(r,\frac{1}{\Psi}\right)+S(r,\Psi)$$
$$\leq 2T(r,f)+\overline{N}_1(r)+\overline{N}_0(r)+\overline{N}\left(r,\frac{1}{\Psi}\right)+S(r,f)+S(r,g).$$

Combining (3.3) with the above inequality, we can obtain

$$(n-2)T(r,f)\leq T\left(r,\frac{1}{L(g)}\right)+\overline{N}\left(r,\frac{1}{\Psi}\right)+S(r,f)+S(r,g). \tag{3.6}$$

Case (i). If $L(g)=g^{(k)}(z)$, from Lemma 2.2(1), then we obtain

$$(n-2)T(r,f)-(k+1)T(r,g)\leq \overline{N}\left(r,\frac{1}{\Psi}\right)+S(r,f)+S(r,g). \tag{3.7}$$

Let $\Phi(z):=g(z)^nL(f)-a(z)$. We also can get

$$(n-2)T(r,g)-(k+1)T(r,f)\leq \overline{N}\left(r,\frac{1}{\Phi}\right)+S(r,f)+S(r,g). \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$(n-k-3)[T(r,f)+T(r,g)]\leq \overline{N}\left(r,\frac{1}{\Phi}\right)+\overline{N}\left(r,\frac{1}{\Psi}\right)+S(r,f)+S(r,g). \tag{3.9}$$

Thus, at least one of $f(z)^n g^{(k)}(z) - a(z)$ and $g(z)^n f^{(k)}(z) - a(z)$ have infinitely many zeros if $n \geq k+4$. The following three cases can be proved in a similar way as the above, we will not give the details.

Case (ii). If $L(h) = h(z+c)$, from Lemma 2.2(2), then we get that at least one of $f(z)^n g(z+c) - a(z)$ and $g(z)^n f(z+c) - a(z)$ have infinitely many zeros when $n \geq 4$.

Case (iii). If $L(h) = h(z+c) - h(z)$, from Lemma 2.2(3), then we get that at least one of $f(z)^n [g(z+c) - g(z)] - a(z)$ and $g(z)^n [f(z+c) - f(z)] - a(z)$ have infinitely many zeros when $n \geq 5$.

Case (iv). If $L(h) = h^{(k)}(z+c)$, from Lemma 2.2(4), then we get that at least one of $f(z)^n g^{(k)}(z+c) - a(z)$ and $g(z)^n f^{(k)}(z+c) - a(z)$ have infinitely many zeros when $n \geq k+4$.

The conclusions for entire functions in Theorem 1.1 can be obtained similarly by applying the corresponding inequalities in Lemma 2.2. □

Proof of Theorem 1.4. Case (i). Let $F = f(z)^n g^{(k)}(z)$ and $G = g(z)^n f^{(k)}(z)$. Thus, we have F and G share $a(z)$ CM. From Lemma 2.8(i), Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}
&(n-k-1)[T(r,f) + T(r,g)] \\
&\leq 2N_2\left(r, \frac{1}{f^n g^{(k)}}\right) + 2N_2\left(r, \frac{1}{g^n f^{(k)}}\right) \\
&\quad + 2N_2\left(r, f^n g^{(k)}\right) + 2N_2\left(r, g^n f^{(k)}\right) + S(r,f) + S(r,g) \\
&\leq 2\left[2N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g^{(k)}}\right) + 2N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{f^{(k)}}\right)\right] \\
&\quad + 2(2N(r,f) + 2N(r,g)) + 2(2N(r,g) + 2N(r,f)) + S(r,f) + S(r,g) \\
&\leq 6N\left(r, \frac{1}{f}\right) + 6N\left(r, \frac{1}{g}\right) + (2k+8)N(r,f) + (2k+8)N(r,g) \\
&\quad + S(r,f) + S(r,g) \\
&\leq (2k+14)(T(r,f) + T(r,g)) + S(r,f) + S(r,g).
\end{aligned}$$

Since $n \geq 3k+16$, Lemma 2.8(i) cannot occur. So (ii), (iii) of Lemma 2.8 happen. Thus, $f(z)^n g^{(k)}(z) = g(z)^n f^{(k)}(z)$ or $f(z)^n g^{(k)}(z) g(z)^n f^{(k)}(z) = a(z)^2$ follows.

Case (ii). Let $F = f(z)^n g(z+c)$ and $G = g(z)^n f(z+c)$. Since F and G share $a(z)$ CM, from Lemma 2.8(i) and Lemma 2.4, we have

$$\begin{aligned}
&(n-1)[T(r,f) + T(r,g)] \\
&\leq 2N_2\left(r, \frac{1}{f^n g(z+c)}\right) + 2N_2\left(r, \frac{1}{g^n f(z+c)}\right) \\
&\quad + 2N_2\left(r, f^n g(z+c)\right) + 2N_2\left(r, g^n f(z+c)\right) + S(r,f) + S(r,g)
\end{aligned}$$

$$\leq 2\left[2N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g(z+c)}\right)+2N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{f(z+c)}\right)\right]$$
$$+2(2N(r,f)+2N(r,g))+2(2N(r,g)+2N(r,f))+S(r,f)+S(r,g)$$
$$\leq 6N\left(r,\frac{1}{f}\right)+6N\left(r,\frac{1}{g}\right)+8N(r,f)+8N(r,g)+S(r,f)+S(r,g)$$
$$\leq 14(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Since $n \geq 16$, the above inequality is impossible. So (ii) and (iii) of Lemma 2.8 happen. Thus, $f(z)^n g(z+c) = g(z)^n f(z+c)$ or $f(z)^n g(z+c) g(z)^n f(z+c) = a(z)^2$.

Case (iii). Let $F = f(z)^n(g(z+c)-g(z))$ and $G = g(z)^n(f(z+c)-f(z))$. Since F and G share $a(z)$ CM, from Lemma 2.8(i) and Lemma 2.5, we also can have

$$(n-2)[T(r,f)+T(r,g)] \leq 16(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which is impossible for $n \geq 19$. Thus, we have $f(z)^n(g(z+c)-g(z)) = g(z)^n(f(z+c)-f(z))$ or $f(z)^n(g(z+c)-g(z))g(z)^n(f(z+c)-f(z)) = a(z)^2$.

Case (iv). Let $F = f(z)^n g^{(k)}(z+c)$ and $G = g(z)^n f^{(k)}(z+c)$. Thus, F and G share $a(z)$ CM, from Lemma 2.8(i) and Lemma 2.6, we also can have

$$(n-k-1)[T(r,f)+T(r,g)] \leq (2k+14)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which is impossible for $n \geq 3k+16$. Thus, we have $f(z)^n g^{(k)}(z+c) = g(z)^n f^{(k)}(z+c)$ or $f(z)^n g^{(k)}(z+c) g(z)^n f^{(k)}(z+c) = a(z)^2$. □

Proof of Theorem 1.5. The conclusions can be obtained similarly by applying the method in the proof of Theorem 1.4 and the corresponding inequalities for entire functions in Lemma 2.3–Lemma 2.6. We just give the proof of Case (i). Let $F = f(z)^n g^{(k)}(z)$ and $G = g(z)^n f^{(k)}(z)$. Thus, we have F and G share $a(z)$ CM. From Lemma 2.8(i), Lemma 2.1 and Lemma 2.3, we have

$$(n-1)[T(r,f)+T(r,g)]$$
$$\leq 2N_2\left(r,\frac{1}{f^n g^{(k)}}\right)+2N_2\left(r,\frac{1}{g^n f^{(k)}}\right)+S(r,f)+S(r,g)$$
$$\leq 2\left[2N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g^{(k)}}\right)+2N\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{f^{(k)}}\right)\right]+S(r,f)+S(r,g)$$
$$\leq 6N\left(r,\frac{1}{f}\right)+6N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g)$$
$$\leq 6(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Thus, Lemma 2.8(i) cannot occur for $n \geq 8$. So (ii), (iii) of Lemma 2.8 happen. Hence, $f(z)^n g^{(k)}(z) = g(z)^n f^{(k)}(z)$ or $f(z)^n g^{(k)}(z) g(z)^n f^{(k)}(z) = a(z)^2$ follows. □

Proof of Corollary 1.6. From Theorem 1.5(i), if $f(z)^n g'(z) = g(z)^n f'(z)$, then

$$\frac{f'(z)}{f(z)^n} = \frac{g'(z)}{g(z)^n}. \tag{3.10}$$

Integrating the above equation, we have

$$\left(\frac{1}{f(z)}\right)^{n-1} - \left(\frac{1}{g(z)}\right)^{n-1} = A, \tag{3.11}$$

where A is a constant. Since $n \geq 8$, $A \equiv 0$ follows by a classic result on Fermat type equations (3.11), see Gross [5]. Then, $f(z)^{n-1} = g(z)^{n-1}$, it implies that $f(z) = tg(z)$ and $t^{n-1} = 1$.

If $f(z)^n g'(z) g(z)^n f'(z) = a^2$, from [20, Theorem 3], then $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$ where c, c_1, c_2 are constants and $(c_1 c_2)^{n+1} c^2 = a^2$. □

Proof of Corollary 1.8. From Theorem 1.5(ii), we have

$$f(z)^n g(z+c) = g(z)^n f(z+c) \text{ or } f(z)^n g(z+c) g(z)^n f(z+c) = a^2.$$

If $f(z)^n g(z+c) = g(z)^n f(z+c)$, then $H(z)^n = H(z+c)$ by defining $H(z) = \frac{f(z)}{g(z)}$ and $H(z) \in \mathcal{M}'$. Thus, we have

$$nT(r, H(z)) = T(r, H(z+c)) = T(r, H(z)) + S(r, H(z)),$$

which is impossible for $n \geq 8$ except that $H(z)$ is a constant c_1, so $c_1^{n-1} = 1$. If $f(z)^n g(z+c) g(z)^n f(z+c) = a^2$, then we have $M(z)^n M(z+c) = a^2$ by defining $M(z) = f(z)g(z)$ and $M(z) \in \mathcal{E}'$. Thus, we also have

$$nT(r, M(z)) = T(r, M(z+c)) = T(r, M(z)) + S(r, M(z)),$$

which is impossible for $n \geq 8$ except that $M(z)$ is a constant c_2, so $c_2^{n+1} = a^2$. □

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