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PARABOLIC QUATERNIONIC MONGE-AMPÈRE EQUATION ON COMPACT MANIFOLDS WITH A FLAT HYPERKÄHLER METRIC

Jiaogen Zhang

Abstract. The quaternionic Calabi conjecture was introduced by Alesker-Verbitsky, analogous to the Kähler case which was raised by Calabi. On a compact connected hypercomplex manifold, when there exists a flat hyperKähler metric which is compatible with the underlying hypercomplex structure, we will consider the parabolic quaternionic Monge-Ampère equation. Our goal is to prove the long time existence and C^∞ convergence for normalized solutions as $t \to \infty$. As a consequence, we show that the limit function is exactly the solution of quaternionic Monge-Ampère equation, this gives a parabolic proof for the quaternionic Calabi conjecture in this special setting.

1. Introduction

Suppose that (M, I, J, K, Ω) is a compact HKT (stands for hyperKähler with torsion) manifold, where I, J, K is a triple of complex structures satisfying the imaginary quaternion relations and Ω is a smooth HKT form. Let $f : M \to \mathbb{R}$ be a given smooth function. In the spirit of the famous Calabi-Yau theorem [31] on the compact Kähler manifold, in 2010 Alesker-Verbitsky [5] posed the following quaternionic Calabi conjecture which has been studied extensively.

Conjecture 1.1. *Let (M, I, J, K, Ω) be a compact HKT manifold of real dimension $4n$. Then there exists a constant $A > 0$ such that the equation*

$$(\Omega + \partial\partial_J \varphi)^n = Ae^f \Omega^n, \ \Omega + \partial\partial_J \varphi > 0, \tag{1}$$

admits a unique solution $\varphi \in C^\infty(M, \mathbb{R})$.

One can deduce that the equation (1) is a fully nonlinear elliptic equation of second order when $n \geq 2$. In this paper, we are interested in the HKT manifold because it belongs to the realm of quaternionic geometries and also intimately

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connected with the 2-dimensional sigma models with (4,0) supersymmetry arise in theoretical physics, see [16, 18] and the references therein.

In the pioneer work [5], the authors also conjectured that the equation (1) admits a unique solution when there exists a non-vanishing I-holomorphic $(2n, 0)$ form Θ on the complex manifold (M, I) such that

$$\int_M (1 - Ae^f)\Omega^n \wedge \overline{\Theta} = 0.$$

They proved the uniqueness theorem and also provided the C^0 estimate under the existence of such Θ, by using the classical Moser iteration technique used by Yau [31]. Aside from these, they also gave a geometrical explanation for the equation (1) analogous to the complex case, see also [29] by Verbitsky. Precisely, given an HKT form Ω and a complex volume form denoted by $e^f\Omega^n$ for some smooth function f, we can find another HKT form $\Omega_\varphi := \Omega + \partial\partial_J\varphi$ whose volume form is exactly the prescribed form $e^f\Omega^n$.

When the hypercomplex structure is locally flat[1], using a similar argument of Błocki [7] in Kähler manifolds, Alesker-Shelukhin [2] were able to give the C^0 estimate. Furthermore, under a more stringent assumption, when we assume M is a compact HKT manifold with a flat hyperKähler metric which is compatible with the underlying hypercomplex structure, the conjecture was already confirmed by Alesker [1]. In [3], Alesker-Shelukhin systematically considered the C^0 estimate without further assumptions. Very recently, Sroka [23] provided a simpler proof, by using a similar procedure as in [11,17,25,27,32] for the Hermitian case. Let us remark that when the paper was being reviewed, the same result was posted on arxiv [6], we wish to thank the referee for pointing out it to us.

Motivated by the work of Alesker [1], our goal of this paper is to consider the parabolic quaternionic Monge-Ampère equation for the unknown function φ, which can be written in the following form

$$\begin{cases} \partial_t\varphi = \log\frac{(\Omega+\partial\partial_J\varphi)^n}{\Omega^n} - f, \\ \Omega + \partial\partial_J\varphi > 0, \\ \varphi(\cdot, 0) = 0. \end{cases} \tag{2}$$

Similar flows have been extensively studied in [10, 12, 15, 19, 24] and references therein in the Riemannian, Kähler, Hermitian and almost Hermitian settings.

We now state our main result.

Theorem 1.2. *Suppose (M, I, J, K, Ω) is a compact HKT manifold of real dimension $4n$. Assume there exists a flat hyperKähler metric which is compatible with the underlying hypercomplex structure. Let $f \in C^\infty(M, \mathbb{R})$. Then the solution φ for the flow (2) exists for all time.*

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[1] Locally, the hypercomplex structures I, J and K are pull backs of standard hypercomplex structures in $\mathbb{H}^n$. For more details, see [2, 22].

Moreover, if we normalize Ω *by a constant such that* $\int_M \Omega^n \wedge \bar{\Omega}^n = 1$. *Let*

$$\tilde{\varphi} = \varphi - \int_M \varphi \Omega^n \wedge \bar{\Omega}^n. \tag{3}$$

Then $\tilde{\varphi}$ *converges to* φ_∞ *smoothly as* $t \to \infty$, *and there exists a constant* $b \in \mathbb{R}$ *such that*

$$(\Omega + \partial\partial_J \varphi_\infty)^n = e^{f+b}\Omega^n. \tag{4}$$

This gives a parabolic proof of the quaternionic Calabi conjecture in a special setting, based on the solution of Alesker toward this conjecture [1].

The organization of this paper as follows: In Section 2, we will collect some basic concepts in hypercomplex manifolds, especially those notions we have used repeatedly in this paper. The C^0 and C^2 estimates for the flow (2) will be provided in Section 3 and Section 4, respectively. In Section 5, we will prove the long time existence for the flow (2). Using the parabolic Harnack inequality established in Section 6, we can obtain the convergence of the flow (2) in Section 7.

2. Preliminaries

In this section, to avoid confusions, we give a short review of some concepts in the hypercomplex manifolds which will be used repeatedly in this paper.

2.1. HKT manifolds

We start by recalling the definition of HKT manifolds. Let M be a smooth manifold of real dimension $4n$ endowed with a triple of endomorphisms I, J, K on TM satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -Id.$$

Moreover, if those I, J, K are integrable complex structures, then we call (M, I, J, K) is a *hypercomplex* manifold, which was introduced explicitly by Boyer [9]. In what follows, we also suppose that I, J, K act on the right on the tangent bundle TM of M, since the left action is similar.

Let G be a Riemannian metric on the hypercomplex manifold (M, I, J, K). We say that G is quaternionic Hermitian if I, J, K are G-orthogonal, i.e.,

$$G(X, Y) = G(X \cdot I, Y \cdot I) = G(X \cdot J, Y \cdot J) = G(X \cdot K, Y \cdot K).$$

In this case, we say (M, I, J, K, G) is a *hyperhermitian* manifold. Another equivalent definition states that G is invariant with respect to the group $SU(2)$ $\subset \mathbb{H}^*$ of unitary quaternionic, that is, $G(X \cdot q, Y \cdot q) = G(X, Y)$ for all real vector fields X, Y and all $q \in \mathbb{H}$ with $\|q\| = 1$. For simplicity, we denote S_M by whole sphere of complex structures, i.e.,

$$S_M = \big\{aI + bJ + cK \mid a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1\big\}.$$

Given a quaternionic Hermitian metric G on the hypercomplex manifold (M, I, J, K), let us consider the following Hodge type (2,0) form

$$\Omega = \omega_J - \sqrt{-1}\omega_K$$

with respect to I, where $\omega_N = G(\cdot N, \cdot)$ for each $N \in S_M$. Let ∂ be a Dolbeault differential operator on (M, I), we say (M, I, J, K, Ω) is an HKT manifold if

$$\partial\Omega = 0.$$

In this case, the metric G is referred to as an HKT metric, which was first introduced by Howe and Papadopoulos [18]. Sometimes, we also call Ω is an HKT form with respect to the HKT metric G.

Remark 2.1. It is well-known that G is a hyperKähler metric when we further assume $\bar{\partial}\Omega = 0$. Notice that in [2], Alesker provided an example of HKT manifold (such as the quaternionic torus: quotient of $\mathbb{H}^n$ by a lattice) which admits a flat hyperKähler metric while the original metric is not hyperKähler.

2.2. ∂_J operator and quaternionic Hessian

Now we are ready to recall the definition of ∂_J operator, which was first introduced by Verbitsky [28].

Let (M, I, J, K) be a hypercomplex manifold of real dimension $4n$, and $\Lambda_I^{p,q}(M)$ be the vector bundle of (p, q) forms on (M, I). By the abuse of notations, in what follows we shall use the same symbol $\Lambda_I^{p,q}(M)$ to denote the space of smooth sections of this bundle.

Let $\partial : \Lambda_I^{p,q}(M) \longrightarrow \Lambda_I^{p+1,q}(M)$ be the usual ∂-differential on differential forms on the complex manifold (M, I). We set $\partial_J = J^{-1} \cdot \bar{\partial} \cdot J$. First, consider $\partial_J : C^\infty \to \Lambda_I^{1,0}(M)$ which maps f to $J^{-1}(\bar{\partial} f)$, where $\bar{\partial} : C^\infty \to \Lambda_I^{0,1}(M)$ is the standard Dolbeault differential operator on (M, I). Using the Leibniz rule, we can extend it to

$$\partial_J : \Lambda_I^{p,q}(M) \to \Lambda_I^{p+1,q}(M).$$

One can verify that $\partial\partial_J = -\partial_J\partial$.

For any $p = 0, 1, \ldots, n$, we say a form $\omega \in \Lambda_I^{2p,0}(M)$ is real if $\overline{J \cdot \omega} = \omega$, where the conjugate is in the quaternionic sense. Let us denote the space of C^∞-smooth $(2p, 0)$ real forms on (M, I) by $\Lambda_{I,\mathbb{R}}^{2p,0}(M)$. We have the following lemma.

Lemma 2.2 ([1, Lemma 0.5])**.** *Let (M, I, J, K) be a hypercomplex manifold. For $v \in C^\infty(M, \mathbb{R})$, then $\partial\partial_J v \in \Lambda_{I,\mathbb{R}}^{2,0}(M)$ and we call it the quaternionic Hessian of v.*

Let $S_{\mathbb{H}}M$ be a vector bundle over M whose fiber at $q \in M$ is exactly the set of hyperhermitian forms on the tangent space T_qM. Recall the isomorphism

$$\tau : \Lambda_{I,\mathbb{R}}^{2,0}(M) \to S_{\mathbb{H}}M \tag{5}$$

which is defined by

$$\tau(\eta)(V, V) := \eta(V, V \cdot J)$$

for any real vector field V on M.

For each quaternion $q \in \mathbb{H}$, we write it in the following standard form

$$q = t \cdot 1 + x \cdot i + y \cdot j + z \cdot k,$$

where t, x, y, $z \in \mathbb{R}$ and i, j, k satisfy the usual quaternionic relations. For each $\mathbb{H}$-valued function F, its Cauchy-Riemann-Fueter derivatives $\frac{\partial}{\partial \bar{q}}$ were given as

$$\frac{\partial F}{\partial \bar{q}} := \frac{\partial F}{\partial t} + i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z},$$

and its quaternion conjugate $\frac{\partial}{\partial q}$ were defined by

$$\frac{\partial F}{\partial q} := \frac{\partial F}{\partial t} - \frac{\partial F}{\partial x}i - \frac{\partial F}{\partial y}j - \frac{\partial F}{\partial z}k.$$

In the higher dimensional case, we can also define $\frac{\partial}{\partial \bar{q}_j}$ and $\frac{\partial}{\partial q_i}$ in the similar way. One can obtain the following commute relationship

$$\Big[\frac{\partial}{\partial q_i}, \frac{\partial}{\partial \bar{q}_j}\Big] = 0.$$

Let $X \subset \mathbb{H}^n$ be an open subset. Now we recall another version of Hessian for real valued functions on X. For any $F \in C^\infty(X, \mathbb{R})$, the matrix $\left(\frac{\partial^2 F}{\partial \bar{q}_i \partial q_j}\right) \in S_{\mathbb{H}}X$ is hyperhermitian. In what follows we will denote it by $\mathrm{Hess}_{\mathbb{H}} F$ for convenience. Sometimes, we also call it as the quaternionic Hessian of F without confusion occurs. Furthermore, using the τ-isomorphism defined in (5), one can arrive that

$$\tau(\partial\partial_J F) = \kappa \cdot \Big(\frac{\partial^2 F}{\partial \bar{q}_i \partial q_j}\Big), \tag{6}$$

where $\kappa > 0$ is a normalizing constant and we may choose $\kappa = 1$ (see e.g. [4]).

Definition 2.3. Let (M, I, J, K, Ω) be an HKT manifold of real dimension $4n$. We say a form $\eta \in \Lambda^{2,0}_{I,\mathbb{R}}(M)$ is strictly positive (resp. positive) if $\tau(\eta) > 0$ (resp. $\tau(\eta) \geq 0$). Moreover, we say a form $\Theta \in \Lambda^{2n,0}_{I,\mathbb{R}}(M)$ is strictly positive if

$$\Theta = e^{\phi}\Omega^n$$

for some function $\phi \in C^\infty(M, \mathbb{R})$.

Notice that the positivity of Θ above is independent of the choice of Ω, since the canonical bundle $\Lambda^{2n,0}_{I,\mathbb{R}}(M)$ is orientable. Analogous to the complex case, we recall the notion of plurisubharmonic functions.

Definition 2.4. We say $F \in C^2(M, \mathbb{R})$ is strictly quaternionic plurisubharmonic (resp. quaternionic plurisubharmonic) if $\partial\partial_J F \in \Lambda^{2,0}_{I,\mathbb{R}}(M)$ is strictly positive (resp. positive).

Let us remark that we can extend this terminology to general real valued continuous functions, see [4, Definition 7.1] for more details.

For a hyperhermitian matrix A, one can define the so called *Moore determinant* $\det(A)$ (see [2, Difinition 17] and references therein). The following lemma is a quiet useful tool in many places.

Lemma 2.5 ([4, Corollary 4.6])**.** *Let $F : X \to \mathbb{R}$ be a smooth function. Then there exists a dimensional constant $c_n > 0$ such that*

$$(\partial\partial_J F)^n = c_n \det(\mathrm{Hess}_{\mathbb{H}} F)\Omega_n, \tag{7}$$

where $\Omega_n \in \Lambda^{2n,0}_{I,\mathbb{R}}(X)$ is a standard strictly positive form in $\mathbb{H}^n$.

3. C^0 estimate

Proposition 3.1. *Let φ solve the flow (2). There is a positive constant C depending on all the allowed data and $\|f\|_{L^\infty(M)}$ such that*

$$\sup_{M_T} |\tilde{\varphi}| \le \sup_{t\in[0,T)} \big(\sup_{x\in M} \varphi(x,t) - \inf_{x\in M} \varphi(x,t)\big) \le C, \tag{8}$$

where $[0,T)$ is a maximal time interval of the flow (2).

Proof. Differentiating the flow (2) by $\frac{\partial}{\partial t}$, we can deduce the following heat type equation:

$$\partial_t\big(\partial_t\varphi\big) = \frac{n\partial\partial_J\partial_t\varphi \wedge (\Omega + \partial\partial_J\varphi)^{n-1}}{(\Omega + \partial\partial_J\varphi)^n} =: \Box\big(\partial_t\varphi\big). \tag{9}$$

Clearly, $\Box$ is an operator of second order and elliptic.

Applying the parabolic maximum principle for (9), we know that $\partial_t\varphi$ attains its maximum at $t = 0$. Thus,

$$\sup_{M_T} \big|\partial_t\varphi\big| \le \sup_{M\times\{0\}} \big|\partial_t\varphi\big| \le \big\|f\big\|_{L^\infty(M)}. \tag{10}$$

Let us denote $\tilde{f} = f + \partial_t\varphi$. Then this is a smooth bounded function on M for each $t \in [0,T)$. Moreover,

$$\sup_{M_T} |\tilde{f}| \le \|f\|_{L^\infty(M)} + \sup_{M_T} \Big|\partial_t\varphi(x,t)\Big| \le 2\|f\|_{L^\infty(M)}. \tag{11}$$

Since φ solves the equation,

$$(\Omega + \partial\partial_J\varphi)^n = e^{\tilde{f}}\Omega^n.$$

According to the main Theorem in [2] (see also [3, 23] for more general case), there is a constant C depending on the allowed data and on $\sup_{M_T} |\tilde{f}|$ (hence on $\|f\|_{L^\infty(M)}$) such that

$$\sup_{t\in[0,T)} \big(\sup_{x\in M} \varphi(x,t) - \inf_{x\in M} \varphi(x,t)\big) \le C,$$

which completes the proof. $\Box$

4. C^2 estimate

In this section we mainly prove the following theorem:

Theorem 4.1. *Suppose (M, I, J, K, Ω) is a compact HKT manifold of real dimension $4n$ with a flat hypercomplex structure. Let us assume in addition that M admits a metric $\tilde{G}$ which is parallel to with respect to the Obata connection ∇^{Ob}.*[2] *If φ solves the flow (2), then there exists a constant C depending on (M, I, J, K), Ω, $\tilde{G}$ such that*

$$\sup_{M\times[0,T)} |\mathrm{Hess}_{\mathbb{H}}\varphi|_{\tilde{G}} \leq C. \tag{12}$$

Proof. Let us define a Laplacian operator: for each $h \in C^2(M, \mathbb{R})$,

$$\tilde{\Delta}h = \mathrm{Tr}(\tilde{G}^{-1} \cdot \mathrm{Hess}_{\mathbb{H}}h).$$

We consider the following quantity

$$Q = 2\sqrt{\mathrm{Tr}(\tilde{G}^{-1} \cdot (G + \mathrm{Hess}_{\mathbb{H}}\varphi))} - \tilde{\varphi}.$$

For any $T' < T$, we may assume Q achieves its maximum at (x_0, t_0) in $M \times [0, T']$. Let $g \in C^\infty$ be the local potential function of the metric G (see e.g. [4, Proposition 1.14]), whence $u = g + \varphi$ is a strictly plursubharmonic function and we denote $U = \mathrm{Hess}_{\mathbb{H}}u$.

Around (x_0, t_0), we can pick up a proper locally flat coordinates $(q_1, \ldots, q_n)$ such that $\tilde{G} = I_n$ in a small neighborhood and U is diagonal at (x_0, t_0). Based on this notation we have

$$Q = 2(\tilde{\Delta}u)^{\frac{1}{2}} - \tilde{\varphi}.$$

We also recall another Laplacian operator: for each $h \in C^2(M, \mathbb{R})$,

$$\Delta_\varphi h := \mathrm{Tr}\Big((G + \mathrm{Hess}_{\mathbb{H}}\varphi)^{-1} \cdot \mathrm{Hess}_{\mathbb{H}}h\Big).$$

For simplicity, we denote $\partial_t\varphi$ by φ_t. At (x_0, t_0), by the maximal principle,

$$\begin{aligned} 0 &\leq \big(\frac{\partial}{\partial t} - \Delta_\varphi\big)Q \\ &= (\tilde{\Delta}u)^{-\frac{1}{2}}\tilde{\Delta}\varphi_t - \varphi_t + \int_M \varphi_t\Omega^n \wedge \bar{\Omega}^n + \Delta_\varphi\varphi - 2\Delta_\varphi(\tilde{\Delta}u)^{\frac{1}{2}} \\ &= (\tilde{\Delta}u)^{-\frac{1}{2}}\tilde{\Delta}\varphi_t - \varphi_t + \int_M \varphi_t\Omega^n \wedge \bar{\Omega}^n + n - \sum_i \frac{1}{u_{i\bar{i}}} - 2\Delta_\varphi(\tilde{\Delta}u)^{\frac{1}{2}}. \end{aligned} \tag{13}$$

Now in this local coordinates around (x_0, t_0) we have $\det U = \exp\{f + \varphi_t\}$. It follows [1, Proposition 3.6] that

$$2\Delta_\varphi(\tilde{\Delta}u)^{\frac{1}{2}} \geq (\tilde{\Delta}u)^{-\frac{1}{2}}\big(\tilde{\Delta}f + \tilde{\Delta}\varphi_t\big). \tag{14}$$

[2]As pointed in [1, p. 204], M admits a flat hyperKähler metric χ compatible with the hypercomplex structure is equivalent to say $\tilde{G}$ is parallel with respect to the Obata connection.

In light of (13) to (14) and Proposition 3.1, we can deduce that

$$\sum_i \frac{1}{u_{i\bar{i}}} + (\tilde{\Delta} u)^{-\frac{1}{2}} \tilde{\Delta} f \leq C_1.$$

Let us set $C_2 := \|\tilde{\Delta} f\|_{C^0(M)}$. It follows that

$$\sum_i \frac{1}{u_{i\bar{i}}} \leq C_1 + C_2 \big(\sum_i u_{i\bar{i}} \big)^{-\frac{1}{2}}. \tag{15}$$

We may assume $u_{1\bar{1}} = \min_{1\leq i\leq n} u_{i\bar{i}}$. Therefore,

$$\frac{1}{u_{1\bar{1}}} \leq C_1 + C_2 u_{1\bar{1}}^{-\frac{1}{2}}, \tag{16}$$

which implies $u_{i\bar{i}} \geq u_{1\bar{1}} \geq C_3^{-1}$ for all $1 \leq i \leq n$. Since

$$\prod_i u_{i\bar{i}} = \exp\{f + \varphi_t\} \leq C_4,$$

so, $u_{i\bar{i}} \leq C_4 C_3^{n-1}$ for all i. We conclude that

$$\frac{1}{C_5} \leq \tilde{\Delta} u = \sum_i u_{i\bar{i}} \leq C_5, \tag{17}$$

which gives a uniform estimate for the Laplacian of φ. □

5. Long time existence of the solution

In this section, we will give the proof of the long time existence of Theorem 1.2. To this end, we need the following theorem.

Theorem 5.1. *Let φ solve the flow* (2) *and $[0, T)$ be the maximal time interval. For each $\varepsilon \in (0, T)$ and for each $k \in \mathbb{N}$, there exists a constant $C_{\varepsilon,k}$ depending on the allowed data, ε and k such that*

$$\sup_{M\times[\varepsilon,T)} |\nabla^k \varphi| \leq C_{\varepsilon,k}. \tag{18}$$

Proof. To prove (18), it suffices to show that $G + \mathrm{Hess}_{\mathbb{H}}\varphi$ is Hölder continuous. Indeed, given the Hölder bound for the metric $\mathrm{Hess}_{\mathbb{H}}\varphi$ and the second order estimate for φ, differentiating the flow and then using the Schauder estimates and the standard bootstrapping arguments, we can get the higher order estimates.

The estimate of $\big[\mathrm{Hess}_{\mathbb{H}}\varphi\big]_{C^\alpha(M\times[\varepsilon,T))}$ is standard, we split it into the next proposition and a sketch of proof will be included. □

Proposition 5.2. *Let φ solve the flow* (2) *and $[0, T)$ be the maximal time interval. For each $\varepsilon \in (0, T)$, there exist $\alpha \in (0, 1)$ and a constant C_ε depending only on the initial data and ε such that*

$$\big[\mathrm{Hess}_{\mathbb{H}}\varphi\big]_{C^\alpha(M\times[\varepsilon,T))} \leq C_\varepsilon. \tag{19}$$

Proof. The proof is local since M is locally flat. Let $\mathcal{O} \subset \mathbb{H}^n$ be an arbitrary open subset. For each $\alpha \in (0,1)$, on $\mathcal{O}_T := \mathcal{O} \times [0,T)$, we define

$$\left[\varphi\right]_{\alpha,(x,t)} := \sup_{(y,s)\in\mathcal{O}_T\setminus(x,t)} \frac{|\varphi(y,s)-\varphi(x,t)|}{(|y-x|+\sqrt{|s-t|})^\alpha};$$

$$\left[\varphi\right]_{\alpha,\mathcal{O}_T} := \sup_{(x,t)\in\mathcal{O}_T} \left[\varphi\right]_{\alpha,(x,t)}.$$

As mentioned before, the metric G can be locally represented by a potential g. That is, $G = \mathrm{Hess}_{\mathbb{H}} g$ on $\mathcal{O}$ when we shrink $\mathcal{O}$ if necessary, whence $u = g + \varphi$ is a strictly quaternionic plursubharmonic function and we denote $U = \mathrm{Hess}_{\mathbb{H}} u$. Let us define an operator on the hyperhermitian matrix $A = (a_{i\bar{j}})$ by

$$\Phi(A) := \log\det(A); \tag{20}$$

moreover,

$$\Phi^{i\bar{j}}(A) := \frac{\partial \Phi(A)}{\partial a_{i\bar{j}}} = A^{i\bar{j}}. \tag{21}$$

Now we can rewrite (2) as

$$\partial_t \varphi = \Phi(U) - F, \ U > 0, \tag{22}$$

where $F = f + \log\det G$. By the concavity of Φ, for all $(x,t_1),(y,t_2) \in \mathcal{O} \times [0,T)$, we have

$$\begin{aligned} &\sum U^{i\bar{j}}(y,t_2)(U_{i\bar{j}}(x,t_1) - U_{i\bar{j}}(y,t_2)) \\ &\geq \partial_t\varphi(x,t_1) - \partial_t\varphi(y,t_2) - F(x) + F(y) \\ &\geq \partial_t u(x,t_1) - \partial_t u(y,t_2) - C\|x-y\| \end{aligned} \tag{23}$$

for some constant C depending on $\|F\|_{C^1}$.

Notice that the quaternionic Hessian U has eigenvalues in (λ,Λ) with $0 < \lambda < \Lambda < \infty$ by Theorem 4.1. Analogous to the real and complex settings in [7, 13, 14], we need the following lemma which is from linear algebra.

Lemma 5.3 ([1, Lemma 4.9]). *There exist a constant N, unit vectors ξ^α $(1 \leq \alpha \leq N)$, and constants $0 < \lambda_* < \Lambda_* < \infty$ depending on n, λ, Λ such that*

$$U^{-1} = \sum_{\alpha=1}^{N} \mu_\alpha \xi^\alpha \otimes (\xi^\alpha)^*,$$

where $\lambda_* \leq \mu_\alpha \leq \Lambda_*$ and $\xi^1, \dots, \xi^N$ containing an orthonormal basis of $\mathbb{H}^n$.

For any unit vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{H}^n$, we denote by Δ_ξ the Laplacian on any translate of the quaternionic line spanned by ξ. Then we have

$$\mathrm{Tr}((\xi \otimes \xi^*)(u_{i\bar{j}})) = \mathrm{Tr}(\xi(u_{i\bar{j}})\xi^*) = \Delta_\xi u. \tag{24}$$

For convenience, we denote $\mu_0 = 1$ and $\Delta_{\xi^0} = -\frac{\partial}{\partial t}$. By (23) we obtain

$$\sum_{\beta=0}^{N} \mu_\beta \big(\Delta_{\xi^\beta} u(y, t_2) - \Delta_{\xi^\beta} u(x, t_1)\big) \le C\|x - y\|. \tag{25}$$

Lemma 5.4. *For any $\beta = 0, 1, \ldots, N$, there exists a bounded function h (depending on U) such that*

$$\partial_t \Delta_{\xi^\beta} u \le U^{i\bar{j}}\big(\Delta_{\xi^\beta} U_{i\bar{j}}\big) + h. \tag{26}$$

Proof. For $\beta = 0$. Applying ∂_t to (22), then

$$\partial_t\big(\partial_t u\big) = U^{i\bar{j}} \partial_t\big(U_{i\bar{j}}\big)$$

and the (26) follows.

For other $\beta \ge 1$, write $\xi^\beta = (\xi_1^\beta, \ldots, \xi_n^\beta)$, we can differentiate (22) along ξ_p^β twice and take sum for index p, then

$$\begin{aligned} \partial_t \Delta_{\xi^\beta} u &= U^{i\bar{j}}\big(\Delta_{\xi^\beta} U_{i\bar{j}}\big) + \sum_{p=1}^{n} \frac{\partial^2 \Phi(U)}{\partial a_{i\bar{j}} \partial a_{k\bar{l}}} U_{i\bar{j}\xi_p^\beta} U_{k\bar{l}\xi_p^\beta} - \Delta_{\xi^\beta} F \\ &\le U^{i\bar{j}}\big(\Delta_{\xi^\beta} U_{i\bar{j}}\big) - \Delta_{\xi^\beta} F, \end{aligned}$$

by the concavity of Φ. Then the lemma follows. □

Fixing $\hat{t} \in [\varepsilon, T)$, we can find a constant $1 > r > 0$ sufficient small such that $10r^2 \le \hat{t}$. Define two types of parabolic cylinders

$$P_r := \big\{(x, t) \in \mathcal{O}_T : \|x\| \le r, \hat{t} - 5r^2 \le t \le \hat{t} - 4r^2\big\};$$

and

$$Q_r := \big\{(x, t) \in \mathcal{O}_T : \|x\| \le r, \hat{t} - r^2 \le t \le \hat{t}\big\}.$$

For any $\beta = 0, 1, \ldots, N$, let us denote

$$M_{\beta,r} := \sup_{Q_r} \Delta_{\xi^\beta} u, \;\; m_{\beta,r} := \inf_{Q_r} \Delta_{\xi^\beta} u;$$

$$\eta(r) := \sum_{\beta=0}^{N} (M_{\beta,r} - m_{\beta,r}).$$

To prove (19), it suffices to prove there exist a constant C (only depending on ε) and $0 < \delta < 1$ such that

$$\eta(r) \le C r^\delta.$$

Lemma 5.5 ([1, Lemma 4.6])**.** *The operator $v \to Tr(U^{-1} \cdot Hess_{\mathbb{H}} v)$ can be written in the following divergence form*

$$\mathcal{D}v = \sum_{s,t} D_s(a_{st} D_t v), \tag{27}$$

where s, t run over all the real variables, $(a_{st})_{4n\times 4n}$ is a symmetric matrix with C^2-smooth coefficients satisfy uniform elliptic estimates $\lambda\|\xi\|^2 \le \sum_{s,t} a_{st}\xi_s\xi_t \le \Lambda\|\xi\|^2$ for $0 < \lambda < \Lambda < \infty$ and $\xi \in \mathbb{R}^{4n}$.

The following weak parabolic Harnack inequality is crucial.

Lemma 5.6 ([21, Theorem 7.37]). *If $v \in W^{2,1}_{2n+1}$ is a nonnegative function and satisfies*

$$-\frac{\partial v}{\partial t} + \sum_{s,t} D_s(a_{st} D_t v) \leq h \text{ on } Q_{4r},$$

where h is a bounded function and the matrix (a_{st}) as in Lemma 5.5. Then there exist positive constants C, p depending on n, λ, Λ such that

$$\frac{1}{r^{4n+2}}\Big(\int_{P_r} v^p\Big)^{\frac{1}{p}} \leq C\big(\inf_{B_r} v + r^{\frac{4n}{4n+1}} \|h\|_{L^{2n+1}}\big). \tag{28}$$

For each $\beta = 0, 1, \ldots, N$, let us denote $v_\beta := M_{\beta,2r} - \Delta_{\xi^\beta} u$. Then $v_\beta \in W^{2,1}_{2n+1}$ is a nonnegative function. Moreover, each v_β satisfies

$$-\partial_t v_\beta + \mathrm{Tr}(U^{-1} \cdot \mathrm{Hess}_{\mathbb{H}} v_\beta) \leq h$$

since $\Delta_{\xi^\beta} u_{i\bar{j}} = (\Delta_{\xi^\beta} u)_{i\bar{j}}$ on $\mathcal{O}_T$. Then by Lemmas 5.4 and 5.6,

$$\frac{1}{r^{4n+2}}\Big(\int_{P_r} (M_{\beta,2r} - \Delta_{\xi^\beta} u)^p\Big)^{\frac{1}{p}} \leq C\big(M_{\beta,2r} - M_{\beta,r} + r^{\frac{4n}{4n+1}}\big), \tag{29}$$

where C is a constant as in Lemma 5.6. On the other hand, let $(x, t_1), (y, t_2) \in Q_{2r}$, it then follows from (25) that

$$\mu_\beta\big(\Delta_{\xi^\beta} u(y, t_2) - \Delta_{\xi^\beta} u(x, t_1)\big) \leq Cr + \sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq \beta}} \mu_\gamma\big(\Delta_{\xi^\gamma} u(x, t_1) - \Delta_{\xi^\gamma} u(y, t_2)\big).$$

Recall the definition of $M_{\beta,r}$ and $m_{\beta,r}$, for each $\epsilon > 0$, choose a point $(x, t_1) \in Q_{2r}$ properly such that

$$m_{\beta,2r} \leq \Delta_{\xi^\beta} u(x, t_1) + \epsilon.$$

As a consequence, after dividing a uniform constant μ_β,

$$\Delta_{\xi^\beta} u(y, t_2) - m_{\beta,2r} \leq Cr + C \sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq \beta}} (M_{\gamma,2r} - \Delta_{\xi^\gamma} u(y, t_2)),$$

by the arbitrariness of ϵ. Integrating for (y, t_2) on P_r, and using the fundamental inequality $\|a + b\|_p \leq \|a\|_p + \|b\|_p$ for $p > 1$,

$$\begin{aligned}
&\frac{1}{r^{4n+2}}\Big(\int_{P_r} \big(\Delta_{\xi^\beta} u(y, t_2) - m_{\beta,2r}\big)^p\Big)^{\frac{1}{p}} \\
&\leq \frac{C}{r^{4n+2}}\Big(\int_{P_r} \Big[r + \sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq \beta}} (M_{\gamma,2r} - \Delta_{\xi^\gamma} u(y, t_2))\Big]^p\Big)^{\frac{1}{p}} \\
&\leq Cr + \frac{C}{r^{4n+2}} \sum_{\substack{0 \leq \gamma \leq N \\ \gamma \neq \beta}} \Big(\int_{P_r} [M_{\gamma,2r} - \Delta_{\xi^\gamma} u(y, t_2)]^p\Big)^{\frac{1}{p}}
\end{aligned} \tag{30}$$

$$\overset{(29)}{\leq} C \sum_{\substack{0\leq\gamma\leq N\\ \gamma\neq\beta}} (M_{\gamma,2r} - M_{\gamma,r}) + Cr^{\frac{4n}{4n+1}},$$

where we have used the fact $0 < r < 1$ in the last inequality. In light of (29) to (30), and the inequality $\|a + b\|_p \leq \|a\|_p + \|b\|_p$, we obtain

$$\begin{aligned} M_{\beta,2r} - m_{\beta,2r} &\leq \frac{C}{r^{4n+2}}\Big(\int_{P_r}(M_{\beta,2r} - \Delta_{\xi^\beta}u)^p\Big)^{\frac{1}{p}} + \frac{C}{r^{4n+2}}\Big(\int_{P_r}(\Delta_{\xi^\beta}u - m_{\beta,2r})^p\Big)^{\frac{1}{p}} \\ &\leq C\sum_{\gamma=0}^{N}(M_{\gamma,2r} - M_{\gamma,r}) + Cr^{\frac{4n}{4n+1}}. \end{aligned}$$

Summing over β, then we deduce

$$\eta(2r) \leq C\sum_{\gamma=0}^{N}(M_{\gamma,2r} - M_{\gamma,r}) + Cr^{\frac{4n}{4n+1}}.$$

By the definition, $m_{\cdot,s}$ is non-increasing about s, whence

$$\begin{aligned} \eta(2r) &\leq C\sum_{\gamma=0}^{N}((M_{\gamma,2r} - m_{\gamma,2r}) - M_{\gamma,r} + m_{\gamma,r}) + Cr^{\frac{4n}{4n+1}} \\ &= C(\eta(2r) - \eta(r)) + Cr^{\frac{4n}{4n+1}}. \end{aligned}$$

Equivalently,

$$\eta(r) \leq \Big(1 - \frac{1}{C}\Big)\eta(2r) + Cr^{\frac{4n}{4n+1}}.$$

Now we apply a standard iteration technique (see [14, Chapter 8] for more details), there exists a dimensional constant δ with $1 > \delta > 0$ such that $\eta(r) \leq Cr^{\delta}$. This completes the proof of Proposition 5.2. □

6. Parabolic Harnack inequality

First of all, we have the following proposition.

Proposition 6.1. *For each $k \in \mathbb{N}$, there exists a constant C_k depending on the allowed data and k such that*

$$\sup_{M\times[0,\infty)} |\nabla^k\varphi| \leq C_k, \tag{31}$$

where ∇ is the Levi-Civita connection with respect to $\tilde{G}$.

Proof. Suppose that $[0, T)$ is the maximal time interval of the flow (2) and $T < \infty$. By (10), there exists a uniform constant C such that

$$|\varphi| \leq T \sup_{M\times[0,T)} |\partial_t\varphi| \leq CT, \text{ on } M \times [0, T). \tag{32}$$

We know that φ is actually smooth on $M \times [0, T)$ by (18). Together with short time existence, one can extend the flow to $[0, T+\varepsilon_0)$ with $\varepsilon_0 > 0$, which yields a contradiction. The interested reader can find more details about this standard discussion in the proof of [26, Theorem 3.1], see also in [8, 30] and references therein. Then the proposition is a consequence of Theorem 5.1. ☐

Let us denote $\phi = \partial_t\varphi$. By (9) we know

$$\partial_t\phi - \Box\phi = 0.$$

We have the following parabolic Harnack inequality:

Proposition 6.2. *Let $0 < t_1 < t_2 < T$ with $[0,T)$ be the maximal time interval. Then there exist constants C_i $(i = 1, 2, 3)$ depending only on (M, I, J, K), Ω and f such that*

$$\sup_{x\in M}\phi(x,t_1) \leq \inf_{x\in M}\phi(x,t_2)\Big(\frac{t_2}{t_1}\Big)^{C_1}\exp\Big(\frac{C_2}{t_2-t_1} + C_3(t_2-t_1)\Big). \tag{33}$$

Proof. With the Lemmas A.2 and A.3 below at hand, then we can apply the procedure of [15, Lemma 6.3] or [12, Proposition 7.4] verbatim. ☐

7. Convergence of parabolic flow

Proof of Theorem 1.2. For each $m \in \mathbb{N}$, we define

$$\check{\phi}_m(x,t) := \sup_{x\in M}\phi(x, m-1) - \phi(x, m-1+t);$$

$$\hat{\phi}_m(x,t) := \phi(x, m-1+t) - \inf_{x\in M}\phi(x, m-1).$$

We are able to verify that

$$(\partial_t - \Box)\phi = (\partial_t - \Box)\hat{\phi}_m = (\partial_t - \Box)\check{\phi}_m = 0.$$

Applying the parabolic Harnack inequality (33), this yields

$$\sup_{x\in M}\hat{\phi}_m(x,t_1) \leq C\inf_{x\in M}\hat{\phi}_m(x,t_2); \tag{34}$$

$$\sup_{x\in M}\check{\phi}_m(x,t_1) \leq C\inf_{x\in M}\check{\phi}_m(x,t_2). \tag{35}$$

Choosing $t_1 = \frac{1}{2}$, $t_2 = 1$ in (34) and (35), we get

$$\sup_{x\in M}\phi(x, m-\frac{1}{2}) - \inf_{x\in M}\phi(x, m-1) \leq C\big(\inf_{x\in M}\phi(x,m) - \inf_{x\in M}\phi(x, m-1)\big); \tag{36}$$

$$\sup_{y\in M}\phi(y, m-1) - \inf_{x\in M}\phi(x, m-\frac{1}{2}) \leq C\big(\sup_{y\in M}\phi(y, m-1) - \sup_{x\in M}\phi(x,m)\big). \tag{37}$$

In light of (36) and (37), let

$$\theta(t) := \sup_{y\in M}\phi(y,t) - \inf_{y\in M}\phi(y,t).$$

Then we have

$$\theta(m-1) + \theta(m-\frac{1}{2}) \leq C\big(\theta(m-1) - \theta(m)\big), \tag{38}$$

which implies that

$$\theta(m) \le e^{-\delta}\theta(m-1)$$

for some $\delta := -\log(1-\frac{1}{C}) > 0$. By induction, we know that

$$\theta(t) \le Ce^{-\delta t}.$$

While $\int_M \partial_t\tilde{\varphi}\omega^m = 0$, by the mean value theorem, there exists a point $x_t \in M$ such that $\partial_t\tilde{\varphi} = 0$ at (x_t, t). Therefore,

$$\begin{aligned}\Big|\partial_t\tilde{\varphi}(x,t)\Big| &= \Big|\partial_t\tilde{\varphi}(x,t) - \partial_t\tilde{\varphi}(x_t,t)\Big| \\ &\le \sup_{y\in M}\partial_t\tilde{\varphi}(y,t) - \inf_{y\in M}\partial_t\tilde{\varphi}(y,t) \\ &\le \sup_{y\in M}\partial_t\varphi(y,t) - \inf_{y\in M}\partial_t\varphi(y,t) \le Ce^{-\delta t},\end{aligned}$$

which implies that $\tilde{\varphi} + \frac{C}{\delta}e^{-\delta t}$ (resp. $\tilde{\varphi} - \frac{C}{\delta}e^{-\delta t}$) is non-increasing (resp. non-decreasing) with respect to t. It then follows from Proposition 6.1 that

$$\lim_{t\to\infty}\tilde{\varphi} = \varphi_\infty$$

in the C^∞ topology. Besides, $\tilde{\varphi}$ satisfies

$$\partial_t\tilde{\varphi} = \log\frac{(\Omega + \partial\partial_J\tilde{\varphi})^n}{\Omega^n} - f - \int_M \partial_t\varphi\Omega^n \wedge \bar{\Omega}^n.$$

Letting $t \to \infty$ and then we have

$$\log\frac{(\Omega + \partial\partial_J\varphi_\infty)^n}{\Omega^n} = f + b,$$

where

$$b = \int_M \Big(\log\frac{(\Omega + \partial\partial_J\varphi_\infty)^n}{\Omega^n} - f\Big)\Omega^n \wedge \bar{\Omega}^n$$

By Proposition 3.1, we can indeed obtain this limit. Then the proof is completely. □

Appendix A.

Let (M, I, J, K) be a hypercomplex manifold. For each $p \in M$, we can choose an open neighborhood $D \ni p$. Furthermore, (D, I) is biholomorphic to an open subset of $\mathbb{C}^{2n}$. We need the following observation.

Proposition A.1 ([3, Proposition 3.1.1]). *Let $D \subset M$ as above with complex coordinates $z_1, \dots, z_{2n}$. At a point $z \in D$, for each real valued function $u \in C^2(M)$, the $(2,0)$-form $\partial\partial_J u(z)$ depends only on the second derivatives u and on the complex structure $J(z)$ at z. Precisely,*

$$\partial\partial_J u(z) = \frac{\partial^2 u}{\partial\bar{z}_j\partial z_i}(J^{-1})^j_k dz_i \wedge dz_k. \tag{39}$$

Now we consider the following Li-Yau [20] type equation

$$(\Box - \partial_t)u = 0,\ u > 0. \tag{40}$$

Let u solve (40) and denote $h = \log u$, $h_t = \partial_t h$. Write $\Omega_\varphi = \Omega + \partial\partial_J\varphi$ for short. We consider the quantity

$$H = t(|\partial h|^2 - \alpha h_t),$$

where α is a constant satisfying $1 < \alpha < 2$ and

$$|\partial h|^2 := \frac{\partial h \wedge \partial_J h \wedge \Omega_\varphi^{n-1}}{\Omega_\varphi^n}.$$

Plugging $u = e^h$ into (40) we have

$$\Box h - h_t = -|\partial h|^2. \tag{41}$$

Lemma A.2. *There exists a constant $C > 0$ such that*

$$\begin{aligned}(\Box - \partial_t)H \geq \frac{t}{4n}\big(|\partial h|^2 - h_t\big)^2 - 2\mathrm{Re}\langle \partial h, \partial H\rangle \\ - \big(|\partial h|^2 - \alpha h_t\big) - tC|\partial h|^2 - Ct,\end{aligned} \tag{42}$$

where $\langle\cdot,\cdot\rangle$ is an inner product defined by $\langle \partial f, \partial g\rangle := \frac{\partial f\wedge\partial_J g\wedge\Omega_\varphi^{n-1}}{\Omega_\varphi^n}$.

Proof. The proof is local. For each $z \in M$, we can find an I-complex coordinates $z_1, \dots, z_{2n}$ on a local chart $D \ni z$. Assume $f \in C^2(M)$, let $f_j = \frac{\partial f}{\partial z_j}$ be the ordinary derivative, we have

$$\partial_J f = J^{-1}(\bar\partial f) = f_{\bar j} J^{-1}(d\bar z_j) = f_{,\bar k} dz_k;$$

$$\partial\partial_J f = \partial(f_{\bar j}(J^{-1})^j_k)dz_k = f_{,i\bar k} dz_i \wedge dz_k.$$

Here we use the notation

$$f_{,\bar k} := f_{\bar j}(J^{-1})^j_k,\ \ f_{,i\bar k} := f_{i\bar j}(J^{-1})^j_k + f_{\bar j}[(J^{-1})^j_k]_i. \tag{43}$$

For each $f, g \in C^2(M, \mathbb{R})$, it is easy to verify

$$(fg)_{,\bar k} = fg_{,\bar k} + gf_{,\bar k}; \tag{44}$$

$$(fg)_{,i\bar k} = fg_{,i\bar k} + gf_{,i\bar k} + f_i g_{,\bar k} + g_i f_{,\bar k}. \tag{45}$$

Using (41) we know that

$$H = -t\Box h - t(\alpha - 1)h_t. \tag{46}$$

It then gives us

$$t\partial_t\big(\Box h\big) = \frac{1}{t}H - \partial_t H - t(\alpha - 1)h_{tt}. \tag{47}$$

For each k, s, we define

$$\hat\chi^{k\bar s} := \frac{dz_k \wedge dz_s \wedge \Omega_\varphi^{n-1}}{\Omega_\varphi^n}; \tag{48}$$

$$\chi^{k\bar{s}} := \frac{dz_k \wedge (J^{-1})^l_s dz_l \wedge \Omega_\varphi^{n-1}}{\Omega_\varphi^n}. \tag{49}$$

By a straightforward computation we arrive that

$$\begin{aligned} -\partial_t H = &- \big(|\partial h|^2 - \alpha h_t\big) - 2t\mathrm{Re}\big\langle \partial h, \partial h_t\big\rangle \\ &+ t\alpha h_{tt} + t\partial_t(\chi^{i\bar{j}})h_i h_{\bar{j}}; \end{aligned} \tag{50}$$

$$\Box H = t\Box(|\partial h|^2) - t\alpha\Box h_t. \tag{51}$$

Now we compute

$$\begin{aligned} &\Box(|\partial h|^2) \\ &= n\partial\partial_J(|\partial h|^2)\wedge \Omega_\varphi^{n-1}/\Omega_\varphi^n \\ &= n\Big[\partial\partial_J(\chi^{s\bar{l}})\cdot h_s h_{\bar{l}} + \partial(\chi^{s\bar{l}})\wedge\partial_J(h_s h_{\bar{l}}) + \partial_J(\chi^{s\bar{l}})\wedge\partial(h_s h_{\bar{l}}) \\ &\qquad + \chi^{s\bar{l}}\partial\partial_J(h_s h_{\bar{l}})\Big]\wedge\Omega_\varphi^{n-1}/\Omega_\varphi^n \\ &= n\Big[\partial\partial_J(\chi^{s\bar{l}})h_s h_{\bar{l}} + \partial(\chi^{s\bar{l}})\wedge\partial_J(h_s h_{\bar{l}}) + \partial_J(\chi^{s\bar{l}})\wedge\partial(h_s h_{\bar{l}})\Big]\wedge\Omega_\varphi^{n-1}/\Omega_\varphi^n \\ &\quad + n\chi^{s\bar{l}}\Big(\frac{\partial h_s\wedge\partial_J h_{\bar{l}}\wedge\Omega_\varphi^{n-1}}{\Omega_\varphi^n} + \frac{\partial h_{\bar{l}}\wedge\partial_J h_s\wedge\Omega_\varphi^{n-1}}{\Omega_\varphi^n}\Big) \\ &\quad + \chi^{s\bar{l}}h_s\Box(h_{\bar{l}}) + \chi^{s\bar{l}}\Box(h_s)h_{\bar{l}}. \end{aligned} \tag{52}$$

Notice that Ω_φ has uniform bounded C^k norms for every k by Theorem 5.1. Hence, analogous to the (almost) Hermitian case [12, 15], we deduce

$$|n\partial\partial_J(\chi^{s\bar{l}})h_s h_{\bar{l}}\wedge\Omega_\varphi^{n-1}/\Omega_\varphi^n| \le C|\partial h|^2. \tag{53}$$

Let us define

$$|\partial\partial_J h|^2 := \chi^{i\bar{k}}\chi^{s\bar{l}}h_{s\bar{k}}h_{i\bar{l}},\ \ |D^2 h|^2 := \chi^{i\bar{k}}\chi^{s\bar{l}}h_{si}h_{\bar{l}\bar{k}}.$$

For each $0 < \varepsilon < 1$, using the Cauchy-Schwarz inequality we see that

$$\begin{aligned} &\big|n\partial(\chi^{s\bar{l}})\wedge\partial_J(h_s h_{\bar{l}})\wedge\Omega_\varphi^{n-1}/\Omega_\varphi^n| + |n\partial_J(\chi^{s\bar{l}})\wedge\partial(h_s h_{\bar{l}})\wedge\Omega_\varphi^{n-1}/\Omega_\varphi^n| \\ \le\ &\frac{Ct}{\varepsilon}|\partial h|^2 + 2t\varepsilon|D^2 h|^2 + 2t\varepsilon|\partial\partial_J h|^2. \end{aligned} \tag{54}$$

By definition, we know that

$$n\chi^{s\bar{l}}\frac{\partial h_s\wedge\partial_J h_{\bar{l}}\wedge\Omega_\varphi^{n-1}}{\Omega_\varphi^n} = |D^2 h|^2; \tag{55}$$

$$n\chi^{s\bar{l}}\frac{\partial h_{\bar{l}}\wedge\partial_J h_s\wedge\Omega_\varphi^{n-1}}{\Omega_\varphi^n} = |\partial\partial_J h|^2. \tag{56}$$

For every smooth real valued function v, $\Box v = \hat{\chi}^{i\bar{k}} v_{,i\bar{k}}$. Then

$$
\begin{aligned}
&(\Box h)_s - (\Box h_s) \\
&= (\hat{\chi}^{i\bar{k}} h_{,i\bar{k}})_s - \hat{\chi}^{i\bar{k}} h_{s,i\bar{k}} \\
&= \hat{\chi}^{i\bar{k}} \big((h_{,i\bar{k}})_s - h_{s,i\bar{k}} \big) + (\hat{\chi}^{ik})_s h_{,i\bar{k}} \\
&= \hat{\chi}^{i\bar{k}} \big(h_j [(J^{-1})^j_k]_{is} + h_{i\bar{j}} [(J^{-1})^j_k]_s \big) + (\hat{\chi}^{i\bar{k}})_s h_{,i\bar{k}}.
\end{aligned} \tag{57}
$$

It follows that

$$
\begin{aligned}
&\chi^{s\bar{l}} h_s \Box(h_{\bar{l}}) + \chi^{s\bar{l}} \Box(h_s) h_{\bar{l}} - 2\mathrm{Re}\big\langle \partial h, \partial \Box h \big\rangle \\
&= \chi^{s\bar{l}} h_s \big(\Box(h_{\bar{l}}) - (\Box h)_{\bar{l}} \big) + \chi^{s\bar{l}} \big(\Box(h_s) - (\Box h)_s \big) h_{\bar{l}} \\
&= \chi^{s\bar{l}} h_s \big(\hat{\chi}^{i\bar{k}} \big(h_j [(J^{-1})^j_k]_{i\bar{l}} + h_{i\bar{j}} [(J^{-1})^j_k]_{\bar{l}} \big) + (\hat{\chi}^{i\bar{k}})_{\bar{l}} h_{,i\bar{k}} \big) \\
&\quad + \chi^{s\bar{l}} \big(\hat{\chi}^{i\bar{k}} \big(h_j [(J^{-1})^j_k]_{is} + h_{i\bar{j}} [(J^{-1})^j_k]_s \big) + (\hat{\chi}^{i\bar{k}})_s h_{,i\bar{k}} \big) h_{\bar{l}} \\
&\geq -\frac{C}{\varepsilon} |\partial h|^2 - \varepsilon |\partial \partial_J h|^2,
\end{aligned} \tag{58}
$$

where the last inequality we have used Cauchy-Schwarz inequality. Meanwhile,

$$
\begin{aligned}
&2t\mathrm{Re}\big\langle \partial h, \partial \Box h \big\rangle \\
&\overset{(46)}{=} -2\mathrm{Re}\big\langle \partial h, \partial H \big\rangle - 2t(\alpha-1)\mathrm{Re}\langle \partial h, \partial h_t \rangle \\
&\overset{(50)}{=} -2\mathrm{Re}\big\langle \partial h, \partial H \big\rangle - (\alpha-1)\partial_t H + (\alpha-1)\big(|\partial h|^2 - \alpha h_t \big) \\
&\quad + t(\alpha-1)\partial_t(\chi^{i\bar{j}}) h_i h_{\bar{j}} - t\alpha(\alpha-1) h_{tt} \\
&\geq -2\mathrm{Re}\big\langle \partial h, \partial H \big\rangle - (\alpha-1)\partial_t H + (\alpha-1)\big(|\partial h|^2 - \alpha h_t \big) \\
&\quad - Ct|\partial h|^2 - t\alpha(\alpha-1) h_{tt}.
\end{aligned} \tag{59}
$$

It follows from (58) and (59) that

$$
\begin{aligned}
&t\big(\chi^{s\bar{l}} h_s \Box(h_{\bar{l}}) + \chi^{s\bar{l}} \Box(h_s) h_{\bar{l}} \big) \\
&\geq -2\mathrm{Re}\big\langle \partial h, \partial H \big\rangle - (\alpha-1)\partial_t H + (\alpha-1)\big(|\partial h|^2 - \alpha h_t \big) \\
&\quad - Ct|\partial h|^2 - t\alpha(\alpha-1) h_{tt} - \frac{t}{\varepsilon} |\partial h|^2 - t\varepsilon |\partial \partial_J h|^2 - t\varepsilon |D^2 h|^2.
\end{aligned} \tag{60}
$$

Using the Cauchy-Schwarz inequality, at z, we deduce

$$
\begin{aligned}
-\alpha t \Box h_t &= -\alpha t \partial_t(\Box h) + \alpha t \partial_t(\chi^{i\bar{j}}) h_{i\bar{j}} \\
&\overset{(47)}{=} -\frac{\alpha}{t} H + \alpha \partial_t H + t\alpha(\alpha-1) h_{tt} + \alpha t \partial_t(\chi^{i\bar{j}}) h_{i\bar{j}} \\
&\geq -\frac{\alpha}{t} H + \alpha \partial_t H + t\alpha(\alpha-1) h_{tt} - \frac{Ct}{\varepsilon} - t\varepsilon |\partial \partial_J h|^2,
\end{aligned} \tag{61}
$$

where the last inequality we have used the Proposition 6.1 (notice that Proposition 6.1 implies $-C\chi^{i\bar{j}} \leq \partial_t(\chi^{i\bar{j}}) \leq C\chi^{i\bar{j}}$ for a uniform constant C).

Plugging (53) to (61) into (51), then we get

$$\begin{aligned}\Box H \geq & -Ct|\partial h|^2 - \big(\frac{2t}{\varepsilon}|\partial h|^2 + 2t\varepsilon|D^2h|^2\big) - \big(\frac{2t}{\varepsilon}|\partial h|^2 + 2t\varepsilon|\partial\partial_J h|^2\big)\\ & - 2\mathrm{Re}\langle \partial h, \partial H\rangle - (\alpha-1)\partial_t H + (\alpha-1)\big(|\partial h|^2 - \alpha h_t\big)\\ & - Ct|\partial h|^2 - t\alpha(\alpha-1)h_{tt} - \frac{t}{\varepsilon}|\partial h|^2\\ & - t\varepsilon(|\partial\partial_J h|^2 + |D^2h|^2) + t|D^2h|^2 + t|\partial\partial_J h|^2\\ & - \frac{\alpha}{t}H + \alpha\partial_t H + t\alpha(\alpha-1)h_{tt} - \frac{Ct}{\varepsilon} - t\varepsilon|\partial\partial_J h|^2\\ \geq & \; t(1-4\varepsilon)|\partial\partial_J h|^2 + t(1-3\varepsilon)|D^2h|^2 - t(C+\frac{3}{\varepsilon})|\partial h|^2\\ & - \big(|\partial h|^2 - \alpha h_t\big) + \partial_t H - 2\mathrm{Re}\langle \partial h, \partial H\rangle - \frac{Ct}{\varepsilon}.\end{aligned}$$

Therefore, if $\frac{1}{16} \leq \varepsilon \leq \frac{1}{8}$, we have

$$(\Box - \partial_t)H \geq \frac{t}{2}|\partial\partial_J h|^2 - Ct|\partial h|^2 - \big(|\partial h|^2 - \alpha h_t\big) - 2\mathrm{Re}\langle \partial h, \partial H\rangle - Ct. \tag{62}$$

Now we apply the arithmetic-geometric mean inequality, and by (41),

$$|\partial\partial_J h|^2 \geq \frac{1}{2n}(\Box h)^2 = \frac{1}{2n}\big(h_t - |\partial h|^2\big)^2. \tag{63}$$

Plugging into (62), one can obtain

$$\begin{aligned}(\Box - \partial_t)H \geq & \frac{t}{4n}\big(h_t - |\partial h|^2\big)^2 - Ct|\partial h|^2\\ & - \big(|\partial h|^2 - \alpha h_t\big) - 2\mathrm{Re}\langle \partial h, \partial H\rangle - Ct.\end{aligned} \tag{64}$$

By the arbitrariness of z, this proves (42). □

Using the parabolic maximum principle, we can prove the following lemma.

Lemma A.3. *On $M \times (0,T)$, we have*

$$|\partial h|^2 - \alpha h_t \leq \frac{8n\alpha^2}{t} + \sqrt{8n\alpha^2\Big(C + \frac{nC^2\alpha^2}{2(\alpha-1)^2}\Big)}. \tag{65}$$

Proof. Let us fix an arbitrary time $t_0 \in (0,T)$. Suppose $H(x,t)$ achieves its maximum at the point $(\hat{q},\hat{t}) \in M \times [0,t_0]$, we may assume $\hat{t} > 0$. Otherwise, $|\partial h|^2 - \alpha h_t \leq 0$ on $M \times [0,t_0]$ and we are done. It follows that

$$H(\hat{q},\hat{t}) \geq H(\hat{q},0) = 0.$$

At $(\hat{q},\hat{t})$, using maximum principle, we deduce $(\Box - \partial_t)H \leq 0$ and $\partial H = 0$. Put these into (42). Hence,

$$\frac{\hat{t}^2}{4n}\big(|\partial h|^2 - h_t\big)^2 - C\hat{t}^2|\partial h|^2 - H \leq C\hat{t}^2. \tag{66}$$

Notice that at $(\hat{q}, \hat{t})$,

$$\begin{aligned} \hat{t}^2\big(|\partial h|^2 - h_t\big)^2 &= \frac{\hat{t}^2}{\alpha^2}\big(|\partial h|^2 - \alpha h_t + (\alpha-1)|\partial h|^2\big)^2 \\ &= \frac{H^2}{\alpha^2} + \Big(\frac{\alpha-1}{\alpha}\Big)^2 \hat{t}^2|\partial h|^4 + \frac{2(\alpha-1)\hat{t}H}{\alpha^2}|\partial h|^2 \\ &\geq \frac{H^2}{\alpha^2} + \Big(\frac{\alpha-1}{\alpha}\Big)^2 \hat{t}^2|\partial h|^4, \end{aligned} \tag{67}$$

where we have used the fact that H is nonnegative at $(\hat{q}, \hat{t})$. Using the elementary inequality $ax^2 + bx \geq -\frac{b^2}{4a}$, we get

$$\frac{1}{4n}\Big(\frac{\alpha-1}{\alpha}\Big)^2 \hat{t}^2|\partial h|^4 - \hat{t}^2 C|\partial h|^2 \geq -\frac{nC^2\alpha^2}{2(\alpha-1)^2}\hat{t}^2. \tag{68}$$

Plugging (67) and (68) into (66), so

$$\frac{H^2}{4n\alpha^2} \leq H + C\hat{t}^2 + \frac{nC^2\alpha^2}{2(\alpha-1)^2}\hat{t}^2; \tag{69}$$

we can deduce

$$H(\hat{q}, \hat{t}) \leq 8n\alpha^2 + \sqrt{8n\alpha^2\Big(C + \frac{nC^2\alpha^2}{2(\alpha-1)^2}\Big)}\hat{t}. \tag{70}$$

Hence, at each point $q \in M$,

$$H(q, t_0) \leq H(\hat{q}, \hat{t}) \leq 8n\alpha^2 + \sqrt{8n\alpha^2\Big(C + \frac{nC^2\alpha^2}{2(\alpha-1)^2}\Big)}t_0. \tag{71}$$

Consequently, at (q, t_0),

$$|\partial h|^2_\varphi - \alpha h_t \leq \frac{8n\alpha^2}{t_0} + \sqrt{8n\alpha^2\Big(C + \frac{nC^2\alpha^2}{2(\alpha-1)^2}\Big)}.$$

Then the lemma follows by arbitrariness of t_0. □

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Jiaogen Zhang
School of Mathematical Sciences
University of Science and Technology of China
Hefei 230026, P. R. China
Email address: zjgmath@ustc.edu.cn