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SYNCHRONIZED COMPONENTS OF A SUBSHIFT

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Abstract. We introduce the notion of a minimal synchronizing word; that is a synchronizing word whose proper subwords are not synchronized. This has been used to give a new shorter proof for a theorem in [6]. Also, the common synchronized components of a subshift and its derived set have been characterized.

1. Introduction

For a subshift, two approximations may be considered: (1) From outside, that is, X is the intersection of X_k's where $X_{k+1} \subseteq X_k$, $k \in \mathbb{N}$. In this case, $h(X)$ the entropy of X, is exactly $\lim_{k\to\infty} h(X_k)$ [4, Proposition 4.4.6]. (2) From inside, that is, $X = \overline{\cup X_k}$, $X_k \subseteq X_{k+1}$. Then, $h(X) \geq \lim_{k\to\infty} h(X_k)$ and equality occurs in special cases. For instance, when X is sofic, or more generally when X is specified, one can find X_k's all SFT so that equality is satisfied [4, page 452]. In this respect, Thomsen in [5] considers a synchronized component of a general subshift and investigates the approximation of entropy from inside of this synchronized component by some certain SFT's. In fact, many results in [5] are based on this result. To be more specified, suppose $W(X)$ is the set of admissible blocks of X and $W_n(X)$ the set of admissible blocks of length n or so called n-blocks. Thomsen proves that $\lim_{k\to\infty} h(X_k) = h_{syn}(X)$ where

$$h_{syn}(X) := \limsup_n \frac{1}{n} \log \left(\text{cardinal}\left\{a \in W_n(X) : mam \in W(X)\right\}\right), \tag{1}$$

where m is an arbitrary synchronizing block in $W(X)$ and X_k's are SFT approaching X from inside. In Section 3, we give a new proof for this result, and in particular, our approach is a constructive approach. Our main tool is that we consider the minimal synchronizing blocks, i.e., synchronizing blocks whose any pure subblock are not synchronizing. Then, in the sequel we will investigate synchronized components of systems with finite minimal synchronizing blocks.

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Now let $\mathrm{Per}(X)$ be the set of periodic points of X and set $R(X) = \overline{\mathrm{Per}(X)}$. Also let $S(X)$ denote the set of synchronizing blocks for $R(X)$. Set

$$\partial X := \{x \in R(X) : w \subseteq x \Rightarrow w \notin S(X)\} \tag{2}$$

called the *derived shift space* of X. Then, $\partial(X)$ plays an important role in the dynamics of the system. As an example, a result in [6] states that in synchronized systems

$$h(X) = \max\{h_{\mathrm{syn}}(X), h(\partial(X))\}.$$

Note that ∂X is empty for an SFT and it is sofic whenever X is sofic [5, Theorem 6.6]. In fact, for specified systems (containing sofic shifts) $h(X) = h_{\mathrm{syn}}(X)$ [3, page 16] and elsewhere easy examples can be established so that $h(X) = h(\partial(X))$. For a synchronized system which is irreducible by definition, no transitive points will be in $\partial(X)$ and so $\partial(X)$ lies in the complement of a residual set; however, it may be dynamically interesting. Generally, ∂X is more interesting for non-specified systems and in particular for reducible systems. For instance, it may happen that as a subshift, X and ∂X have a common synchronized component. We will characterize this situation in Section 3.

2. Background and definitions

This section is devoted to basic definitions for our later work. The notations have been taken from [2], [4] and [5] for the relevant concepts.

First, we present some elementary concepts from [4]. Let $\mathcal{A}$ be a set of non-empty finite symbols called *alphabet*. The full $\mathcal{A}$-shift, denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. A *block* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty block* which is denoted by ε. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a block of length $j - i + 1$ by $x_{[i,\,j]} = x_i x_{i+1} \cdots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$. The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$. By our topology, σ is a homeomorphism. Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$. The complement of $\mathcal{F}$ is the set of *admissible blocks* or just blocks in X. For a full shift $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block from $\mathcal{F}$. A *shift space* or a *subshift* is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden blocks.

Let $W_n(X)$ denote the set of all admissible n-blocks. The *language* of X is the collection $W(X) = \cup_n W_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks u, $v \in W(X)$ there is a block $w \in W(X)$ so that $uwv \in W(X)$. It is *mixing* if for every ordered pair u, $v \in W(X)$, there is an N such that for each $n \geq N$ there is a block $w \in W_n(X)$ such that $uwv \in W(X)$. A shift space X is called a *shift of finite type* (SFT) if there is a finite set $\mathcal{F}$ of forbidden blocks such that $X = X_{\mathcal{F}}$. A shift of *sofic* is the image of an SFT by a factor code (an onto sliding block code). Every SFT is sofic [4, Theorem 3.1.5], but the converse is not true. For instance, if $\mathcal{F} = \{10^{2n+1}1 : n \in \mathbb{N} \cup \{0\}\}$, then $X_{\mathcal{F}}$ is called the even shift which is not SFT but it is sofic [4, page 67].

Let G be a graph with edge set $\mathcal{E} = \mathcal{E}(G)$ and the set of vertices $\mathcal{V} = \mathcal{V}(G)$. The *edge shift* X_G is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ defined by

$$X_G = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\}.$$

Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A labeled graph is a pair $\mathcal{G} = (G, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. For a path $\pi = \pi_0 \cdots \pi_k$, $\mathcal{L}(\pi) = \mathcal{L}(\pi_0) \cdots \mathcal{L}(\pi_k)$ is the label of π.

Let $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G and set

$$X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or a *cover* for $X = \overline{X_{\mathcal{G}}}$. In particular, X is sofic if and only if $X = X_{\mathcal{G}}$ for a finite graph G [4, Proposition 3.2.10]. A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels.

In this part we collect some information from [2]. Let X be a shift space and $w \in W(X)$. The follower set $F(w)$ of w is defined by $F(w) = \{v \in W(X) : wv \in W(X)\}$. Let $x \in X$. Then, $x_+ = (x_i)_{i\in Z^+}$ (resp. $x_- = (x_i)_{i\leq 0}$) is called a right (resp. left) infinite X-ray. For a left infinite X-ray, say x_-, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_-x_+ \in X\}$. Consider the collection of all follower sets $w_+(x_-)$ as the set of vertices of a graph. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that x_-a is an X-ray and $I_1 = w_+(x_-), I_2 = w_+(x_-a)$. This labeled graph is called the *Krieger graph* for X. A block $m \in W(X)$ is *synchronizing* if whenever um and mv are in $W(X)$, we have $umv \in W(X)$. An irreducible shift space X is a *synchronized system* if it has a synchronizing block, or equivalently, if and only if it admit a countable generating graph G such that $\mathcal{L}_\infty(X_G)$ is residual in X [2, Theorem 1.1].

If X is a synchronized system with synchronizing m, the irreducible component of the Krieger graph containing the vertex $w_+(m)$ is denoted by X_0^+ and is called the *Fischer cover* of X. If for some $m \in W(X)$ there is a unique vertex I such that $m \in F_-(m)$, then m is called a *magic* block for the Fischer cover.

For the last part of this section we bring some concepts from [5]. Let X be a shift space. Set $R(X) = \overline{\mathrm{Per} X}$ and let $S(X)$ denote the set of synchronizing blocks for $R(X)$. For $s, t \in S(X)$ we write $s \sim t$ when there are blocks $u, v \in W(R(X))$ such that sut, $tvs \in W(R(X))$. Then, $\sim$ is an equivalence relation in $S(X)$. Note that $s \sim t$ if only if there is an $x \in R(X)$ such that s, $t \subseteq x$.

Consider an element $\alpha \in S(X)/\sim$. Let $X_{(\alpha,0)}$ denote the set of elements $x \in R(X)$ for which

$$\sup_{i\in\mathbb{Z}} \big\{\inf\big\{(j-i) \geq 0 : \exists w \in \alpha,\, w \subseteq x_{[i,j]}\big\}\big\} \tag{3}$$

is finite. Here, we use the convention that $\inf \emptyset = \infty$. We can associate to $X_{(\alpha,0)}$ an irreducible graph Γ_α. For each $m \in \alpha$, let $F(m) = \{u \in W(X_{(\alpha,0)}) : mu \in W(X_{(\alpha,0)})\}$. The vertices of Γ_α consist of $\{F(m) : m \in \alpha\}$; and there is an edge labeled a from $F(m)$ to $F(m')$ when $a \in F(m)$ and $F(ma) = F(m')$. Then, Γ_α is a *cover* of $\overline{X_{(\alpha,0)}}$. Let $\partial(X)$ be as in (2) and denote the set of synchronizing blocks for $R(\partial^k(X))$ by $S(\partial^k(X))$. Define the *depth* of X to be $\mathrm{Depth}(X) = \sup\{k \in \mathbb{N} : \partial^k(X) \neq \emptyset\}$. If $\alpha \in S(\partial^k(X))/\sim$, then $(\partial^k(X))_{(\alpha,0)}$ will be denoted by $X_{(\alpha,k)}$.

Let X be a shift space. The *entropy* of X is defined by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |W_n(X)|.$$

A shift space X is *almost sofic* if there are sofic shifts $X_n \subseteq X$ such that

$$\lim_{n\to\infty} h(X_n) = h(X).$$

3. Synchronized components

In this section we introduce the notion of minimal synchronizing blocks and will exploit them to identify synchronized components.

Definition 3.1. A block $m \in S(X)$ is a *minimal synchronizing block*, if whenever $w \subsetneq m$, then w is not synchronizing. If a shift space X has finitely many minimal synchronizing blocks and $S(X) \neq \emptyset$, then we say that X is an *FmSyn* system; otherwise, it is called an *ImSyn* system.

Example 3.2.
(i) Every sofic space is FmSyn. This fact can be seen by the fact that any sofic has a finite cover, say its Fischer cover. An easy consequence is that Fischer cover of an ImSyn system must be infinite.
(ii) The block 1 is minimal synchronizing for any S-gap shift $X(S)$ and no other minimal synchronizing block exists which means that this system is FmSyn even when it is not sofic. See [1] for criteria on S to have $X(S)$ non-sofic.
(iii) Let G be the graph as in Figure 1 and $X = X_G = R(X)$. Then all blocks in $A = \{2, 101, 10^3 1, 10^5 1, \ldots\}$ are synchronizing blocks. However, blocks in

$$\{1, 0, 100, 1000, \ldots\} \cup \{01, 001, 0001, \ldots\}$$

are not. Therefore, no blocks in A has a synchronizing subblock and so X is ImSyn.

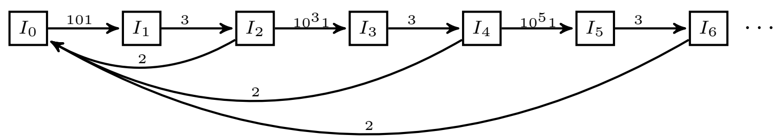

FIGURE 1. Graph G for the cover of an ImSyn.

Definition 3.3. Let X be a shift space. The subsystem Y is called a *synchronized component* of X whenever Y is a synchronized subsystem of X and if Z is any synchronized subsystem with $Y \subseteq Z \subseteq X$, then $Y = Z$.

Note that if X is a synchronized system, then $Y = X$.

Let $x \in X$ and $p \leq s$ for integers p and s. Set the *gap* between two blocks $x_{[p,\, q]}$ and $x_{[s,\, t]}$ to be 0 when $s \leq q$ and $s - q$ otherwise. Denote this gap by $\mathrm{gap}(x_{[p,\, q]},\, x_{[s,\, t]})$.

Definition 3.4. Let $p < s$ and $q \leq t$ and let $u = x_{[p,\, q]}, v = x_{[s,\, t]} \in \alpha \in S(X)/\sim$ be two minimal synchronizing blocks. If the only minimal synchronizing blocks in $x_{[p,\, t]}$ are u and v, then call u and v the *consecutive minimal pairs of* α *in* x.

Let $x \in X_{(\alpha,\, 0)}$. By (3),

$$\{\mathrm{gap}(u,\, v) : u,\, v \text{ are the consecutive minimal pairs of } \alpha \text{ in } x\}$$

is bounded and we will denote the maximum by $\mathrm{maxgap}(x, \alpha)$.

Lemma 3.5. *Let* $x \in X_{(\alpha,\, 0)}$. *Then,*

(i) *For all* $i \in \mathbb{Z}$ *there are* $u_i,\, v_i \in \alpha$ *such that* $u_i \subseteq x_{(-\infty,\, i]},\, v_i \subseteq x_{[i,\, +\infty)}$.
(ii) *There is* $M > 0$ *such that if* $u,\, v$ *are the consecutive minimal pairs of* α *in* x, *then* $\mathrm{gap}(u,\, v) \leq M$.
(iii) $A = \{w \subset x : w \text{ is a minimal synchronizing}\} \cap \alpha$ *is finite.*

Conversely, if (i), (ii) *and* (iii) *hold for* $x \in R(X) = \overline{PerX}$, *then* $x \in X_{(\alpha,\, 0)}$. *In particular, if* $x \in R(X)$ *is a periodic point with a subblock in* α, *then* $x \in X_{(\alpha,\, 0)}$.

Proof. Let $x \in X_{(\alpha,\, 0)}$ and for all $i \in \mathbb{Z}$ set

$$M_i := \inf\{j - i \geq 0 : \exists\, w \in \alpha,\, w \subseteq x_{[i,j]}\}. \tag{4}$$

Hence $x \in X_{(\alpha,\, 0)}$ if and only if $M_x := \sup\{M_i : i \in \mathbb{Z}\} < \infty$.

(i) Pick $i_0 \in \mathbb{Z}$. If $\{u \in \alpha : u \subseteq x_{[i_0,\, +\infty)}\} = \emptyset$, then the infimum in (3) would be taken over an empty set and so (3) will not be satisfied. Thus there is $j_0 \in \mathbb{N}$ such that $x_{[i_0,\, j_0]} \in \alpha$.

Now let $\{u \in \alpha : u \subseteq x_{(-\infty,\, i_0]}\} = \emptyset$. Then, for all $n \in \mathbb{N}$, $x_{[i_n,\, i_0]} \notin \alpha$ where $i_n := i_0 - n$. Thus

$$M_{i_n} = \inf\{j - i_n \geq 0 : \exists\, w \in \alpha,\, w \subseteq x_{[i_n,\, j]}\} > i_0 - i_n = n$$

and so $M_x = \sup\{M_{i_n} : n \in \mathbb{N}\} = \infty$ that is absurd. Hence there is $n \in \mathbb{N}$ such that $x_{[i_n,\, i_0]} \in \alpha$.

(ii) Let $u := x_{[p,\, q]}$, $v := v_{[s,\, t]}$ be the consecutive minimal pairs of α in x. If $s \le q$, then by definition $\mathrm{gap}(u,\, v) = 0$. So let $q < s$. Then,

$$\begin{aligned} M_q &= \inf\{j - q \ge 0 : \exists w \in \alpha,\, w \subseteq x_{[q,\, j]}\} \\ &= t - q = s - q + t - s = \mathrm{gap}(u,\, v) + |v| - 1 \end{aligned}$$

and so $\mathrm{gap}(u,\, v) \le \mathrm{gap}(u,\, v) + |v| - 1 = M_q \le M_x$.

(iii) Suppose x has infinitely many minimal synchronizing blocks $w_1,\, w_2,\, \ldots$. Then we may write $|w_1| < |w_2| < \cdots$ and $w_i := x_{[j_i,\, j_i + |w_i| - 1]}$. Hence for all $i \in \mathbb{N}$, $M_{j_i} = |w_i| - 1$ and so $M_x = \sup\{M_{j_1},\, M_{j_2},\, \ldots\} = \infty$ that is absurd.

For the converse set

$$M' := \max\{|w| : w \subseteq x \text{ is minimal synchronizing}\}.$$

By (iii) such a maximum exists. Let $i \in \mathbb{Z}$. By (i) there is a $v \in \alpha$ such that v is the terminal segment of $x_{[i,\, j]}$ ($x_{[i,\, j)} \cap S(X) = \emptyset$). Suppose u, v are the consecutive minimals of α in x (Not v and u). Then by (ii), $j - i \le |u| + |v| + \mathrm{gap}(u,\, v) \le 2M' + M$ and so $x \in X_{(\alpha,\, 0)}$. □

Based on the conclusions of Lemma 3.5, we give a new shorter and simpler proof for a theorem of Thomsen [5, Theorem 3.2].

Proposition 3.6 (Thomsen)**.** *Let X be a shift space. Then, there is a sequence $A_{\alpha,\, 1} \subseteq A_{\alpha,\, 2} \subseteq \cdots$ of irreducible SFT's in X such that*

$$X_{(\alpha,\, 0)} = \bigcup_n A_{\alpha,\, n}, \tag{5}$$

and

$$\begin{aligned} \lim_n h(A_{\alpha,\, n}) &= \sup\{h(A) : A \subseteq X_{(\alpha,\, 0)} \text{ is an irreducible SFT}\} \\ &= h(\Gamma_\alpha) = h_{syn}(\overline{X_{(\alpha,\, 0)}}). \end{aligned} \tag{6}$$

Proof. Let $T = \{m \in \alpha : m \text{ is a minimal synchronizing of } R(X)\} = \{m_1, m_2, \ldots\}$ and pick $x^{(1)} \in X(\alpha,\, 0)$ so that $m_1, m_2 \subseteq x^{(1)}$. Since $x^{(1)} \in \overline{\mathrm{Per}X}$, there is $y^{(1)} = v^\infty$ such that $m_1, m_2 \subseteq v$ and by Lemma 3.5, $y^{(1)} \in X_{(\alpha,\, 0)}$. If necessary, by a rearrangement of the elements of T, let $m_1, m_2, \ldots, m_{j_1}$ be the elements of T appearing as subblocks of $y^{(1)}$ and let $M_1 = \mathrm{maxgap}(y^{(1)},\, \alpha)$. Suppose A_1 is the set of all $x \in X_{(\alpha,\, 0)}$ with $\mathrm{maxgap}(x,\, \alpha) \le M_1$ and whenever $m_i \subseteq x$, then $i \le j_1$. Since $m_1,\, m_2, \ldots,\, m_{j_1},\, m_{j_1+1} \in \alpha$, by irreducibility of $X_{(\alpha,\, 0)}$ there exists $y^{(2)} = u^\infty \in \mathrm{Per}X$ such that $m_1,\, m_2, \ldots,\, m_{j_1+1} \subseteq u$. Now let

$$m_1,\, m_2, \ldots,\, m_{j_1},\, m_{j_1+1}, \ldots,\, m_{j_2}$$

be subblocks of $y^{(2)}$ lying in T and let $M_2 = \max\{M_1 + 1,\, \mathrm{maxgap}(y^{(2)},\, \alpha)\}$. Similar to A_1, A_2, for any $n \in \mathbb{N}$, set

$$A_n := \{x \in X(\alpha,\, 0) : \mathrm{maxgap}(x,\, \alpha) \le M_n \text{ and if } m_i \subseteq x,\ \text{then } i \le j_n\}.$$

Then, $A_n \subseteq A_{n+1}$ and $y^{(n)} \in A_n$. We claim that

(i) $\overline{A_1} \subseteq \overline{A_2} \subseteq \cdots$.
(ii) $\overline{A_n}$ is an irreducible shift space.
(iii) $\overline{A_n}$ is SFT.
(iv) $\cup \overline{A_n} = X_{(\alpha,\,0)}$.

Observe that by setting $A_{\alpha,\,n} := \overline{A_n}$, we will have (5).

Validity of (i) and the fact that $\overline{A_n}$ is a shift space is trivial. For irreducibility of $\overline{A_n}$, let $a,\, b \in W(\overline{A_n})$. Then, there are $x,\, z \in A_n$ such that $a \subseteq x$ and $b \subseteq z$. Let j_n be the integer provided by the definition of $y^{(n)}$. There are $i \le j \le j_n$ and $a',\, b' \in W(X)$ such that $aa'm_i \subseteq x, m_jb'b \subseteq z$ or $aa'm_i, m_jb'b \in W(A_n)$ for some $m_i,\, m_j \in T$. Also, $y^{(n)} \in A_n$ is a periodic point having all synchronizing m_i, $i \le j_n$. Therefore, there is w with $m_iwm_j \subseteq y^{(n)}$ which this in turn implies that $aa'm_iwm_jb'b \in W(A_n)$ and so $\overline{A_n}$ is irreducible.

For (iii), let $u \in W(\overline{A_n})$ where $|u| \ge M_n + 2\max\{|m_1|, \ldots, |m_{j_n}|\} = M$. By definition of A_n, there must be at least one $m_i \subseteq u$ and such a u is essentially synchronizing. As a result, any block of length M in $\overline{A_n}$ is synchronizing and so $\overline{A_n}$ is SFT [4, Theorem 2.1.8].

To prove (iv), clearly $\cup \overline{A_n} \subseteq X_{(\alpha,\,0)}$. For the other side let $x \in X_{(\alpha,\,0)}$ and using Lemma 3.5(iii), let $m_{i_1}, \ldots, m_{i_r}$ be the elements of α occurring as some subblocks in x. Then, there is k such that $x \in \overline{A_k}$ and we are done with our claim.

Now we set up to prove (6). Let $\epsilon > 0$ and

$$t_0 := \sup\{h(A) : A \subseteq X_{(\alpha,\,0)} \text{ is an irreducible } SFT\}.$$

Choose $r > 0$ and $A \subseteq X_{(\alpha,\,0)}$, an irreducible SFT such that $t_0 - \epsilon < r < h(A) = h_{\text{syn}}(A)$; where the last equality is satisfied because A is SFT [5, Lemma 3.1]. Let $m \in \alpha \cap W(A)$ and let $|m|$ be so large that m is a synchronizing block of A. Utilizing (1), there is $N \ge 1$ such that

$$\frac{1}{N} \log |\{a \in W_N(A) : (am)^\infty \in A\}| \ge t_0 - \epsilon. \tag{7}$$

Also there exists $k \ge 1$ such that for all $a \in W_N(A)$ if $(am)^\infty \in A$, then $(am)^\infty \in \overline{A_k}$. Thus,

$$\frac{1}{N} \log \left|\{a \in W_N(\overline{A_k}) : (am)^\infty \in \overline{A_k}\}\right| \ge \frac{1}{N} \log |\{a \in W_N(A) : (am)^\infty \in A\}| \,.$$

Hence $t_0 \le \epsilon + \frac{1}{N} \log \left|\{a \in W_N(\overline{A_k}) : (am)^\infty \in \overline{A_k}\}\right|$. Thus $t_0 \le \epsilon + h_{\text{syn}}(\overline{A_k}) = \epsilon + h(\overline{A_k})$. So

$$t_0 \le \lim(h(\overline{A_k})) \tag{8}$$

and then in fact $\lim h(\overline{A_k}) = t_0$.

Now we show that Γ_α is the Fischer cover of $\overline{X_{(\alpha,\,0)}}$; that is, we prove that Γ_α is a right resolving and follower separated graph with a magic block [2, Theorem 2.16]. By definition Γ_α is right resolving. It is also follower separated

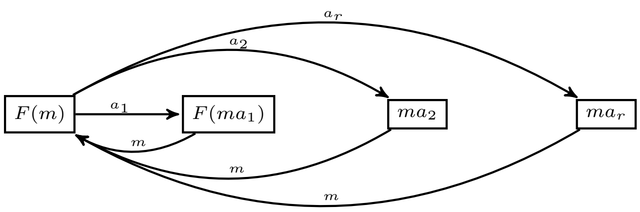

FIGURE 2. The subgraph H of Γ_α.

for let $F(m) \neq F(m')$. Then, for $u \in F(m) \setminus F(m')$, $mu \in W(X_{(\alpha,0)})$ but $m'u \notin W(X_{(\alpha,0)})$. Choose $x \in X_{(\alpha,0)}$ such that $mu = x_{[-|m|,\,|u|-1]}$. Hence $x_+ \in w_+(F(m))$ and $x_+ \notin w_+(F(m'))$ and so $w_+(F(m)) \neq w_+(F(m'))$. This means that Γ_α is follower separated.

To this end, we look for a magic block for Γ_α. Let $m \in \alpha$ and choose $u \in W(X)$ such that $mum \in W(X_{(\alpha,0)})$. Since $F(mum) = F(m)$, so $m \in F_-(F(m))$ where $F(m)$ is a vertex of Γ_α. If $m \in F_-(F(m'))$, then there is $m'' \in \alpha$ such that $F(m') = F(m''m) = F(m)$. Thus m is a magic block for the Γ_α [2, page 147]. Hence Γ_α is a Fischer cover of $\overline{X_{(\alpha,0)}}$ and so $h(\Gamma_\alpha) = h_{\mathrm{syn}}(\overline{X_{(\alpha,0)}})$ [3, Section 5].

It remains to prove that $h(\Gamma_\alpha) = t_0$. Let $\epsilon > 0$ and set

$$\{a_1, \dots, a_r\} := \{a \in W_N(\overline{A_k}) : (am)^\infty \in \overline{A_k}\}.$$

A graph consisting of the path labeled ma_im $(1 \leq i \leq r)$ is a finite subgraph H of Γ_α as in Figure 2. So $h(\Gamma_\alpha) \geq h(H) \geq \frac{1}{N+|m|}\log(r)$ and by (7) we have

$$h(\Gamma_\alpha) \geq \lim_N \frac{1}{N+|m|}\log(r) = \lim_N \frac{1}{N}\log\left|\{a \in W_N(\overline{A_k}) : (am)^\infty \in \overline{A_k}\}\right| \geq t_0 - \epsilon.$$

Also $h(\Gamma_\alpha) \leq t_0$ is trivial and we are done. □

Corollary 3.7. *Let X be a synchronized system such that $h_{syn}(X) = h(X)$. Then, X is almost sofic.*

Proof. In this case there is one element, say α in $S(X)/\sim$ where $X = \overline{X_{(\alpha,0)}}$ and so $h_{\mathrm{syn}}(\overline{X_{(\alpha,0)}}) = h(X)$. Now the conclusion follows from Proposition 3.6. □

Lemma 3.8. *Let $x \in R(X)$ and suppose that there is $s \in \alpha \in S(X)/\sim$ such that $s \subseteq x$. Then, $x \in \overline{X(\alpha,0)}$.*

Proof. Let $x \in R(X)$ and suppose $s \in \alpha \in S(X)/\sim$ with $s \subseteq x$. Then, for sufficiently large $n \geq 1$, $s \subseteq x_{[-n,n]} \subseteq x \in R(X)$. Since s and $x_{[-n,n]}$

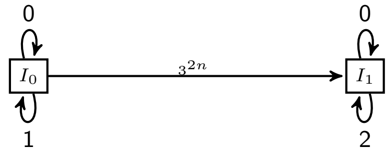

FIGURE 3. A graph for G_n; $X := X(\cup_{n\in\mathbb{N}} G_n)$.

are in $S(X)$, $s \sim x_{[-n,\,n]}$ and this implies that there is $y \in X_{(\alpha,\,0)}$ such that $\{s,\, x_{[-n,\,n]}\} \subseteq y$. But n was arbitrary large and so $x \in \overline{X_{(\alpha,\,0)}}$. □

Now the following is immediate.

Proposition 3.9. *Let $s \in \alpha$ and let $Y \subseteq R(X)$ be an irreducible shift space such that $s \in W(Y)$. Then, $Y \subseteq \overline{X_{(\alpha,\,0)}}$.*

Corollary 3.10. *$\overline{X_{(\alpha,\,0)}}$ is a synchronized component of X.*

Proof. Not that $\overline{X_{(\alpha,\,0)}}$ is a synchronized system. Now let Y be a synchronized subsystem of X with $\overline{X_{(\alpha,\,0)}} \subseteq Y$. Since $Y \subseteq R(X)$, Proposition 3.9 shows that $\overline{X_{(\alpha,\,0)}} = Y$. □

Corollary 3.11. *If C is a synchronized component of X, then either there is a unique $\alpha_0 \in S(X)/\sim$ such that $C = \overline{X_{(\alpha_0,\,0)}}$ or C is a synchronized component of $\partial(X)$.*

Proof. Suppose there is $\alpha_0 \in S(X)/\sim$ such that $C \cap X_{(\alpha_0,\,0)} \neq \emptyset$. Then, $C \nsubseteq \partial(X)$ and Proposition 3.9 shows that $\overline{X_{(\alpha_0,\,0)}} = C$. If $\overline{X_{(\alpha_0,\,0)}} = \overline{X_{(\beta,\,0)}}$, then $\alpha_0 \cap \beta \neq \emptyset$ and so $\alpha_0 = \beta$. Thus α_0 is unique.

Now let for each $\alpha \in S(X)/\sim$, $C \cap X_{(\alpha,\,0)} = \emptyset$. If $C \nsubseteq \partial(X)$, then there are $\alpha \in S(X)/\sim$ and $s \in \alpha$ such that $s \in W(C)$. Thus $C \cap X_{(\alpha,\,0)} \neq \emptyset$ that is absurd and so $C \subseteq \partial(X)$. □

Note that for $\beta \in S(\partial^i(X))/\sim$, $\overline{X_{(\beta,\,i)}}$ is a synchronized component of $\partial^i(X)$ which may not be a synchronized component for X. Now we investigate the cases where a synchronized component of $\partial^i(X)$ is a synchronized component of X as well. First an example:

Example 3.12. Let $X := X(\cup_n G_n)$ where G_n is the graph as in Figure 3. Then, the synchronized components of X are $\{0,\,1\}^\infty$, $\{0,\,2\}^\infty$ and $\{3^\infty\}$ while the synchronized component of $\partial(X)$ is $\{0^\infty\}$.

Proposition 3.13. *Let $Y \subseteq \partial(X)$ be a synchronized component of $\partial(X)$. Then, Y is a synchronized component of X if and only if for all $\alpha \in S(X)/\sim$, $Y \nsubseteq (X_{(\alpha,\,0)})' = \overline{X_{(\alpha,\,0)}} \setminus X_{(\alpha,\,0)}$.*

Proof. Suppose $Y \subseteq \partial(X)$ is a synchronized component of $\partial(X)$ but not a synchronized component of X. Then, there is a synchronized subsystem Z such that $Y \subsetneqq Z \subseteq X$. By the fact that Y is a synchronized component of $\partial(X)$, so $Z \nsubseteq \partial(X)$. Also Proposition 3.9 implies that there is α such that $Y \subseteq Z \subseteq \overline{X_{(\alpha,0)}}$. However $Y \subseteq \partial(X)$, hence $Y \cap X_{(\alpha,0)} = \emptyset$ and so $Y \subseteq (X_{(\alpha,0)})'$.

Conversely, let $Y \subseteq (X_{(\alpha,0)})'$. If Y is a synchronized component of X, then $Y = \overline{X_{(\alpha,0)}}$. That implies $X_{(\alpha,0)} \subseteq \partial(X)$ which is absurd. □

Remark 3.14. One may use Γ_α to visualize $X_{(\alpha,0)}$. To sort this out, let $X = R(X)$. Then, Γ_α is the Fischer cover of X and for any synchronizing block such as m, there is a unique "magic" vertex in Γ_α which is the terminal of any path labeled m. Now if $x \in R(X)$ and satisfies (3), then since x will have infinitely many synchronizing blocks in past and future, there must be a bi-infinite path labeled x, say π_x in Γ_α passing through m and the magic vertex.

In fact by Lemma 3.5, if π_y is a bi-infinite path visiting some finite magic blocks infinitely many in bounded times in past and future, then $y \in X_{(\alpha,0)}$.

By the above remark, an equivalent statement to (3) is

$$\sup_{i\in\mathbb{Z}} \left\{\inf\{(j-i) \geq 0 : x_{[i,j]} \text{ is synchronizing, } \pi_x \in X_0^+\}\right\}. \tag{9}$$

4. Finite minimal synchronizing in depths

If X is sofic, then ∂X is sofic, $\mathrm{Depth}(X)$ is finite and X has finitely many synchronized components [5].

In this section we introduce certain subshifts with finite depths and we will show that they also have finitely many synchronized components.

Definition 4.1. A shift space X is called *finite minimal synchronizing in depths* (FmSynID), if $\mathrm{Depth}(X) = n < \infty$ and all X, $R(X)$, $R(\partial(X)), \ldots$, $R(\partial^n(X))$ are FmSyn as defined in Definition 3.1.

By definition, if X is an irreducible FmSynID, then X is a synchronized system.

Example 4.2. Sofics and S-gap shifts are FmSynID. For a set of examples of non-FmSynID with finite depth let $\mathfrak{F}$ be a countable family of pairwise disjoint subsets of $\mathbb{N}$ with at least two elements and set $X_i := X(S_i)$, $X := \overline{\cup_i X_i}$ for $S_i \in \mathfrak{F}$. If there are at least two 1's appearing in $x \in X$, then there must be a $k_i \in S_i$ such that $10^{k_i}1 \in X_i$ is a synchronizing word for X_i as well as X. Hence, $\partial X = \{0^\infty, 0^\infty 10^\infty\}$ and so $\mathrm{Depth}(X) = 2$. Now if $|\mathfrak{F}| = \infty$, then X will have infinitely many minimal synchronizing blocks and so X cannot be FmSynID.

Let X be a shift and let X^{-1} be the shift space consisting of points $x^{-1} := \ldots x_2 x_1 x_0 x_{-1} \ldots$ whenever $x = \ldots x_{-1} x_0 x_1 x_2 \ldots \in X$.

Proposition 4.3. *(i) X is FmSynID if and only if X^{-1} is FmSynID.*
(ii) If X is FmSynID, then X has finitely many synchronized components.

Proof. (i) Note that $u = u_0 \cdots u_k$ is a synchronizing block for X if and only if $u^{-1} = u_k \cdots u_0$ is a synchronizing block for X^{-1}. So $\mathrm{Depth}(X) = \mathrm{Depth}(X^{-1})$, $R(X^{-1}) = (R(X))^{-1}$, $\partial(X^{-1}) = (\partial X)^{-1}, \ldots$ and $\partial^n(X^{-1}) = (\partial^n X)^{-1}$. This completes the proof of (i).

(ii) By applying Corollary 3.11 to higher depths, if Y is a synchronized component of X, then there are $i \in \{0, 1, 2, \ldots, n\}$ and $\alpha \in S(R^i(X))/\sim$ such that $Y = \overline{X_{(\alpha,\, i)}}$. But for each i, $R(\partial^i(X))$ is FmSyn or $S(\partial^i(X))/\sim$ is finite. This implies $|\{\overline{X_{(\alpha,\, i)}}\}_{\alpha \in S(\partial^i(X))/\sim}| < \infty$. □

Proposition 4.4. *Suppose X is FmSynID, with Depth$(X) = n$ and for each $0 \leq i \leq n$,*

(i) $h(\partial^i X) = h(\overline{Per(\partial^i X)})$,
(ii) $h(\overline{X_{(\alpha,\, i)}}) = h_{syn}(\overline{X_{(\alpha,\, i)}})$.

Then, X is almost sofic.

Proof. Let $S(\partial^n(X))/\sim = \{\alpha_{(n,1)}, \alpha_{(n,2)}, \ldots, \alpha_{(n,k_n)}\}$. Since $R(\partial^{n+1}(X)) = \emptyset$,

$$\overline{Per(\partial^n(X))} = R(\partial^n(X)) = \bigcup_{i=1}^{k_n} \overline{X_{(\alpha_{(n,i)},\, n)}}.$$

Hence, $h(R(\partial^n(X))) = \max\left\{h\left(\overline{X_{(\alpha_{(n,i)},\, n)}}\right) : 1 \leq i \leq k_n\right\}$ and so by (i) in our hypothesis,

$$h(\partial^n(X)) = \max\left\{h\left(\overline{X_{(\alpha_{(n,i)},\, n)}}\right) : 1 \leq i \leq k_n\right\}.$$

Since $|S(\partial^{n-1}(X))/\sim| = k_{n-1} < \infty$,

$$\overline{\mathrm{Per}(\partial^{n-1}(\mathrm{X}))} = R(\partial^{n-1}(X)) = \left(\bigcup_{i=1}^{k_{n-1}} \overline{X_{\left(\alpha_{(n-1,i)},\, n-1\right)}}\right) \cup \partial^n(X)$$

and so

$$h(\partial^{n-1}(X)) = \max\left\{h\left(\overline{X_{\left(\alpha_{(n-1,1)},\, n-1\right)}}\right), \ldots, h\left(\overline{X_{\left(\alpha_{(n-1,\, k_{n-1})},\, n-1\right)}}\right), h(\partial^n(X))\right\}.$$

Continuing this way, there will be $\alpha \in S(\partial^i(X))/\sim$ for some i such that

$$h(X) = h(\partial^0(X)) = h\left(\overline{\mathrm{Per}(\partial^0(X))}\right) = h\left(\overline{X_{(\alpha,\, i)}}\right).$$

Therefore by (ii), $h(X) = h_{\mathrm{syn}}\left(\overline{X_{(\alpha,\, i)}}\right)$. Now the result follows from Corollary 3.7. □

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