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GORENSTEIN SEQUENCES OF HIGH SOCLE DEGREES

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Abstract. In [4], the authors showed that if an h-vector $(h_0, h_1, \ldots, h_e)$ with $h_1 = 4e-4$ and $h_i \le h_1$ is a Gorenstein sequence, then $h_1 = h_i$ for every $1 \le i \le e-1$ and $e \ge 6$. In this paper, we show that if an h-vector $(h_0, h_1, \ldots, h_e)$ with $h_1 = 4e-4$, $h_2 = 4e-3$, and $h_i \le h_2$ is a Gorenstein sequence, then $h_2 = h_i$ for every $2 \le i \le e-2$ and $e \ge 7$. We also propose an open question that if an h-vector $(h_0, h_1, \ldots, h_e)$ with $h_1 = 4e-4$, $4e-3 < h_2 \le (h_1)_{(1)}|^{+1}_{+1}$, and $h_2 \le h_i$ is a Gorenstein sequence, then $h_2 = h_i$ for every $2 \le i \le e-2$ and $e \ge 6$.

1. Introduction

We consider a standard graded Artinian algebra $A = R/I$, where $R = \Bbbk[x_0, x_1, \ldots, x_n]$, I is a homogeneous ideal of R, and $\Bbbk$ is a field of any characteristic. The *h-vector* of A is $\mathbf{H} = (h_0, h_1, \ldots, h_e)$, where $h_i = \dim_{\Bbbk} A_i$ and e is the last index such that $\dim_{\Bbbk} A_e \neq 0$. The *socle* of A is the annihilator of the maximal homogeneous ideal $\mathfrak{m} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n) \subset A$, i.e., $\operatorname{soc}(A) = \{a \in A \mid a \cdot \mathfrak{m} = 0\}$. We define a *socle vector* of $s_A = (s_0, s_1, \ldots, s_e)$, where $s_i = \dim_{\Bbbk} \operatorname{soc}(A)_i$. Note that $s_e = h_e$. The integer e is called the *socle degree* of A (or of $\mathbf{H}$). If $s_A = (0, \ldots, 0, s_e = s)$, we say that A is an Artinian level algebra of type s. Moreover, if $s = 1$, then A is an Artinian Gorenstein algebra, and $\mathbf{H}$ is a Gorenstein sequence (or Gorenstein h-vector). In this paper, we show that the non-unimodals satisfying certain conditions do not occur (see Question 1.1).

Recall that an h-vector $\mathbf{H} = (h_0, h_1, \ldots, h_e)$ is defined to be an SI-sequence if it is symmetric and its first half is differentiable, namely, $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor \frac{e}{2} \rfloor} - h_{\lfloor \frac{e}{2} \rfloor - 1})$ satisfies Macaulay's theorem. It is well known that every SI-sequence can be a Gorenstein h-vector. By a result of P. Marocia [22] there is a length s smooth punctual scheme $Z \subset \mathbb{P}^n$ having Hilbert function agreeing with the first half of the SI-sequence. Define $\tau(Z)$ the first degree in which $h_i(Z) = |Z|$. If we get Z, having the Hilbert function in the

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first part of $\mathbf{H} = (h_0, h_1, \ldots, h_{\lfloor \frac{e}{2} \rfloor}, \ldots)$, followed by $s, s, \ldots$, then we can take a generic element F in $((I_Z)_j)^{\perp}$, and its annihilator $\mathrm{Ann}(F)$ contains the original ideal of Z, and have the symmetrized Hilbert function (see [19, Theorem 5.21A, Theorem 5.3], [11], and [22]).

Recall that a sequence of integers is *unimodal* if it does not strictly increase after a strict decrease. It is known that if a Gorenstein h-vector has a codimension ≤ 3, then it is unimodal ([10]). Furthermore, there are nonunimodal Gorenstein h-vectors of codimension ≥ 5 ([5, 7, 18]) and it is still unknown if there exists a nonunimodal Gorenstein sequence of codimension 4 ([12, 20, 24, 28]). In [4, 26], the authors classified Gorenstein h-vectors of small socle degree. In particular, in [26], the authors showed that nonunimodal Gorenstein h-vectors of socle degree 4 (respectively, 5) and codimension r exist if and only if $r \geq 13$ (respectively $r \geq 17$). In [4], the authors also considered Gorenstein h-vectors of general socle degree.

There has been a flurry of papers devoted to classifying possible unimodal or nonunimodal Artinian Gorenstein sequences (see [1–5, 8, 11, 12, 15–20, 23, 25, 28, 30]).

Let n and i be positive integers. The i-binomial expansion of n is

$$n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots \geq j \geq 1$. We call $n_i, n_{i-1}, \ldots, n_j$ the *Macaulay coefficients of* $n_{(i)}$ (see [9, page 160]). Following [6], we define, for any integers a and b,

$$(n_{(i)})\big|_a^b = \binom{n_i + b}{i + a} + \binom{n_{i-1} + b}{i - 1 + a} + \cdots + \binom{n_j + b}{j + a},$$

where we set $\binom{m}{q} = 0$ for $m < q$ or $q < 0$. We also use a notation $(n_{(i)})_a^b$ instead of $(n_{(i)})\big|_a^b$ for convenience.

The key ingredients in this paper are two important theorems, so called, Macaulay's theorem [21] and Green's theorems [14]. Together with these two theorems, we often use another theorem of Migliore, Nagel, and Zanello, namely, if an h-vector $\mathbf{H} = (h_0, h_1, \ldots, h_e)$ is a Gorenstein sequence, then

$$h_{i+1} \geq (h_i)_{(e-i)}\big|_{-1}^{-1} + (h_i)_{(e-i)}\big|_{-(e-2i-1)}^{-(e-2i)} \tag{1.1}$$

for $1 \leq i \leq \frac{e}{2}$ ([25]). It is a nice formula to determine if an h-vector is a Gorenstein sequence, though there are infinite series of non-Gorenstein sequences having the lower bound in equation (1.1) (see [3, 8]). Macaulay's theorem [21] and Green's theorem [14] play an important role in the study of Hilbert functions of standard graded Gorenstein algebras. In particular, Macaulay's theorem regulates the possible growth of the Hilbert function from one degree to the next, and Green's theorem regulates the possible Hilbert functions of the restriction modulo a general linear form.

In [4], the authors considered interesting Gorenstein h-vectors of higher socle degree using Macaulay's theorem, Green's theorem, and Gotzmann's theorem [13], namely, if $\mathbf{H} = (h_0, h_1, \ldots, h_e)$ with $h_1 = 4e-4$, $e \geq 6$, and $h_i \leq h_1$ for $1 \leq i \leq e-1$, is a Gorenstein sequence, then $h_1 = h_i$ for such i. Moreover, they constructed nonunimodal Gorenstein sequences $\mathbf{H} = (h_0, h_1, \ldots, h_e)$ with $h_1 = 4e-3$, $h_i = h_2 = 4e-4$, for $e \geq 6$ and $2 \leq i \leq e-2$.

Here, we have an open question as follows.

Question 1.1. Let $\mathbf{H} = (h_0, h_1, \ldots, h_e)$ with $h_1 = h_{e-1} = 4e-4$, $4e-3 \leq h_2 \leq (h_1)_{(1)}|^{+1}_{+1}$, $h_i \leq h_2$ for $2 \leq i \leq e-2$ and $e \geq 6$. Is $h_i = h_2$ for such i if $\mathbf{H}$ is a Gorenstein h-vector?

In this paper, we give a complete answer to Question 1.1 when $h_2 = 4e-3$ with $e \geq 7$. In other words, we show that non-unimodal Gorenstein sequences satisfying the conditions in Question 1.1 don't exist. However, it is still open when $h_2 = 4e-3$ and $e = 6$. In Section 2, we introduce some preliminary definitions, and notations. In Section 3, we introduce the main theorem of this paper and the proofs of Question 1.1 for two cases. In particular, we consider a Gorenstein h-vector of socle degree 12 in Subsection 3.1 and another Gorenstein h-vectors of high socle degrees $e \geq 16$ in Subsection 3.2. For the other cases of Question 1.1 when $h_2 = 4e-3$, $7 \leq e \leq 15$, and $e \neq 12$, we show all proofs in the Appendix.

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2. Preliminaries

First, we recall the results of Macaulay's theorem and Green's hyperplane restriction theorem ([14, 21]) which provide the upper bound for the Hilbert function of the quotient of a given graded algebra (not necessarily Artinian).

Theorem 2.1 ([14, 21])**.** *Let h_d be the entry of degree d of the Hilbert function of R/I and let ℓ_d be the degree d entry of the Hilbert function of $R/(I, L)$ where L is a general linear form of R. Then, we have the following inequalities.*

(a) Macaulay's Theorem: $h_{d+1} \leq \left((h_d)_{(d)}\right)\big|^{+1}_{+1}$.

(b) Green's Hyperplane Restriction Theorem: $\ell_d \leq \left((h_d)_{(d)}\right)\big|^{-1}_{\ 0}$.

Lemma 2.2 ([25, Proposition 8])**.** *If $(1, r, h_2, \ldots, r, 1)$ is a Gorenstein h-vector, then $(1, r+1, h_2+1, \ldots, r+1, 1)$ is also a Gorenstein h-vector.*

Lemma 2.3 ([29])**.** *Let $A = R/I$ be an Artinian Gorenstein algebra, and let $L \notin I$ be a linear form of R. Then the h-vector of A can be written as*

$$\mathbf{H} := (h_0, h_1, \ldots, h_e) = (1, b_1 + \ell_1, \ldots, b_{e-1} + \ell_{e-1}, b_e = 1),$$

where

$$b = (b_1, b_2, \ldots, b_{e-1}, b_e) \quad \text{with} \quad b_1 = b_e = 1$$

is the h-vector of $R/(I:L)(1)$ (with the indices shifted by 1), which is a Gorenstein algebra, and

$$\ell = (\ell_0, \ell_1, \ldots, \ell_{e-1}) \quad \text{with} \quad \ell_0 = 1$$

is the h-vector of $R/(I, L)$.

Notation. With notation as in Lemma 2.3, we shall simply call the following diagram

$$\begin{array}{cccccc} h_0 & h_1 & h_2 & \cdots & h_{e-1} & h_e \\ & b_1 & b_2 & \cdots & b_{e-1} & b_e \\ \hline \ell_0 & \ell_1 & \ell_2 & \cdots & \ell_{e-1} & \end{array} \tag{2.1}$$

the decomposition of the Hilbert function **H**. Moreover, we denote an h-vector $(b_1, b_1, \ldots, b_e)$ by b and an h-vector $(\ell_0, \ell_1, \ell_2, \ldots, \ell_{e-1})$ by ℓ for the rest of this paper.

3. Gorenstein sequences

In this section, we introduce the main theorem (Theorem 3.18) of this paper and prove two cases of socle degrees $e = 12$ and $e \geq 16$ only. For the rest of the cases, when $7 \leq e \leq 15$ and $e \neq 12$, we arrange the statements (Propositions 3.10∼3.17) only in Subsection 3.3 and place the proofs in [27, Appendix] because these cases can be proved using analogous ideas and methods with the decomposition tricks in equation (2.1) for Gorenstein sequences.

3.1. A Gorenstein sequence of socle degree 12

Before we prove Proposition 3.4, we introduce the following 3 lemmas first.

Lemma 3.1. *Suppose that an h-vector* $\mathbf{H} = (h_0, h_1, \ldots, h_e)$ *of socle degree* e *with* $h_1 = 4e - 2$ *satisfies one of the following.*

(1) $h_1 \geq h_2 + 2$ *and* $e \geq 9$, *or*
(2) $h_1 > h_2 > h_3$ *and* $e \geq 10$.

Then **H** *is not a Gorenstein sequence.*

Proof. Assume that there exists a Gorenstein Artinian algebra R/I with Hilbert function **H**.

(1) We suppose that the Hilbert function **H** is of the form

$$\mathbf{H} = (1, 4e-2, 4e-4-a, \ldots, 4e-4-a, 4e-2, 1).$$

Note that for $e \geq 9$,

$$(4e-2)_{(e-1)} = \binom{e}{e-1} + \binom{e-1}{e-2} + \binom{e-2}{e-3} + \binom{e-3}{e-4} + \binom{e-5}{e-5} + \binom{e-6}{e-6} + \binom{e-7}{e-7} + \binom{e-8}{e-8}.$$

By Green's theorem, $\ell_{e-1} \le 4$. So the decomposition of $\mathbf{H}$ is of the form

						(e−1)-st	
$\mathbf{H}$	:	1	$4e-2$	$4e-4-a$	$\cdots$	$4e-2$	1
b	:		1	$4e-6+\alpha$	$\cdots$	$4e-6+\alpha$	1
ℓ	:	1	$4e-3$	$2-\alpha-a$	$\cdots$	$4-\alpha$	

Then $\ell_2 \le 2$ and $\ell_2 < \ell_{e-1}$, that is, ℓ is not an O-sequence.

(2) If $h_1 \ge h_2+2$, then by (1) it holds. So we suppose that $h_2 = 4e-3 > h_3$, and the decomposition of $\mathbf{H}$ is

								(e−1)-st	
$\mathbf{H}$	:	1	$4e-2$	$4e-3$	$4e-4$	$\cdots$	$4e-3$	$4e-2$	1
b	:		1	$4e-6$	$4e-3-\beta$	$\cdots$	$4e-3-\beta$	$4e-6$	1
ℓ	:	1	$4e-3$	3	$\beta-1$	$\cdots$	β	4	

If $\beta \le 4$, then $\ell_3 \le 3$ and $\ell_3 = \beta - 1 < \beta = \ell_{e-2}$. In other words, ℓ is not an O-sequence. If $\beta \ge 5$, then by (1) b is not a Gorenstein sequence.

This completes the proof. □

Lemma 3.2. *The h-vector*

$$\mathbf{H} = (1, 41, 41, 40, h_3, \ldots, h_7, 40, 41, \overset{(e-1)\text{-st}}{41}, 1)$$

with $h_3 \le 39$ *and* $e \ge 11$ *is not a Gorenstein sequence.*

Proof. Suppose there exists a Gorenstein Artinian algebra with Hilbert function $\mathbf{H}$. First, if $e \ge 12$, then by equation (1.1), $h_3 \ge 40$, and so $\mathbf{H}$ is not a Gorestein sequence. So we assume that $e = 11$. By Green's theorem, $\ell_8 \le 6$, $\ell_9 \le 5$, and $\ell_{10} \le 4$. The decomposition of $\mathbf{H}$ is

											10-th	
$\mathbf{H}$	:	1	41	41	40	$39-a$	$\cdots$	$39-a$	40	41	41	1
b	:		1	$41-\ell_{10}$	$41-\ell_9$	$40-\ell_8$	$\cdots$	$39-\ell_7-a$	$40-\ell_8$	$41-\ell_9$	$41-\ell_{10}$	1
ℓ	:	1	40	ℓ_{10}	ℓ_9-1	ℓ_8-a-1	$\cdots$	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	

If $\ell_9 \le 4$, then $\ell_3 = \ell_9 - 1 \le 3$ and $\ell_3 = \ell_9 - 1 < \ell_9$. So ℓ is not an O-sequence, i.e., $\ell_9 = 5$. Moreover, if $a > 0$, then $\ell_4 = \ell_8 - a - 1 \le 4$, and $\ell_4 = \ell_8 - a - 1 < \ell_8$, that is, ℓ is not an O-sequence. Hence we rewrite the decomposition of $\mathbf{H}$ as

											10-th	
$\mathbf{H}$	:	1	41	41	40	39	$\cdots$	39	40	41	41	1
b	:		1	$41-\ell_{10}$	36	$40-\ell_8$	$\cdots$	$39-\ell_7$	$40-\ell_8$	36	$41-\ell_{10}$	1
ℓ	:	1	40	ℓ_{10}	4	ℓ_8-1	$\cdots$	ℓ_7	ℓ_8	5	ℓ_{10}	

(1) Suppose $\ell_{10} \le 2$. Then $(\ell_2, \ell_3) = (\ell_{10}, \ell_3) = (\le 2, 4)$, i.e., ℓ is not an O-sequence.

(2) Assume $\ell_{10} = 3$. Then $(b_2, b_3) = (38, 36)$, and so, by Lemma 3.1, b is not a Gorenstein sequence.

(3) Assume $\ell_{10} = 4$. If $\ell_8 \le 5$, then $\ell_4 = \ell_8 - 1 \le 4$, and so $\ell_4 = \ell_8 - 1 < \ell_8$. Thus ℓ is not an O-sequence. If $\ell_8 = 6$, then $(b_2, b_3, b_4) = (37, 36, 34)$, i.e., by [4, Lemma 3.8], b is not a Gorenstein sequence.

This completes the proof. □

Lemma 3.3. *The h-vector*

$$\mathbf{H} = (1, 41, 40, 40, 39, 37, 37, 39, 40, 40, \overset{\text{10-th}}{41}, 1)$$

is not a Gorenstein sequence.

Proof. Suppose there exists a Gorenstein Artinian algebra with Hilbert function $\mathbf{H}$. By Green's theorem, $\ell_6 \le 8$, $\ell_7 \le 8$, $\ell_8 \le 6$, $\ell_9 \le 5$, and $\ell_{10} \le 4$. The decomposition of $\mathbf{H}$ is

												10-th	
$\mathbf{H}$	:	1	41	40	40	39	37	37	39	40	40	41	1
b	:		1	$41-\ell_{10}$	$40-\ell_9$	$40-\ell_8$	$39-\ell_7$	$37-\ell_6$	$39-\ell_7$	$40-\ell_8$	$40-\ell_9$	$41-\ell_{10}$	1
ℓ	:	1	40	$\ell_{10}-1$	ℓ_9	ℓ_8-1	ℓ_7-2	ℓ_6	ℓ_7	ℓ_8-1	ℓ_9	ℓ_{10}	

Since $\ell_5 = \ell_7 - 2 \ge 6$, we have $\ell_7 \ge 8$, i.e., $\ell_7 = 8$, and so $\ell_6 = 7$. Moreover, $\ell_2 = \ell_{10} - 1 \ge 3$, and thus $\ell_{10} = 4$. It follows that $\ell_3 = \ell_9 = 4$ and $\ell_4 = \ell_8 - 1 = 5$. Hence we have $(b_2, b_3, b_4) = (37, 36, 34)$, and so by [4, Lemma 3.8], b is not a Gorenstein sequence. This completes the proof. □

Proposition 3.4 ($e = 12$). *Let $\mathbf{H} = (h_0, h_1, h_2, \dots, h_{10}, h_{11}, h_{12})$ be a symmetric sequence with*

$$h_1 = 44, \quad h_2 = 45, \quad \textit{and} \quad h_i \le h_2 \textit{ for all } i \ge 3.$$

Then $\mathbf{H}$ is a Gorenstein sequence if and only if $h_i = h_2 = 45$ for every $2 \le i \le 10$.

Proof. Suppose there is an Artinian Gorenstein algebra R/I with Hilbert function $\mathbf{H}$. From equation (1.1), there are 55 possible nonunimodal h-vectors (see [27, Appendix]). We shall show that all 55-cases cannot be Gorenstein sequences.

We shall prove this by 4-cases for (h_3, h_4), namely,

$$(h_3, h_4) = (44, 44), (44, 45), (45, 44), (45, 45).$$

By Green's theorem, we have

$$\ell_{10} \le 5 \quad \text{and} \quad \ell_{11} \le 4.$$

(1) We first consider the case $(h_3, h_4) = (44, 44)$, i.e.,

$$\mathbf{H} = (1, 44, 45, 44, 44, h_5, h_6, h_7, 44, 44, 45, 44, 1).$$

Note that $\ell_8 \le 8$, $\ell_9 \le 5$, $\ell_{10} \le 5$, and $\ell_{11} \le 4$.

Assume the decomposition of $\mathbf{H}$ is

$\mathbf{H}$	:	1	44	45	44	44	h_5	h_6	h_7	44	44	45	44	1
b	:		1	$44-\ell_{11}$	$45-\ell_{10}$	$44-\ell_9$	–	–	–	$44-\ell_8$	$44-\ell_9$	$45-\ell_{10}$	$44-\ell_{11}$	1
ℓ	:	1	43	$1+\ell_{11}$	$\ell_{10}-1$	ℓ_9	–	–	–	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	

Since ℓ is an O-sequence, we have

$$\begin{array}{llll} \ell_3 = \ell_{10} - 1 \geq 4, & \text{i.e.,} \quad \ell_{10} = 5, & (\text{by Green's theorem } \ell_{10} \leq 5), & \text{and} \\ \ell_4 = \ell_9 \geq 5, & \text{so} \quad \ell_9 = 5, & (\text{since } \ell_{10} = 5). & \end{array}$$

Hence the decomposition of $\mathbf{H}$ is

$\mathbf{H}$:	1	44	45	44	44	42	41	42	44	44	45	44	1
b :		1	$44-\ell_{11}$	40	39	−	−	−	$44-\ell_8$	39	40	$44-\ell_{11}$	1
ℓ :	1	43	$1+\ell_{11}$	4	5	−	−	−	ℓ_8	5	5	ℓ_{11}	

Since ℓ is an O-sequence, one can see that $\ell_2 = \ell_{11}+1 \geq 3$, i.e., $\ell_{11} \geq 2$.

(a) If $\ell_{11} = 2$, then $(b_2, b_3) = (42, 40)$. By equation (1.1), b is not a Gorenstein sequence (see also Lemma 3.1(1)).

(b) If $\ell_{11} = 3$, then $(b_2, b_3, b_4) = (41, 40, 39)$. By [4, Lemma 3.8(b)], b is not a Gorenstein sequence.

(c) If $\ell_{11} = 4$, then $(b_2, b_3, b_4) = (40, 40, 39)$. By [4, Proposition 3.14], b is not a Gorenstein sequence.

(2) We consider the case $(h_3, h_4) = (44, 45)$, i.e.,

$$\mathbf{H} = (1, 44, 45, 44, 45, h_5, h_6, h_7, 45, 44, 45, 44, 1).$$

Note that $\ell_8 \leq 9$, $\ell_9 \leq 5$, $\ell_{10} \leq 5$, and $\ell_{11} \leq 4$.

Assume the decomposition of $\mathbf{H}$ is

$\mathbf{H}$:	1	44	45	44	45	h_5	h_6	h_7	45	44	45	44	1
b :		1	$44-\ell_{11}$	$45-\ell_{10}$	$44-\ell_9$	−	−	−	$45-\ell_8$	$44-\ell_9$	$45-\ell_{10}$	$44-\ell_{11}$	1
ℓ :	1	43	$1+\ell_{11}$	$\ell_{10}-1$	ℓ_9+1	−	−	−	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	

Since ℓ is an O-sequence and $\ell_{10} - 1 = \ell_3 < \ell_{10}$, we have

$$\begin{array}{llll} \ell_3 = \ell_{10} - 1 \geq 4, & \text{i.e.,} \quad \ell_{10} = 5, & (\text{by Green's theorem } \ell_{10} \leq 5), & \text{and} \\ \ell_4 = \ell_9 + 1 \geq 5, & \text{so} \quad \ell_9 \geq 4, \ \text{i.e.,}\ \ell_9 = 5, & (\text{since } \ell_{10} = 5). & \end{array}$$

But, then $(\ell_3, \ell_4) = (4, 6)$ is not an O-sequence.

(3) We consider the case $(h_3, h_4) = (45, 44)$, i.e.,

$$\mathbf{H} = (1, 44, 45, 45, 44, h_5, h_6, h_7, 44, 45, 45, 44, 1).$$

Note that $\ell_8 \leq 8$, $\ell_9 \leq 6$, $\ell_{10} \leq 5$, and $\ell_{11} \leq 4$.

Assume the decomposition of $\mathbf{H}$ is

$\mathbf{H}$:	1	44	45	45	44	h_5	h_6	h_7	44	45	45	44	1
b :		1	$44-\ell_{11}$	$45-\ell_{10}$	$45-\ell_9$	−	−	−	$44-\ell_8$	$45-\ell_9$	$45-\ell_{10}$	$44-\ell_{11}$	1
ℓ :	1	43	$1+\ell_{11}$	ℓ_{10}	ℓ_9-1	−	−	−	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	

Since ℓ is an O-sequence and $\ell_9 - 1 = \ell_4 < \ell_9$, we have

$$\begin{array}{llll} \ell_4 = \ell_9 - 1 \geq 5, & \text{i.e.,} \quad \ell_9 = 6, & (\text{by Green's theorem } \ell_9 \leq 6), & \text{and} \\ \ell_{10} = 4, 5. & & & \end{array}$$

$\mathbf{H}$:	1	44	45	45	44	h_5	h_6	h_7	44	45	45	44	1
b :		1	$44-\ell_{11}$	$45-\ell_{10}$	39	−	−	−	$44-\ell_8$	39	$45-\ell_{10}$	$44-\ell_{11}$	1
ℓ :	1	43	$1+\ell_{11}$	ℓ_{10}	5	−	−	−	ℓ_8	6	ℓ_{10}	ℓ_{11}	

(a) If $\ell_{10} = 4$, i.e., $b_3 = 41$, then by equation (1.1), $(b_3, b_4) = (41, 39)$ is not a Gorenstein sequence.

(b) Let $\ell_{10} = 5$.

(i) If $\ell_{11} \leq 1$, then $\ell_2 = \ell_{11} + 1 \leq 2 < 5 = \ell_4$, i.e., ℓ is not an O-sequence.

(ii) If $\ell_{11} = 2$, then by Lemma 3.1, $(b_2, b_3) = (42, 40)$ is not a Gorenstein sequence.

(iii) If $\ell_{11} = 3$, then by [4, Lemma 3.8], $(b_2, b_3, b_4) = (41, 40, 39)$ is not a Gorenstein sequence.

(iv) Suppose $\ell_{11} = 4$. Then by [4, Proposition 3.14], $(b_2, b_3, b_4) = (40, 40, 39)$ is not a Gorenstein sequence.

(4) We consider the case $(h_3, h_4) = (45, 45)$, i.e.,

$$\mathbf{H} = (1, 44, 45, 45, 45, h_5, h_6, h_7, 45, 45, 45, 44, 1).$$

Note that $\ell_8 \leq 9$, $\ell_9 \leq 6$, $\ell_{10} \leq 5$, and $\ell_{11} \leq 4$.

(a) Suppose $42 \leq h_5 \leq 44$. Then the decomposition of $\mathbf{H}$ is

$\mathbf{H}$	:	1	44	45	45	45	h_5	h_6	h_7	45	45	45	44	1
b	:		1	$44-\ell_{11}$	$45-\ell_{10}$	$45-\ell_9$	$45-\ell_8$	–	–	$45-\ell_8$	$45-\ell_9$	$45-\ell_{10}$	$44-\ell_{11}$	1
ℓ	:	1	43	$1+\ell_{11}$	ℓ_{10}	ℓ_9	$h_5+\ell_8-45$	–	–	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	

Since ℓ is an O-sequence and $h_5 \leq 44$, we get that $h_5 + \ell_8 - 45 \leq \ell_8 - 1$, and so we have

$$h_5 + \ell_8 - 45 \geq 6, \quad \ell_8 \geq 51 - h_5 \geq 7, \quad \text{i.e.,} \quad \ell_8 = 7, 8, 9, \quad \text{and}$$
$$\ell_4 = \ell_9 \geq 5, \quad \text{i.e.,} \quad \ell_9 = 5, 6.$$

(i) Assume $\ell_9 = 5$. Since $b_5 = 45 - \ell_8 \leq 38$, we have $(b_4, b_5) = (40, \leq 38)$. However, by equation (1.1), so b is not a Gorenstein sequence.

(ii) If $\ell_9 = 6$ and $\ell_8 = 8, 9$, then $b_4 = 39$ and $b_5 = 45 - \ell_8 \leq 37$. But, by equation (1.1), b is not a Gorenstein sequence.

(iii) Assume $\ell_9 = 6$ and $\ell_8 = 7$. Since ℓ is an O-sequence, one can see that $\ell_5 = h_5 - 38 \geq 6$. Hence $h_5 = 44$. Moreover, by equation (1.1), $b_3 = 45 - \ell_{10} \geq 40$, i.e., $\ell_{10} = 5$. Hence the decomposition of $\mathbf{H}$ is

$\mathbf{H}$	:	1	44	45	45	45	44	h_6	44	45	45	45	44	1
b	:		1	$44-\ell_{11}$	40	39	38	–	–	38	39	40	$44-\ell_{11}$	1
ℓ	:	1	43	$1+\ell_{11}$	5	6	6	–	–	7	6	5	ℓ_{11}	

(A) If $\ell_{11} \leq 2$, then $(\ell_2, \ell_3) = (\leq 3, 5)$ is not an O-sequence.

(B) If $\ell_{11} = 3$, then [4, Lemma 3.8] $(b_2, b_3, b_4) = (41, 40, 39)$ is not a Gorenstein sequence.

(C) If $\ell_{11} = 4$, then by [4, Proposition 3.14], b is not a Gorenstein sequence as well.

(b) We now consider the case with $h_5 = 45$, i.e.,

$$\mathbf{H} = (1, 44, 45, 45, 45, 45, 44, 45, 45, 45, 45, 44, 1).$$

Note that $\ell_7 \leq 9$, $\ell_8 \leq 9$, $\ell_9 \leq 6$, $\ell_{10} \leq 5$, and $\ell_{11} \leq 4$.

Assume the decomposition of $\mathbf{H}$ is

$\mathbf{H}$:	1	44	45	45	45	45	44	45	45	45	45	44	1
b :		1	$44-\ell_{11}$	$45-\ell_{10}$	$45-\ell_9$	$45-\ell_8$	$45-\ell_7$	$45-\ell_7$	$45-\ell_8$	$45-\ell_9$	$45-\ell_{10}$	$44-\ell_{11}$	1
ℓ :	1	43	$1+\ell_{11}$	ℓ_{10}	ℓ_9	ℓ_8	ℓ_7-1	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	ℓ_{11}	

Since ℓ is an O-sequence, we see that

$$\begin{cases} \ell_2 = \ell_{11}+1 \geq 3, \\ \ell_3 = \ell_{10} \geq 4, \\ \ell_4 = \ell_9 \geq 5, \\ \ell_5 = \ell_8 \geq 6, \text{and} \\ \ell_6 = \ell_7 - 1 \geq 7, \end{cases} \quad \text{i.e.,} \quad \begin{cases} \ell_7 = 8, 9, \\ \ell_8 = 6, 7, 8, 9, \\ \ell_9 = 5, 6, \\ \ell_{10} = 4, 5, \text{and} \\ \ell_{11} = 2, 3, 4. \end{cases}$$

(i) Let $\ell_{11} = 2$, i.e., $b_2 = 42$. Then the decomposition of $\mathbf{H}$ is

$\mathbf{H}$:	1	44	45	45	45	45	44	45	45	45	45	44	1
b :		1	42	$45-\ell_{10}$	$45-\ell_9$	$45-\ell_8$	$45-\ell_7$	$45-\ell_7$	$45-\ell_8$	$45-\ell_9$	$45-\ell_{10}$	42	1
ℓ :	1	43	3	ℓ_{10}	ℓ_9	ℓ_8	ℓ_7-1	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	2	

(A) If $\ell_{10} = 4$, then $(b_2, b_3, b_4) = (42, 41, \leq 40)$ by Lemma 3.1(b), b is not a Gorenstein sequence.

(B) If $\ell_{10} = 5$, then $(b_2, b_3) = (42, 40)$, i.e., by Lemma 3.1(a), b is not a Gorenstein sequence as well.

(ii) Let $\ell_{11} = 3$, i.e., $b_2 = 41$. Then the decomposition of $\mathbf{H}$ is

$\mathbf{H}$:	1	44	45	45	45	45	44	45	45	45	45	44	1
b :		1	41	$45-\ell_{10}$	$45-\ell_9$	$45-\ell_8$	$45-\ell_7$	$45-\ell_7$	$45-\ell_8$	$45-\ell_9$	$45-\ell_{10}$	41	1
ℓ :	1	43	4	ℓ_{10}	ℓ_9	ℓ_8	ℓ_7-1	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	3	

(A) Let $(\ell_9, \ell_{10}) = (5, 4)$. Then $(b_2, b_3, b_4, b_5) = (41, 41, 40, \leq 39)$, and thus, by Lemma 3.2, b is not a Gorenstein sequence.

(B) Now assume $(\ell_9, \ell_{10}) = (6, 4)$. Then $(\ell_{10}, \ell_9) = (\ell_3, \ell_4) = (4, 6)$ is not an O-sequence.

(C) So we assume that $(\ell_9, \ell_{10}) = (5, 5)$.

(D) Let $(\ell_7, \ell_8) = (8, 6)$. Then $(b_2, b_3, b_4, b_5, b_6) = (41, 40, 40, 39, 37)$. By Lemma 3.3, b is not a Gorenstein sequence.

(E) If $(\ell_7, \ell_8) = (9, 6)$, then $(\ell_5, \ell_6) = (\ell_8, \ell_7 - 1) = (6, 8)$ is not an O-sequence.

(F) If $\ell_8 \geq 7$, then $(b_4, b_5) = (40, \leq 38)$, and so by equation (1.1), b is not a Gorenstein sequence.

(G) If $(\ell_9, \ell_{10}) = (6, 5)$, then $(b_2, b_3, b_4) = (41, 40, 39)$, and so, by [4, Lemma 3.8], b is not a Gorenstein sequence.

(iii) If $\ell_{11} = 4$, then the decomposition of $\mathbf{H}$ is

$\mathbf{H}$:	1	44	45	45	45	45	44	45	45	45	45	44	1
b :		1	40	$45-\ell_{10}$	$45-\ell_9$	$45-\ell_8$	$45-\ell_7$	$45-\ell_7$	$45-\ell_8$	$45-\ell_9$	$45-\ell_{10}$	40	1
ℓ :	1	43	5	ℓ_{10}	ℓ_9	ℓ_8	ℓ_7-1	ℓ_7	ℓ_8	ℓ_9	ℓ_{10}	4	

(A) Let $(\ell_9, \ell_{10}) = (5, 4)$. If $\ell_8 = 6$, then, by the proof of Proposition 3.14, $(b_2, b_3, b_4, b_5) = (40, 41, 40, 39)$ is not a Gorenstein sequence. If $\ell_8 \geq 7$, then by equation (1.1), $(b_2, b_3, b_4, b_5) = (40, 41, 40, \leq 38)$ is not a Gorenstein sequence.

(B) Let $(\ell_9, \ell_{10}) = (5, 5)$. Since $\ell_8 \geq 6$, we get that $(b_3, b_4, b_5) = (40, 40, \leq 39)$, i.e., by [4, Proposition 3.14] b is not a Gorenstein sequence.

(C) Let $(\ell_9, \ell_{10}) = (6, 4)$. Then $(b_2, b_3, b_4) = (40, 41, 39)$, and by equation (1.1), b is not a Gorenstein sequence (see also Proposition 3.14).

(D) Let $(\ell_9, \ell_{10}) = (6, 5)$. Then $(b_2, b_3, b_4) = (40, 40, 39)$, and by [4, Proposition 3.14], b is not a Gorenstein sequence.

This completes the proof. □

3.2. Gorenstein sequences of socle degrees ≥ 16

We first introduce the following lemma.

Lemma 3.5. *Let $e \geq 20$. Then for every $3 \leq i \leq \frac{e}{2} - 1$,*

$$4e - 3 < \binom{e-i+2}{2} - \binom{e-2i}{2} = \frac{1}{2}(i+2)(2e-3i+1).$$

In particular, the binomial expansion of $4e - 3$ in degree $(e - i)$ is of the form

$$(3.1)\quad (4e-3)_{(e-i)} = \binom{e-i+1}{e-i} + \cdots + \binom{k+1}{k} + \binom{k-1}{k-1} + \cdots + \binom{m}{m},$$

where $e - 2i \leq k \leq e - i + 1$.

Proof. Define a function

$$\begin{aligned} f_e(i) &= (4e-3) - \frac{1}{2}(i+2)(2e-3i+1) \\ &= \frac{1}{2}(3i^2 - (2e-5)i - 8). \end{aligned}$$

Note that for $3 \leq i \leq \frac{e}{2} - 1$, $f_e(i)$ has the maximum value at $i = 3$ and $i = \frac{e}{2} - 1$. Moreover, for $e \geq 20$,

$$\begin{aligned} f_e(3) &= 17 - e \leq 0, \\ f_e(\frac{e}{2} - 1) &= \frac{1}{8}(-e^2 + 22e - 40) \leq 0. \end{aligned}$$

Hence we obtain the binomial expansion of $4e - 3$ in degree $(e - i)$ as in equation (3.1). This completes the proof. □

Proposition 3.6. *For $e \geq 16$, if an O-sequence of socle degree e of the form*

$$(1, 4e-4, 4e-3, h_3, \ldots, h_{e-3}, 4e-3, 4e-4, 1)$$

is a Gorenstein sequence and

$$h_{i+1} = (h_i)_{(e-i)}|_{-1}^{-1} + (h_i)_{(e-i)}|_{-(e-2i-1)}^{-(e-2i)}$$

for $3 \le i \le \frac{e}{2} - 1$*, then*

$$h_i = 4e - 3$$

for every $3 \le i \le e-2$*.*

Proof. By a simple calculation, one can easily show that it holds for $16 \le i \le 19$. So we suppose that $e \ge 20$. By Lemma 3.5, the binomial expansion of $(4e-3)$ in degree $(e-i)$ with $3 \le i \le \frac{e}{2} - 1$ is of the form

$$(4e-3)_{(e-i)} = \binom{e-i+1}{e-i} + \cdots + \binom{k+1}{k} + \binom{k-1}{k-1} + \cdots + \binom{m}{m},$$

where $e - 2i \le k \le e - i + 1$. Hence, for $3 \le i \le \frac{e}{2} - 1$,

$$\begin{aligned} h_{i+1} &= (4e-3)_{(e-i)}|_{-1}^{-1} + (4e-3)_{(e-i)}|_{-(e-2i-1)}^{-(e-2i)} \\ &= \binom{e-i+1}{e-i} + \cdots + \binom{k+1}{k} + \binom{k-1}{k-1} + \cdots + \binom{m}{m}\Big|_{-1}^{-1} \\ &\quad + \binom{e-i+1}{e-i} + \cdots + \binom{k+1}{k} + \binom{k-1}{k-1} + \cdots + \binom{m}{m}\Big|_{-(e-2i-1)}^{-(e-2i)} \\ &= \left[\binom{e-i+1}{e-i} + \cdots + \binom{k+1}{k} + (k-m) - (e-i-k+1)\right] \\ &\quad + (e-i-k+1) \\ &= \binom{e-i+1}{e-i} + \cdots + \binom{k+1}{k} + \binom{k-1}{k-1} + \cdots + \binom{m}{m} \\ &= 4e-3, \end{aligned}$$

as we wished. □

We now introduce a simple way to construct a Gorenstein algebra having certain unimodal h-vectors (see [19] for details).

Theorem 3.7 ([19, Theorem 5.21A, Theorem 5.3])**.** *If* $Z = \{\wp_1, \ldots, \wp_s\}$ *is a finite set of reduced points in* $\mathbb{P}^n$*, then* Z *is an annihilating scheme for* f *if and only if* f *has an additive decomposition*

$$f = c_1 L_{\wp_1}^{[e]} + \cdots + c_s L_{\wp_s}^{[e]},$$

where $L_{\wp_i}$ *is the linear form corresponding to* $\wp_i$*.*

Corollary 3.8. *Let* $\mathbf{H} = (h_0, h_1, \ldots, h_e)$ *be an SI-sequence. Then* $\mathbf{H}$ *is a Gorenstein h-vector.*

Proof. Let $s = \max\{h_i\}$ and τ be the first degree in which $h_i = |Z|$. Assume $Z = \{\wp_1, \dots, \wp_s\}$ is a finite set of reduced s-points in the projective space $\mathbb{P}^n$ with $n = h_1 - 1$ such that the Hilbert function of Z is

$$\mathbf{H}_Z \quad : \quad 1 \quad h_1 \quad \cdots \quad h_\tau \quad \rightarrow .$$

Note that this is always possible since an SI-sequence is a differentiable O-sequence.

Define

$$f = c_1 L_{\wp_1}^{[e]} + \cdots + c_s L_{\wp_s}^{[e]},$$

where $L_{\wp_i}$ is the linear form corresponding to $\wp_i$. By Theorem 3.7, we see that the Hilbert function of $R/\mathrm{Ann}(f)$ is $\mathbf{H}$, as we wished. $\square$

The following corollary is immediate from Corollary 3.8. We omit the proof.

Corollary 3.9. *If* $\mathbf{H} = (1, 4e-4, 4e-3, \dots, 4e-3, 4e-4, 1)$ *is a symmetric h-vector with*

$$h_2 = h_i = 4e - 3$$

for $i = 2, \dots, e-2$ *and* $e \geq 7$, *then* $\mathbf{H}$ *is a Gorenstein h-vector.*

3.3. The main theorem

We shall state the following propositions for the cases of socle degrees $7 \leq e \leq 15$ and $e \neq 12$. Their proofs will be in the Appendix in [27].

Proposition 3.10 ($e = 7$)**.** *Let* $\mathbf{H} = (h_0, h_1, h_2, \dots, h_6, h_7)$ *be a Gorenstein sequence. Assume*

$$h_1 = 24, \quad h_2 = 25, \quad \text{and} \quad h_i \leq h_2 \text{ for all } i \geq 3.$$

Then $\mathbf{H}$ *is a Gorenstein sequence if and only if* $h_i = h_2$ *for* $2 \leq i \leq 5$.

Proposition 3.11 ($e = 8$)**.** *Let* $\mathbf{H} = (h_0, h_1, h_2, \dots, h_7, h_8)$. *Assume*

$$h_1 = 28, \quad h_2 = 29, \quad \text{and} \quad h_i \leq h_2 \text{ for all } i \geq 3.$$

Then $\mathbf{H}$ *is a Gorenstein sequence if and only if* $h_i = h_2 = 29$ *for* $2 \leq i \leq 6$.

Proposition 3.12 ($e = 9$)**.** *Let* $\mathbf{H} = (h_0, h_1, h_2, \dots, h_7, h_8, h_9)$. *Assume*

$$h_1 = 32, \quad h_2 = 33, \quad \text{and} \quad h_i \leq h_2 \text{ for all } i \geq 3.$$

Then $\mathbf{H}$ *is a Gorenstein sequence if and only if* $h_i = h_2 = 33$ *for* $2 \leq i \leq 7$.

Proposition 3.13 ($e = 10$)**.** *Let* $\mathbf{H} = (h_0, h_1, h_2, \dots, h_8, h_9, h_{10})$. *Assume*

$$h_1 = 36, \quad h_2 = 37, \quad \text{and} \quad h_i \leq h_2 \text{ for all } i \geq 3.$$

Then $\mathbf{H}$ *is a Gorenstein sequence if and only if* $h_i = h_2$ *for every* $2 \leq i \leq 8$.

Proposition 3.14 ($e = 11$)**.** *Let* $\mathbf{H} = (h_0, h_1, h_2, \dots, h_9, h_{10}, h_{11})$. *Assume*

$$h_1 = 40, \quad h_2 = 41, \quad \text{and} \quad h_i \leq h_2 \text{ for all } i \geq 3.$$

Then $\mathbf{H}$ *is a Gorenstein sequence if and only if* $h_i = h_2$ *for every* $2 \leq i \leq 9$.

Proposition 3.15 ($e = 13$). *Let* $\mathbf{H} = (h_0, h_1, h_2, \ldots, h_{11}, h_{12}, h_{13})$. *Assume*

$$h_1 = 48, \quad h_2 = 49, \quad \textit{and} \quad h_i \le h_2 \textit{ for all } i \ge 3.$$

Then $\mathbf{H}$ *is a Gorenstein sequence if and only if* $h_i = h_2$ *for every* $2 \le i \le 11$.

Proposition 3.16 ($e = 14$). *Let* $\mathbf{H} = (h_0, h_1, h_2, \ldots, h_{11}, h_{12}, h_{13}, h_{14})$. *Assume*

$$h_1 = 52, \quad h_2 = 53, \quad \textit{and} \quad h_i \le h_2 \textit{ for } 2 \le i \le 12.$$

Then $\mathbf{H}$ *is a Gorenstein sequence if and only if* $h_i = h_2$ *for every* $2 \le i \le 12$.

Proposition 3.17 ($e = 15$). *Let* $\mathbf{H} = (h_0, h_1, h_2, \ldots, h_{13}, h_{14}, h_{15})$. *Assume*

$$h_1 = 56, \quad h_2 = 57, \quad \textit{and} \quad h_i \le h_2 \textit{ for } 2 \le i \le 13.$$

Then $\mathbf{H}$ *is a Gorenstein sequence if and only if* $h_i = h_2$ *for every* $2 \le i \le 13$.

We now introduce the main theorem in this paper.

Theorem 3.18. *For* $e \ge 7$, *if an* O*-sequence*

$$\mathbf{H} = (1, 4e-4, 4e-3, h_3, \ldots, h_{e-2}, 4e-3, 4e-4, 1)$$

with $h_i \le 4e-3$ *for* $2 \le i \le e-2$ *is a Gorenstein h-vector, then*

$$h_i = h_2 = 4e-3$$

for such i.

Proof. (1) For $7 \le e \le 15$, see Propositions 3.10, 3.11, 3.12, 3.13, 3.14, 3.4, 3.15, 3.16, 3.17.

(2) For $e \ge 16$, see Proposition 3.6. □

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