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BOUNDEDNESS AND CONTINUITY FOR VARIATION OPERATORS ON THE TRIEBEL–LIZORKIN SPACES

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Abstract. In this paper, we establish the boundedness and continuity for variation operators for θ-type Calderón–Zygmund singular integrals and their commutators on the Triebel–Lizorkin spaces. As applications, we obtain the corresponding results for the Hilbert transform, the Hermit Riesz transform, Riesz transforms and rough singular integrals as well as their commutators.

1. Introduction

The primary purpose of this paper is to establish the boundedness and continuity for variation operators for θ-type Calderón–Zygmund singular integrals and their commutators on the Triebel–Lizorkin spaces. We now recall some definitions and background. Let $\mathcal{T} = \{T_\epsilon\}_{\epsilon>0}$ be a family of bounded operators satisfying

$$\lim_{\epsilon\to 0} T_\epsilon f(x) = Tf(x)$$

almost everywhere for a certain class of functions f. For $\rho > 2$, the ρ-variation operator of $\mathcal{T}$ is defined by

$$\mathcal{V}_\rho(\mathcal{T})(f)(x) = \sup_{\{\epsilon_i\}\searrow 0} \Big(\sum_{i=1}^{\infty} |T_{\epsilon_i} f(x) - T_{\epsilon_{i+1}} f(x)|^\rho \Big)^{1/\rho},$$

where the supremum runs over all sequences $\{\epsilon_i\}$ of positive numbers decreasing to zero.

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We say that T_K is a θ-type Calderón–Zygmund operator on $\mathbb{R}^n$ if T_K is bounded on $L^2(\mathbb{R}^n)$ and it admits the following representation

$$T_K f(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy \quad \text{for} \quad x \notin \operatorname{supp} f$$

with kernel K satisfying the size condition

$$|K(x,y)| \leq \frac{C_K}{|x-y|^n}$$

and a smoothness condition

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq \theta\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^n}$$

for all $|x-y| > 2|x-z|$, where $\theta : [0,1] \to [0,\infty)$ is a modulus of continuity, that is, θ is a continuous, increasing, subadditive function with $\theta(0) = 0$ and satisfies the following Dini condition:

$$\int_0^1 \theta(t)\frac{dt}{t} < \infty.$$

This type of operator T_K was studied by Lacy [11] and Lerner [13] who proved that T_K is bounded on the weighted Lebesgue space $L^p(w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$. When $\theta(t) = t^\delta$ for some $\delta > 0$, the operator T_K is the classical Calderón–Zygmund singular integral operator.

Formally, the operator T_K can be rewritten as

$$T_K(f)(x) = \lim_{\epsilon \to 0^+} T_{K,\epsilon}(f)(x),$$

where $T_{K,\epsilon}$ is the truncated singular integral operator, i.e.,

$$T_{K,\epsilon}(f)(x) = \int_{|x-y|>\epsilon} K(x,y) f(y) dy.$$

The commutator of T_K with a suitable function b is defined as

$$\begin{aligned} T_{K,b}(f)(x) := [b, T_K](f)(x) &= \int_{\mathbb{R}^n} (b(x) - b(y)) K(x,y) f(y) dy \\ &= \lim_{\epsilon \to 0^+} T_{K,b,\epsilon}(f)(x), \end{aligned}$$

where

$$T_{K,b,\epsilon}(f)(x) := \int_{|x-y|>\epsilon} (b(x) - b(y)) K(x,y) f(y) dy.$$

Denote $T^1_{K,b} = T_{K,b}$. For $m \geq 2$, the m-th iterated commutator $T^m_{K,b}$ is defined by

$$\begin{aligned} T^m_{K,b}(f)(x) := [b, T^{m-1}_{K,b}](f)(x) &= \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x,y) f(y) dy \\ &=: \lim_{\epsilon \to 0^+} T^m_{K,b,\epsilon}(f)(x), \end{aligned}$$

where

$$T^m_{K,b,\epsilon}(f)(x) := \int_{|x-y|>\epsilon} (b(x)-b(y))^m K(x,y)f(y)dy.$$

Let $\mathcal{T} := \{T_{K,\epsilon}\}_{\epsilon>0}$ and $\mathcal{T}^m_{K,b} := \{T^m_{K,b,\epsilon}\}_{\epsilon>0}$. The ρ-variation operator for the families of operators $\mathcal{T}$ and $\mathcal{T}^m_{K,b}$ are defined, respectively, by

$$\mathcal{V}_\rho(\mathcal{T}_K)(f)(x) := \sup_{\varepsilon_i \searrow 0} \Big(\sum_{i=1}^{\infty} \Big| \int_{\varepsilon_{i+1}<|x-y|\le\varepsilon_i} K(x,y)f(y)dy \Big|^\rho \Big)^{1/\rho}, \tag{1.1}$$

$$\begin{aligned} &\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)(x) \\ &:= \sup_{\varepsilon_i \searrow 0} \Big(\sum_{i=1}^{\infty} \Big| \int_{\varepsilon_{i+1}<|x-y|\le\varepsilon_i} (b(x)-b(y))^m K(x,y)f(y)dy \Big|^\rho \Big)^{1/\rho}, \end{aligned} \tag{1.2}$$

where the above supremums are taken over all sequences $\{\varepsilon_i\}$ decreasing to zero.

It should be pointed out that T_K and $T^m_{K,b}$ have some classical models, which are listed as follows:

• When $n=1$ and $K(x,y) = \frac{1}{x-y}$, then T_K (resp., $T^m_{K,b}$) is (resp., the m-th order commutator of) Hilbert transform. We denote $\mathcal{T}_K = \mathcal{H}$ and $\mathcal{T}^m_{K,b} = \mathcal{H}^m_b$ for $m \ge 1$.

• When $n=1$ and $K(x,y) = \mathfrak{R}^\pm(x,y)$, where $\mathfrak{R}^\pm(x,y)$ is a Hermit Riesz kernel whose expression can be found in [21], then T_K (resp., $T^m_{K,b}$) is (resp., the m-th order commutator of) Hermit Riesz transform. We denote $\mathcal{T}_K = \mathcal{R}^\pm$ and $\mathcal{T}^m_{K,b} = \mathcal{R}^m_{\pm,b}$ for $m \ge 1$.

• When $n \ge 2$ and $K(x,y) = R_j(x,y)$, where

$$R_j(x,y) := \Gamma\Big(\frac{n+1}{2}\Big)\pi^{-\frac{n+1}{2}} \frac{x_j-y_j}{|x-y|^{n+1}}$$

for $1 \le j \le n$, then T_K (resp., $T^m_{K,b}$) is (resp., the m-th order commutator of) Riesz transform. We denote $\mathcal{T}_K = \mathcal{R}_j$ and $\mathcal{T}^m_{K,b} = \mathcal{R}^m_{j,b}$ for $m \ge 1$.

• When $n \ge 2$ and $K(x,y) = \frac{\Omega(x-y)}{|x-y|^n}$, where $\Omega \in L^1(\mathrm{S}^{n-1})$ is homogeneous of zero and satisfies $\int_{\mathrm{S}^{n-1}} \Omega(\theta)d\sigma(\theta) = 0$, then T_K (resp., $T^m_{K,b}$) is just the usual (resp., the m-th order commutator of) singular integral operator with rough kernel Ω. We denote $\mathcal{T}_K = \mathcal{T}_\Omega$ and $\mathcal{T}^m_{K,b} = \mathcal{T}^m_{\Omega,b}$ for $m \ge 1$.

The variation inequalities for various operators have been an active topic of current research. This program began with Lépingle [12] who established the first variational inequality for general martingales (see also [20] for a simple proof). Later on, similar variation estimates were obtained by Bourgain [3] for the ergodic averages of a dynamic system. Motivated by the work [3], more and more scholars were devoted to studying variational inequalities for various operators. For the ρ-variation operators of the Calderón–Zygmund singular integrals and their commutators, we can consult [5, 18, 19] for the boundedness on the weighted Lebesgue spaces, [15, 28] for the boundedness on the weighted Morrey spaces, [15] for the boundedness on the Sobolev spaces and [27] for the boundedness and continuity on the Besov spaces. Recently, Wen, Wu and Zhang [23] established the boundedness of ρ-variation operators of the θ-type Calderón–Zygmund singular integrals on the weighted Lebesgue spaces. More precisely, it follows from [23, Theorem 1.1] that:

Theorem A ([23]). *Let $\rho > 2$, K be a θ-type Calderón–Zygmund kernel and $\mathcal{V}_\rho(\mathcal{T}_K)$ be given as in (1.1). If $\mathcal{V}_\rho(\mathcal{T}_K)$ is of type (p_0, p_0) for some $p_0 \in (1, \infty)$, then $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^p(w)$ for all $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$.*

On the other hand, the Triebel–Lizorkin spaces contain many important function spaces, such as Lebesgue spaces, Hardy spaces, Sobolev spaces and Lipschitz spaces. Over the last several years, a considerable amount of attention has been given to study the boundedness for various operators on the above function spaces. For examples, see [2, 4, 6] for singular integrals, [17, 26] for maximal singular integrals, [2, 24, 25] for Marcinkiewicz integrals and [10, 16] for maximal operators. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty\, (p \neq \infty)$. The homogeneous Triebel–Lizorkin spaces $\dot{F}_s^{p,q}(\mathbb{R}^n)$ are defined by

$$\dot{F}_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} < \infty \Big\}, \tag{1.3}$$

where

$$\|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} = \Big\| \Big(\sum_{i\in\mathbb{Z}} 2^{-isq} |\Psi_i * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)},$$

$\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on $\mathbb{R}^n$, $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfies the conditions: $0 \leq \phi(x) \leq 1$; $\mathrm{supp}(\phi) \subset \{x : 1/2 \leq |x| \leq 2\}$; $\phi(x) > c > 0$ if $3/5 \leq |x| \leq 5/3$. The inhomogeneous versions of Triebel–Lizorkin spaces denoted by $F_\alpha^{p,q}(\mathbb{R}^n)$ are obtained by adding the term $\|\Phi * f\|_{L^p(\mathbb{R}^n)}$ to the right hand side of (1.3) with $\sum_{i\in\mathbb{Z}}$ replaced by $\sum_{i\geq 1}$, where $\Phi \in \mathcal{S}(\mathbb{R}^n)$ (the space of Schwartz functions), $\mathrm{supp}(\hat{\Phi}) \subset \{\xi : |\xi| \leq 2\}$, $\hat{\Phi}(x) > c > 0$ if $|x| \leq 5/3$. The following properties are well known (see [8,9,22], for example): for $1 < p,\ q < \infty$ and $\alpha > 0$,

$$\begin{aligned} &\dot{F}_0^{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n), \\ &F_s^{p,q}(\mathbb{R}^n) \sim \dot{F}_s^{p,q}(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n) \quad \text{and} \\ &\|f\|_{F_s^{p,q}(\mathbb{R}^n)} \simeq \|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{1.4}$$

It is natural to ask whether the variation operator $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on the Triebel–Lizorkin spaces. This is the main motivation of this paper. In this paper we shall establish the following result.

Theorem 1.1. *Let $\rho > 2$, K be a θ-type Calderón–Zygmund kernel and $\mathcal{V}_\rho(\mathcal{T}_K)$ be given as in (1.1). Assume that $K(x, y) = K(x - y)$ and $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^{p_0}(\mathbb{R}^n)$ for some $p_0 \in (1, \infty)$. Then for $0 < s < 1$ and $1 < p,\ q < \infty$, the map $\mathcal{V}_\rho(\mathcal{T}_K) : F_s^{p,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ is bounded and continuous.*

In order to establish the corresponding results for commutators, let us introduce the following definition. Let $0 < \gamma \leq 1$. The *homogeneous* Lipschitz space $Lip_\gamma(\mathbb{R}^n)$ is defined as

$$Lip_\gamma(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \text{ continuous} : \|f\|_{Lip_\gamma(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{Lip_\gamma(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f(x+h)-f(x)|}{|h|^\gamma} < \infty.$$

The *inhomogeneous* Lipschitz space $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ is given by

$$\mathrm{Lip}_\gamma(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \text{ continuous} : \|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{Lip_\gamma(\mathbb{R}^n)} < \infty.$$

The second result of this paper can be listed as follows:

Theorem 1.2. *Let $\rho > 2$, $m \geq 1$, $0 < \gamma \leq 1$ and $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$. Let $\mathcal{V}_\rho(\mathcal{T}_K)$ and $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})$ be defined as in* (1.1) *and* (1.2)*, respectively, where K is a θ-type Calderón–Zygmund kernel. Assume that $K(x,y) = K(x-y)$ and $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^{p_0}(\mathbb{R}^n)$ for some $p_0 \in (1,\infty)$. Then for any $0 < s < \gamma$ and $1 < p, q < \infty$, the map $\mathcal{V}_\rho(\mathcal{T}^m_{K,b}) : F^{p,q}_s(\mathbb{R}^n) \to F^{p,q}_s(\mathbb{R}^n)$ is bounded and continuous. Particularly, there exists a constant $C > 0$ independent of b such that*

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)\|_{F^{p,q}_s(\mathbb{R}^n)} \leq C\|b\|^m_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|f\|_{F^{p,q}_s(\mathbb{R}^n)}, \quad \forall f \in F^{p,q}_s(\mathbb{R}^n).$$

As applications of Theorems 1.1 and 1.2, we have:

Corollary 1.3. *Let $\rho > 2$. Assume that one of the following conditions holds*:

(i) *$n = 1$ and $\mathcal{T} = \mathcal{H}$;*
(ii) *$n = 1$ and $\mathcal{T} = \mathcal{R}^\pm$;*
(iii) *$\mathcal{T} = \mathcal{R}_j$, $1 \leq j \leq n$;*
(iv) *$\mathcal{T} = \mathcal{T}_\Omega$, where $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for some $\alpha > 0$.*

Then for any $0 < s < 1$ and $1 < p, q < \infty$, the map $\mathcal{V}_\rho(\mathcal{T}) : F^{p,q}_s(\mathbb{R}^n) \to F^{p,q}_s(\mathbb{R}^n)$ is bounded and continuous.

Corollary 1.4. *Let $m \geq 1$, $\rho > 2$, $0 < \gamma \leq 1$ and $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$. Assume that one of the following conditions holds*:

(i) *$n = 1$ and $\mathcal{T} = \mathcal{H}^m_b$;*
(ii) *$n = 1$ and $\mathcal{T} = \mathcal{R}^m_{\pm,b}$;*
(iii) *$\mathcal{T} = \mathcal{R}^m_{j,b}$, $1 \leq j \leq n$;*
(iv) *$\mathcal{T} = \mathcal{T}^m_{\Omega,b}$, where $\Omega \in \mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for some $\alpha > 0$.*

Then for $0 < s < \gamma$ and $1 < p, q < \infty$, the map $\mathcal{V}_\rho(\mathcal{T}) : F^{p,q}_s(\mathbb{R}^n) \to F^{p,q}_s(\mathbb{R}^n)$ is bounded and continuous. Moreover,

$$\|\mathcal{V}_\rho(\mathcal{T})(f)\|_{F^{p,q}_s(\mathbb{R}^n)} \leq C\|b\|^m_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|f\|_{F^{p,q}_s(\mathbb{R}^n)}, \quad \forall f \in F^{p,q}_s(\mathbb{R}^n).$$

Throughout this paper, we always assume that $\rho > 2$ since the ρ-variation in the case $\rho \leq 2$ is often not bounded (see [1, 3]). The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same at each occurrence but independent of the essential variables. For a cube Q and a function f defined on $\mathbb{R}^n$, we set

$$f_Q = \frac{1}{|Q|} \int_Q f(x) dx.$$

In what follows, we denote by M_{HL} the centered Hardy–Littlewood maximal operator defined on $\mathbb{R}^n$. We set $\mathfrak{R}_n = \{\xi \in \mathbb{R}^n : 1/2 < |\xi| \leq 1\}$. For an arbitrary function f defined on $\mathbb{R}^n$ and $x, \zeta \in \mathbb{R}^n$, we denote $f(x+\zeta) = f_\zeta(x)$. We denote by Δ_ζ the difference of f, i.e., $\Delta_\zeta f(x) = f_\zeta(x) - f(x)$.

2. Preliminaries

2.1. Weights

A weight is a nonnegative, locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. For $1 < p < \infty$, a weight w is said to be in the Muckenhoupt weight class $A_p(\mathbb{R}^n)$ if there exists a positive constant C such that

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \Big(\frac{1}{|Q|} \int_Q w(x) dx \Big) \Big(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \Big)^{p-1} \leq C. \tag{2.1}$$

The smallest constant C in inequality (2.1) is the corresponding A_p constant of w, which is denoted by $[w]_{A_p}$. A weight w is said to be in the Muckenhoupt weight class $A_1(\mathbb{R}^n)$ if

$$M_{HL}(w)(x) \leq C w(x)$$

for almost all $x \in \mathbb{R}^n$, where the smallest constant C is denoted by $[w]_{A_1}$. A weight w is said to be in the Muckenhoupt weight class $A_\infty(\mathbb{R}^n)$ if

$$[w]_{A_\infty} := \sup_{Q \text{ cubes in } \mathbb{R}^n} \frac{1}{w(Q)} \int_Q M_{HL}(w\chi_Q)(x) dx < \infty.$$

It was known that $A_\infty(\mathbb{R}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n)$.

2.2. Sparse family

We now introduce some facts about sparse family, which follows from [23]. Given a cube $Q \subset \mathbb{R}^n$, let $\mathcal{D}(Q)$ be the set of cubes obtained by repeatedly subdividing Q and its descendants into 2^n congruent subcubes. A collection of cubes $\mathcal{D}$ is said to be a dyadic lattice if it satisfies the following properties:

(a) if $Q \in \mathcal{D}$, then every child of Q is also in $\mathcal{D}$;

(b) for every two cubes Q_1, $Q_2 \in \mathcal{D}$, there is a common ancestor $Q \in \mathcal{D}$ such that Q_1, $Q_2 \in \mathcal{D}(Q)$;

(c) for any compact set $K \subset \mathbb{R}^n$, there is a cube $Q \in \mathcal{D}$ such that $K \subset Q$.

A subset $\mathcal{S} \subset \mathcal{D}$ is said to be an η-sparse family with $\eta \in (0,1)$ if for every cube $Q \in \mathcal{S}$, there is a measurable subset $E_Q \subset Q$ such that $\eta|Q| \leq |E_Q|$, and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are mutually disjoint.

The following sparse domination is the main ingredient of proving Theorem 1.1, which follows from [23].

Lemma 2.1 ([23]). *Let $\rho > 2$, K be a θ-type Calderón–Zygmund kernel and $\mathcal{V}_\rho(\mathcal{T}_K)$ be defined in* (1.1). *If $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^{p_0}(\mathbb{R}^n)$ for some $p_0 \in (1,\infty)$, there exist 3^n dyadic lattices $\mathcal{D}^j$ and $\frac{1}{2\cdot 9^n}$-sparse families $\mathcal{S}_j \subset \mathcal{D}^j$ such that for almost every $x \in \mathbb{R}^n$,*

$$\mathcal{V}_\rho(\mathcal{T}_K f)(x) \leq C\sum_{j=1}^{3^n}\sum_{Q\in\mathcal{S}_j}|f|_Q\chi_Q(x).$$

2.3. Some vector-valued inequalities

To proving Theorem 1.1, the following vector-valued inequalities are needed.

Lemma 2.2 ([24]). *For any $1 < p,\, q < \infty$ and $1 \leq r < \min\{p,q\}$, we have*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\|M_{HL}(f_{k,\zeta})\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \leq C\Big\|\Big(\sum_{k\in\mathbb{Z}}\|f_{k,\zeta}\|^q_{L^r(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

Lemma 2.3 ([7]). *Given a family $\mathcal{F}$, suppose that for some $p_0 \in (0,\infty)$ and every $w \in A_\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} f(x)^{p_0}w(x)dx \leq c\int_{\mathbb{R}^n} g(x)^{p_0}w(x)dx$$

for all $(f,g) \in \mathcal{F}$ such that the left hand side is finite, and where $c > 0$ depends only on the $A_\infty(\mathbb{R}^n)$ constant of w. Then for all $0 < p,\, q < \infty$,

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}(f_j)^q\Big)^{1/q}\Big\|_{L^p(w)} \leq C\Big\|\Big(\sum_{j\in\mathbb{Z}}(g_j)^q\Big)^{1/q}\Big\|_{L^p(w)},\ \forall\{(f_j,g_j)\}_{j\in\mathbb{Z}} \subset \mathcal{F}.$$

Applying Lemmas 2.1–2.3, one can get the following vector-valued inequalities for the ρ-variation operator of θ-type Calderón–Zygmund singular integrals, which is the main ingredient of proving Theorem 1.1.

Proposition 2.4. *Let $\rho > 2$, K be a θ-type Calderón–Zygmund kernel and $\mathcal{V}_\rho(\mathcal{T}_K)$ be defined in* (1.1). *If $\mathcal{V}_\rho(\mathcal{T}_K)$ is bounded on $L^{p_0}(\mathbb{R}^n)$ for some $p_0 \in (1,\infty)$, then for $1 < p,\, q < \infty$, we have*

$$\begin{aligned}\Big\|\Big(\sum_{k\in\mathbb{Z}}\|\mathcal{V}_\rho(\mathcal{T}_K)(f_{k,\zeta})\|^q_{L^1(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}&\\ \leq C\Big\|\Big(\sum_{k\in\mathbb{Z}}\|f_{k,\zeta}\|^q_{L^1(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}&\end{aligned} \tag{2.2}$$

for all $\{f_{k,\zeta}(\cdot)\}_{k\in\mathbb{Z}} \in L^p(\ell^q(L^1(\mathfrak{R}_n)),\mathbb{R}^n)$.

Proof. Firstly we shall prove

$$\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}\mathcal{V}_\rho(\mathcal{T}_K)(f_\zeta)(x)d\zeta w(x)dx \le C\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}M_{HL}(f_\zeta)(x)d\zeta w(x)dx. \tag{2.3}$$

Fix $1 \le j \le 3^n$, noting that $\mathcal{S}_j$ is a $\frac{1}{2\cdot 9^n}$-sparse family, then for any $Q \in \mathcal{S}_j$, we have that $|Q \setminus E_Q| \le (1-\frac{1}{2\cdot 9^n})|Q|$. Let $w \in A_\infty(\mathbb{R}^n)$. Then there exists $\beta \in (0,1)$ such that $w(Q \setminus E_Q) \le \beta w(Q)$. This yields that

$$w(E_Q) = w(Q) - w(Q \setminus E_Q) \ge (1-\beta)w(Q).$$

Invoking Lemma 2.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}\mathcal{V}_\rho(\mathcal{T}_K)(f_\zeta)(x)d\zeta w(x)dx\\
&\le C\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}\sum_{j=1}^{3^n}\sum_{Q\in S_j}|f_\zeta|_Q(x)\chi_Q(x)d\zeta w(x)dx\\
&\le C\sum_{j=1}^{3^n}\sum_{Q\in S_j}\int_{\mathfrak{R}_n}\inf_{x\in Q}M_{HL}(f_\zeta)(x)d\zeta w(Q)\\
&\le C\sum_{j=1}^{3^n}\sum_{Q\in S_j}\int_{\mathfrak{R}_n}\inf_{x\in Q}M_{HL}(f_\zeta)(x)d\zeta w(E_Q)\\
&\le C\sum_{j=1}^{3^n}\sum_{Q\in S_j}\int_{E_Q}\int_{\mathfrak{R}_n}M_{HL}(f_\zeta)(x)d\zeta w(x)dx\\
&\le C\int_{\mathbb{R}^n}\int_{\mathfrak{R}_n}M_{HL}(f_\zeta)(x)d\zeta w(x)dx.
\end{aligned}$$

This proves (2.3).

For $R > 0$, we set

$$F_R(f_\zeta)(x) = \min\Big\{\int_{\mathfrak{R}_n}\mathcal{V}_\rho(\mathcal{T}_K)(f_\zeta)(x)d\zeta, R\Big\}\chi_{B(0,R)}(x).$$

Note that $\|F_R(f_\zeta)\|_{L^1(w)} \le Rw(B(0,R)) < \infty$, applying Lemma 2.3 and (2.3), we obtain

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}(F_R(f_{k,\zeta}))^q\Big)^{1/q}\Big\|_{L^p(w)} \le C\Big\|\Big(\sum_{k\in\mathbb{Z}}\|M_{HL}(f_{k,\zeta})\|^q_{L^1(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(w)}.$$

From this and Lemma 2.2, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}\|\mathcal{V}_\rho(\mathcal{T}_K)(f_{k,\zeta})\|^q_{L^1(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(w)}\\
&= \Big\|\Big(\sum_{k\in\mathbb{Z}}(\lim_{R\to\infty}F_R(f_{k,\zeta}))^q\Big)^{1/q}\Big\|_{L^p(w)}
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \Big(\lim_{R\to\infty} \sum_{k\in\mathbb{Z}} (F_R(f_{k,\zeta}))^q \Big)^{1/q} \Big\|_{L^p(w)} \\
&\le \liminf_{R\to\infty} \Big\| \Big(\sum_{k\in\mathbb{Z}} (F_R(f_{k,\zeta}))^q \Big)^{1/q} \Big\|_{L^p(w)} \\
&\le C \Big\| \Big(\sum_{k\in\mathbb{Z}} \|M_{HL}(f_{k,\zeta})\|^q_{L^1(\mathfrak{R}_n)} \Big)^{1/q} \Big\|_{L^p(w)} \\
&\le C \Big\| \Big(\sum_{k\in\mathbb{Z}} \|f_{k,\zeta}\|^q_{L^1(\mathfrak{R}_n)} \Big)^{1/q} \Big\|_{L^p(w)},
\end{aligned}
$$

which gives (2.2) by taking $w \equiv 1$. □

2.4. A criterion

We now end this section by presenting a criterion of continuity for several sublinear operators on the Triebel–Lizorkin spaces.

Proposition 2.5. ([14]). *Assume that T is a sublinear operator and the following conditions hold*:

(i) $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *for some* $p \in (1,\infty)$;

(ii) *For all* x, $\zeta \in \mathbb{R}^n$, *it holds that*

$$|\Delta_\zeta(Tf)(x)| \le |T(\Delta_\zeta(f))(x)|;$$

(iii) *There exist* $\alpha \in (0,1)$ *and* $q \in (1,\infty)$ *such that*

$$\Big\| \Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha} \Big(\int_{\mathfrak{R}_n} |T(\Delta_{2^{-l}\zeta} f)| d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{\dot F^{p,q}_\alpha(\mathbb{R}^n)}.$$

Then T is continuous from $F^{p,q}_s(\mathbb{R}^n)$ to $\dot F^{p,q}_s(\mathbb{R}^n)$.

3. Proofs of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2. Before presenting our proofs, let us introduce some properties for the Triebel–Lizorkin spaces, which play key roles in the main proofs. Let $0 < s < 1$, $1 < p, q < \infty$ and $1 \le r < \min(p,q)$. For a measurable function $g : \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n \to \mathbb{R}$, we define

$$\|g\|_{p,q,r} := \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} |g(x,k,\zeta)|^r d\zeta \Big)^{q/r} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}.$$

In [24], Yabuta observed that if $0 < s < 1$, $1 < p < \infty$, $1 < q \le \infty$ and $1 \le r < \min(p,q)$, then

$$\|f\|_{\dot F^{p,q}_s(\mathbb{R}^n)} \simeq \|\Delta_{2^{-k}\zeta} f\|_{p,q,r}. \tag{3.1}$$

Proof of Theorem 1.1. For any $x,\, h\in\mathbb{R}^n$, it is clear that $\mathcal{V}_\rho(\mathcal{T}_K)(f)(x+h)=\mathcal{V}_\rho(\mathcal{T}_K)(f_h)(x)$. By the sublinearity of $\mathcal{V}_\rho(\mathcal{T}_K)$, one has

$$\begin{aligned}|\Delta_\zeta(\mathcal{V}_\rho(\mathcal{T}_K)(f))(x)|&=|\mathcal{V}_\rho(\mathcal{T}_K)(f)(x+h)-\mathcal{V}_\rho(\mathcal{T}_K)(f)(x)|\\&\le|\mathcal{V}_\rho(\mathcal{T}_K)(\Delta_h(f))(x)|.\end{aligned}\tag{3.2}$$

By Proposition 2.4, we get from (3.1) and (3.2) that

$$\begin{aligned}&\|\mathcal{V}_\rho(\mathcal{T}_K)(f)\|_{\dot{F}^{p,q}_s(\mathbb{R}^n)}\\&\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}_K)(f))|^r d\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_n}|\mathcal{V}_\rho(\mathcal{T}_K)(2^{ks}\Delta_{2^{-k}\zeta}(f))|^r d\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}f|^r d\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\le C\|f\|_{\dot{F}^{p,q}_s(\mathbb{R}^n)}.\end{aligned}\tag{3.3}$$

Combining (3.3) with Theorem A and (1.4) yield the boundedness of $\mathcal{V}_\rho(\mathcal{T}_K)$ on $F^{p,q}_s(\mathbb{R}^n)$. By Propositions 2.4 and 2.5, Theorem A, (1.4), (3.2) and (3.3), one can get the continuity of $\mathcal{V}_\rho(\mathcal{T}_K):F^{p,q}_s(\mathbb{R}^n)\to\dot{F}^{p,q}_s(\mathbb{R}^n)$. This together with the L^p continuity for $\mathcal{V}_\rho(\mathcal{T}_K)$ leads to the continuity of $\mathcal{V}_\rho(\mathcal{T}_K)$ on $F^{p,q}_s(\mathbb{R}^n)$. $\square$

Proof of Theorem 1.2. The proof of Theorem 1.2 will be divided into two steps:

Step 1. Proof of the boundedness part. It was shown in [15] (see [15, (5.16)]) that

$$\begin{aligned}|\Delta_h((\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f))(x)|&\le\sum_{l=0}^m c^l_m|b^{m-l}_h(x)|\mathcal{V}_\rho(\mathcal{T}_K)(b^l_h\Delta_h f)(x)\\&\quad+\sum_{l=1}^m c^l_m\sum_{\ell=0}^l c^\ell_l|\Delta_h b(x)|^\ell\sum_{\mu=0}^{m-l}c^\mu_{m-l}|b^{m-l-\mu}(x)|\\&\quad\times\mathcal{V}_\rho(\mathcal{T}_K)(b^\mu(\Delta_h b)^{l-\ell}f)(x)\\&=:\mathcal{G}(f)(h,x)\end{aligned}\tag{3.4}$$

for any $x,\, h\in\mathbb{R}^n$. Here $C^r_N=\frac{N!}{r!(N-r)!}$ for any $r,\, N\in\mathbb{N}$ with $r\le N$. By (3.1), (3.4) and Minkowski's inequality, we have

$$\begin{aligned}&\|\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)\|_{\dot{F}^{p,q}_s(\mathbb{R}^n)}\\&\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f))|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\&\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathcal{G}(f)(2^{-k}\zeta,x)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\end{aligned}\tag{3.5}$$

$$
\begin{aligned}
&\leq C \sum_{l=0}^{m} c_m^l \\
&\quad \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} |b_{2^{-k}\zeta}^{m-l}| \mathcal{V}_\rho(\mathcal{T}_K)(b_{2^{-k}\zeta}^{l} \Delta_{2^{-k}\zeta} f) d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
&\quad + \sum_{l=1}^{m} c_m^l \sum_{\ell=0}^{l} c_l^\ell \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} b|^\ell \\
&\quad \times \sum_{\mu=0}^{m-l} c_{m-l}^\mu |b^{m-l-\mu}| \mathcal{V}_\rho(\mathcal{T}_K)(b^\mu (\Delta_{2^{-k}\zeta} b)^{l-\ell} f) d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
&=: A_1 + A_2.
\end{aligned}
$$

By Proposition 2.4 and (3.1), one has

$$
\begin{aligned}
A_1 \leq\ & \sum_{l=0}^{m} c_m^l \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l} \\
& \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathcal{V}_\rho(\mathcal{T}_K)(b_{2^{-k}\zeta}^{l} \Delta_{2^{-k}\zeta} f) d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
=\ & \sum_{l=0}^{m} c_m^l \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l} \\
& \times \Big\| \Big(\sum_{k\in\mathbb{Z}} \Big(\int_{\mathfrak{R}_n} \mathcal{V}_\rho(\mathcal{T}_K)(2^{ks} b_{2^{-k}\zeta}^{l} \Delta_{2^{-k}\zeta} f) d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
\leq\ & C \sum_{l=0}^{m} c_m^l \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l} \\
& \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} |b_{2^{-k}\zeta}^{l} \Delta_{2^{-k}\zeta} f)| d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
\leq\ & C \sum_{l=0}^{m} c_m^l \|b\|_{L^\infty(\mathbb{R}^n)}^{m} \\
& \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} f)| d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\
\leq\ & C \|b\|_{L^\infty(\mathbb{R}^n)}^{m} \|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)}.
\end{aligned}
\tag{3.6}
$$

By the L^p bounds for $\mathcal{V}_\rho(\mathcal{T}_K)$,

$$
\begin{aligned}
A_2 \leq\ & \sum_{l=1}^{m} c_m^l \sum_{\mu=0}^{m-l} c_{m-l}^\mu \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l-\mu} \\
& \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathcal{V}_\rho(\mathcal{T}_K)(b^\mu (\Delta_{2^{-k}\zeta} b)^l f) d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}
\end{aligned}
\tag{3.7}
$$

$$+\sum_{l=1}^{m} c_m^l \sum_{\ell=1}^{l} c_l^\ell \sum_{\mu=0}^{m-l} c_{m-l}^\mu \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l-\mu}$$
$$\times \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} b|^\ell \mathcal{V}_\rho(\mathcal{T}_K)(b^\mu(\Delta_{2^{-k}\zeta} b)^{l-\ell} f)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$=: A_{2,1} + A_{2,2}.$$

Fix $1 \le l \le m$ and $0 \le \mu \le m-l$, noting that $s < \gamma$, we then use Proposition 2.4 to obtain that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathcal{V}_\rho(\mathcal{T}_K)(b^\mu(\Delta_{2^{-k}\zeta} b)^l f)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$= \Big\|\Big(\sum_{k\in\mathbb{Z}} \Big(\int_{\mathfrak{R}_n} \mathcal{V}_\rho(\mathcal{T}_K)(2^{ks} b^\mu(\Delta_{2^{-k}\zeta} b)^l f)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\le C\|b\|_{L^\infty(\mathbb{R}^n)}^\mu \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |(\Delta_{2^{-k}\zeta} b)^l f|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\le C\|b\|_{L^\infty(\mathbb{R}^n)}^\mu |\mathfrak{R}_n|$$
$$\times \Big\|\Big(\sum_{k=1}^{\infty} 2^{k(s-l\gamma)q}\|b\|_{Lip_\gamma(\mathbb{R}^n)}^{lq} + 2^{lq}\|b\|_{L^\infty(\mathbb{R}^n)}^{lq} \sum_{k=-\infty}^{0} 2^{ksq}\Big)^{1/q} f\Big\|_{L^p(\mathbb{R}^n)}$$
$$\le C\|b\|_{L^\infty(\mathbb{R}^n)}^\mu \|b\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}^l \|f\|_{L^p(\mathbb{R}^n)}.$$

It follows that

$$A_{2,1} \le C\|b\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}^m \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.8}$$

For $A_{2,2}$, we write

$$A_{2,2} \le \sum_{l=1}^{m} c_m^l \sum_{\ell=1}^{l-1} c_l^\ell \sum_{\mu=0}^{m-l} c_{m-l}^\mu \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l-\mu}$$
$$\times \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} b|^\ell$$
$$\times \mathcal{V}_\rho(\mathcal{T}_K)(b^\mu(\Delta_{2^{-k}\zeta} b)^{l-\ell} f)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$+\sum_{l=1}^{m} c_m^l \sum_{\mu=0}^{m-l} c_{m-l}^\mu \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l-\mu} \tag{3.9}$$
$$\times \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} b|^l \mathcal{V}_\rho(\mathcal{T}_K)(b^\mu f)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}$$
$$\le 2\sum_{l=1}^{m} c_m^l \sum_{\ell=1}^{l-1} c_l^\ell \sum_{\mu=0}^{m-l} c_{m-l}^\mu \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l-\mu+\ell}$$

$$
\begin{aligned}
&\times \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathcal{V}_\rho(\mathcal{T}_K)(b^\mu(\Delta_{2^{-k}\zeta}b)^{l-\ell}f)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&+\sum_{l=1}^{m} c_m^l \sum_{\mu=0}^{m-l} c_{m-l}^\mu \|b\|_{L^\infty(\mathbb{R}^n)}^{m-l-\mu}\\
&\times \Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}b|^l\zeta\Big)^q\Big)^{1/q}\mathcal{V}_\rho(\mathcal{T}_K)(b^\mu f)\Big\|_{L^p(\mathbb{R}^n)}\\
=:\;& A_{2,2,1}+A_{2,2,2}.
\end{aligned}
$$

For $1\le l\le m$, $1\le \ell\le l-1$ and $0\le\mu\le m-1$, we get by Proposition 2.4 that

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathcal{V}_\rho(\mathcal{T}_K)(b^\mu(\Delta_{2^{-k}\zeta}b)^{l-\ell}f)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&=\Big\|\Big(\sum_{k\in\mathbb{Z}} \Big(\int_{\mathfrak{R}_n} \mathcal{V}_\rho(\mathcal{T}_K)(2^{ks}b^\mu(\Delta_{2^{-k}\zeta}b)^{l-\ell}f)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |b^\mu(\Delta_{2^{-k}\zeta}b)^{l-\ell}f)|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\le C\|b\|_{L^\infty(\mathbb{R}^n)}^\mu\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |(\Delta_{2^{-k}\zeta}b)^{l-\ell}|d\zeta\Big)^q\Big)^{1/q}f\Big\|_{L^p(\mathbb{R}^n)}\\
&\le C\|b\|_{L^\infty(\mathbb{R}^n)}^\mu|\mathfrak{R}_n|\|f\|_{L^p(\mathbb{R}^n)}\\
&\quad\times\Big(\Big(\sum_{k=1}^{\infty} 2^{k(s-(l-\ell)\gamma)q}\Big)^{1/q}\|b\|_{Lip_\gamma(\mathbb{R}^n)}^{l-\ell}+\Big(\sum_{k-\infty}^{0} 2^{ksq}\Big)^{1/q}(2\|b\|_{L^\infty(\mathbb{R}^n)})^{l-\ell}\Big)\\
&\le C\|b\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}^{\mu+l-\ell}\|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}
$$

where in the last inequality of the above inequalities we have used the fact that $s<\gamma$ and $1\le l-\ell$. Hence, we get

$$
A_{2,2,1}\le C\|b\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}^m\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.10}
$$

By the L^p boundedness for $\mathcal{V}_\rho(\mathcal{T}_K)$, one has

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}b|^l\zeta\Big)^q\Big)^{1/q}\mathcal{V}_\rho(\mathcal{T}_K)(b^\mu f)\Big\|_{L^p(\mathbb{R}^n)}\\
&\le|\mathfrak{R}_n|\Big(\Big(\sum_{k=1}^{\infty} 2^{k(s-l\gamma)q}\Big)^{1/q}\|b\|_{Lip_\gamma(\mathbb{R}^n)}^{l}+\Big(\sum_{k-\infty}^{0} 2^{ksq}\Big)^{1/q}(2\|b\|_{L^\infty(\mathbb{R}^n)})^{l}\Big)\\
&\quad\times\|\mathcal{V}_\rho(\mathcal{T}_K)(b^\mu f)\Big\|_{L^p(\mathbb{R}^n)}\\
&\le C\|b\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}^l\|b^\mu f\|_{L^p(\mathbb{R}^n)}\le C\|b\|_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}^{l+\mu}\|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

Consequently

$$A_{2,2,2} \le C\|b\|^m_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.11}$$

We get from (3.9)–(3.11) that

$$A_{2,2} \le C\|b\|^m_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.12}$$

Combining (3.12) with (3.7) and (3.8) implies that

$$A_2 \le C\|b\|^m_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.13}$$

It follows from (3.5), (3.6), (3.13) and (1.4) that

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)\|_{\dot{F}^{p,q}_s(\mathbb{R}^n)} \le C\|b\|^m_{\mathrm{Lip}_\gamma(\mathbb{R}^n)}\|f\|_{F^{p,q}_s(\mathbb{R}^n)}. \tag{3.14}$$

On the other hand, by the arguments similar to those used to derive [15, (5.10)], one has

$$\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)(x) \le \sum_{k=0}^m c^k_m|b^{m-k}(x)|\mathcal{V}_\rho(\mathcal{T}_K)(b^k f)(x) \tag{3.15}$$

for all $x \in \mathbb{R}^n$. Using (3.15), the L^p bounds for $\mathcal{V}_\rho(\mathcal{T}_K)$ and Minkowski's inequality, we have

$$\begin{aligned}\|\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)\|_{L^p(\mathbb{R}^n)} &\le \sum_{k=0}^m c^k_m\|b\|^{m-k}_{L^\infty(\mathbb{R}^n)}\|\mathcal{V}_\rho(\mathcal{T}_K)(b^k f)\|_{L^p(\mathbb{R}^n)}\\ &\le C\sum_{k=0}^m c^k_m\|b\|^{m-k}_{L^\infty(\mathbb{R}^n)}\|b^k f\|_{L^p(\mathbb{R}^n)}\\ &\le C\|b\|^m_{L^\infty(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{3.16}$$

Combining (3.16) with (3.14) and (1.4) implies the desired boundedness part.

Step 2. Proof of the continuity part. Let $0 < s < 1$, $1 < p$, $q < \infty$ and $f_j \to f$ in $F^{p,q}_s(\mathbb{R}^n)$ as $j \to \infty$. We know from (1.4) that $f_j \to f$ in $\dot{F}^{p,q}_s(\mathbb{R}^n)$ and in $L^p(\mathbb{R}^n)$ as $j \to \infty$. By the sublinearity of $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})$ and (3.16), we have that $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j) \to \mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$. Hence, it is enough to conclude that

$$\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j) \to \mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f) \ \text{ in } \ \dot{F}^{p,q}_s(\mathbb{R}^n) \ \text{ as } \ j \to \infty. \tag{3.17}$$

Next we shall prove (3.17) by contradiction. Without loss of generality we may assume that there exists $c > 0$ such that

$$\|\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j) - \mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)\|_{\dot{F}^{p,q}_s(\mathbb{R}^n)} > c, \quad \forall j \ge 1. \tag{3.18}$$

Since $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j) \to \mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$, we may assume by extracting a subsequence that $\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j)(x) \to \mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f)$ as $j \to \infty$ for almost every $x \in \mathbb{R}^n$. Hence, $\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j) - \mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f))(x) \to 0$ as

$j \to \infty$ for every $(k, \zeta) \in \mathbb{Z} \times \mathfrak{R}_n$ and almost every $x \in \mathbb{R}^n$. By (3.4) and the sublinearity of $\mathcal{V}_\rho(\mathcal{T}_K)$, we have

$$\begin{aligned} &|\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j) - \mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f))(x)| \\ &\le |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j))(x)| + |\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f))(x)| \\ &\le \mathcal{G}(f_j)(2^{-k}\zeta, x) + \mathcal{G}(f)(2^{-k}\zeta, x) \\ &\le \mathcal{G}(f_j - f)(2^{-k}\zeta, x) + 2\mathcal{G}(f)(2^{-k}\zeta, x). \end{aligned} \tag{3.19}$$

From (3.14) we see that

$$\begin{aligned} &\Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} |\mathcal{G}(f_j - f)(2^{-k}\zeta, x)| d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\ &\le C\|b\|^m_{\mathrm{Lip}_\gamma(\mathbb{R}^n)} \|f_j - f\|_{F^{p,q}_s(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to \infty. \end{aligned} \tag{3.20}$$

Therefore, one can extract a subsequence such that

$$\sum_{j=1}^\infty \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} |\mathcal{G}(f_j - f)(2^{-k}\zeta, x)| d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty. \tag{3.21}$$

For $(k, \zeta, x) \in \mathbb{Z} \times \mathfrak{R}_n \times \mathbb{R}^n$, we set

$$\Gamma(k, \zeta, x) := \sum_{j=1}^\infty \mathcal{G}(f_j - f)(2^{-k}\zeta, x) + 2\mathcal{G}(f)(2^{-k}\zeta, x).$$

By (3.19), we have

$$|\Delta_{2^{-k}\zeta}(\mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f_j) - \mathcal{V}_\rho(\mathcal{T}^m_{K,b})(f))(x)| \le \Gamma(k, \zeta, x) \tag{3.22}$$

for $(k, \zeta, x) \in \mathbb{Z} \times \mathfrak{R}_n \times \mathbb{R}^n$. By (3.20), (3.21) and Minkowski's inequality, we get

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \int_{\mathbb{R}^n} (\Gamma(k, \zeta, x))^p dx d\zeta \Big)^{q/p} \Big)^{1/q} < \infty. \tag{3.23}$$

Using (3.22), (3.23) and the arguments similar to those used to derive Proposition 2.5, one can get a contradiction with (3.18). This completes the proof of Theorem 1.2. □

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