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ALTERNATIVE PROOF OF MARSAGLIA'S METHOD†

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Abstract. We derive an alternative proof of Marsaglia's method for generating a pair of independent standard normal random variables.

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1. Introduction

Standard normal random variables are frequently used in computer science, computational statistics, and in particular, in applications of the Monte Carlo method ([2].)

The Marsaglia's polar method ([3]) is a pseudo-random number sampling method for generating a pair of independent standard normal random variables. The Marsaglia's polar method is a modification of Box-Müller's method that uses the rejection method and it is superior to the Box–Müller's method.

The main objective of this paper is to provide an alternative proof of Marsaglia's method for generating a pair of independent standard normal random variables. However, while polar coordinates are used in Marsaglia's polar method in [3], we do use rectangular coordinates to derive a pair of independent standard normal random variables in this paper.

Let (X_1, X_2) be a random vector. Suppose we know the joint distribution of (X_1, X_2) and we seek the distribution of a transformation of (X_1, X_2).

Let (X_1, X_2) have a jointly continuous distribution with probability density function (pdf) $f_{X_1,X_2}(x_1, x_2)$ and support set $\mathcal{S}$. Suppose the random variables Y_1 and Y_2 are given by $Y_1 = u_1(X_1, X_2)$ and $Y_2 = u_2(X_1, X_2)$, where the functions $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ define a one-to-one transformation that maps the set $\mathcal{S}$ in $\mathbb{R}^2$ onto a (two dimensional) set $\mathcal{T}$ in $\mathbb{R}^2$, where $\mathcal{T}$ is the support of (Y_1, Y_2).

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If we express each of x_1 and x_2 in terms of y_1 and y_2, we can write $x_1 = w_1(y_1, y_2),\ x_2 = w_2(y_1, y_2)$. The determinant of order 2,

$$J = \left|\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \tag{1}$$

is called the *Jacobian* of the transformation and will be denoted by the symbol J. It will be assumed that these first-order partial derivatives are continuous and that the Jacobian J is not identically equal to zero in $\mathcal{T}$.

We can find, by use of a theorem in analysis, the joint probability density function of (Y_1, Y_2). Let A be a subset of $\mathcal{S}$, and let B denote the mapping of A under the one-to-one transformation. Because the transformation is one-to-one, the events $\{(X_1, X_2) \in A\}$ and $\{(Y_1, Y_2) \in B\}$ are equivalent. Hence,

$$P[(Y_1, Y_2) \in B] = P[(X_1, X_2) \in A] = \iint_A f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2.$$

We wish to change variables of integration by writing $y_1 = u_1(x_1, x_2),\ y_2 = u_2(x_1, x_2)$ or $x_1 = w_1(y_1, y_2),\ x_2 = w_2(y_1, y_2)$. It has been proven in analysis, that this change of variables requires

$$\iint_A f_{X_1,X_2}(x_1, x_2)\, dx_1 dx_2 = \iint_B f_{X_1,X_2}(w_1(y_1, y_2), w_2(y_1, y_2))|J|\, dy_1 dy_2.$$

Thus, for every set B in $\mathcal{T}$,

$$P[(Y_1, Y_2) \in B] = \iint_B f_{X_1,X_2}(w_1(y_1, y_2), w_2(y_1, y_2))|J|\, dy_1 dy_2,$$

which implies that the joint probability density function $f_{Y_1,Y_2}(y_1, y_2)$ is

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} f_{X_1,X_2}(w_1(y_1, y_2), w_2(y_1, y_2))|J|, & (y_1, y_2) \in \mathcal{T} \\ 0, & \text{elsewhere.} \end{cases} \tag{2}$$

The following theorem provides a criterion for independence of two random variables.

Theorem 1.1. ([1, 4]) *Let the random variables X_1 and X_2 have the joint probability density function $f_{X_1,X_2}(x_1, x_2)$. Then the random variables X_1 and X_2 are independent if and only if $f_{X_1,X_2}(x_1, x_2)$ can be written as a product of a nonnegative function of x_1 and a nonnegative function of x_2. That is,*

$$f_{X_1,X_2}(x_1, x_2) = g(x_1)h(x_2), \tag{3}$$

where $g(x_1) > 0$, $x_1 \in \mathcal{S}_1$, zero elsewhere, and $h(x_2) > 0$, $x_2 \in \mathcal{S}_2$, zero elsewhere.

2. Alternative proof

Let U be the uniform random variable on $(-1, 1)$, that is, the probability density function of U is

$$f_U(u) = \begin{cases} \dfrac{1}{2}, & -1 < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $X_1 = U$ and $X_2 = U$ be uniform random variables on $(-1, 1)$. Set

$$S = X_1^2 + X_2^2 \tag{4}$$

If $S < 1$, then (X_1, X_2) is uniformly distributed inside the unit circle. Thus, the joint probability density function of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \dfrac{1}{\pi}, & x_1^2 + x_2^2 < 1, \\ 0, & \text{elsewhere.} \end{cases} \tag{5}$$

From now on we use Z_1 and Z_2 instead of using Y_1 and Y_2, since the letter Z has been used to represent the standard normal random variable in Statistics.

Suppose the random variables Z_1 and Z_2 are given by

$$Z_1 = X_1\sqrt{\frac{-2\ln S}{S}}, \qquad Z_2 = X_2\sqrt{\frac{-2\ln S}{S}}.$$

We have the transformation from (x_1, x_2) to (z_1, z_2):

$$\begin{aligned} z_1 &= x_1\sqrt{\frac{-2\ln s}{s}}, \qquad -\infty < z_1 < \infty, \\ z_2 &= x_2\sqrt{\frac{-2\ln s}{s}}, \qquad -\infty < z_2 < \infty, \end{aligned}$$

where $s = x_1^2 + x_2^2$. The functions z_1 and z_2 define a one-to-one transformation that maps the set square onto the two dimensional real plane $\mathbb{R}^2$, where $\mathbb{R}^2$ is the support of (Z_1, Z_2). Then we have

$$\frac{z_1}{z_2} = \frac{x_1}{x_2},$$

which implies

$$x_1 = \frac{z_1}{z_2}x_2, \qquad x_2 = \frac{z_2}{z_1}x_1 \tag{6}$$

and we have

$$z_1^2 + z_2^2 = \left(x_1^2 + x_2^2\right)\frac{-2\ln s}{s} = -2\ln s = -2\ln(x_1^2 + x_2^2),$$

which implies

$$x_1^2 + x_2^2 = e^{-\dfrac{z_1^2 + z_2^2}{2}} \tag{7}$$

By solving (6) and (7) simultaneously for x_1 and x_2, we have the inverse transformation from (z_1, z_2) to (x_1, x_2):

$$x_1 = \frac{z_1}{\sqrt{z_1^2 + z_2^2}} e^{-\frac{z_1^2 + z_2^2}{4}}, \tag{8}$$

$$x_2 = \frac{z_2}{\sqrt{z_1^2 + z_2^2}} e^{-\frac{z_1^2 + z_2^2}{4}}. \tag{9}$$

To find the Jacobian we take partial derivatives of x_1 and x_2 with respect to z_1 and z_2:

$$\begin{aligned}
\frac{\partial x_1}{\partial z_1} &= e^{-\frac{z_1^2 + z_2^2}{4}} \frac{\partial}{\partial z_1}\left(\frac{z_1}{\sqrt{z_1^2 + z_2^2}}\right) + \frac{z_1}{\sqrt{z_1^2 + z_2^2}} \frac{\partial}{\partial z_1}\left(e^{-\frac{z_1^2 + z_2^2}{4}}\right) \\
&= e^{-\frac{z_1^2 + z_2^2}{4}} \left(\frac{\sqrt{z_1^2 + z_2^2} - z_1 \frac{z_1}{\sqrt{z_1^2 + z_2^2}}}{z_1^2 + z_2^2}\right) \\
&\quad + \frac{z_1}{\sqrt{z_1^2 + z_2^2}} \left(-\frac{z_1}{2}\right) \left(e^{-\frac{z_1^2 + z_2^2}{4}}\right) \\
&= e^{-\frac{z_1^2 + z_2^2}{4}} \frac{1}{2(z_1^2 + z_2^2)^{3/2}} \left[2z_2^2 - z_1^2(z_1^2 + z_2^2)\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x_1}{\partial z_2} &= e^{-\frac{z_1^2 + z_2^2}{4}} \frac{\partial}{\partial z_2}\left(\frac{z_1}{\sqrt{z_1^2 + z_2^2}}\right) + \frac{z_1}{\sqrt{z_1^2 + z_2^2}} \frac{\partial}{\partial z_2}\left(e^{-\frac{z_1^2 + z_2^2}{4}}\right) \\
&= e^{-\frac{z_1^2 + z_2^2}{4}} \frac{1}{2(z_1^2 + z_2^2)^{3/2}} \left[-z_1 z_2(2 + z_1^2 + z_2^2)\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x_2}{\partial z_1} &= e^{-\frac{z_1^2 + z_2^2}{4}} \frac{\partial}{\partial z_1}\left(\frac{z_2}{\sqrt{z_1^2 + z_2^2}}\right) + \frac{z_2}{\sqrt{z_1^2 + z_2^2}} \frac{\partial}{\partial z_2}\left(e^{-\frac{z_1^2 + z_2^2}{4}}\right) \\
&= e^{-\frac{z_1^2 + z_2^2}{4}} \frac{1}{2(z_1^2 + z_2^2)^{3/2}} \left[-z_1 z_2(2 + z_1^2 + z_2^2)\right]
\end{aligned}$$

$$\begin{aligned}\frac{\partial x_2}{\partial z_2} &= e^{-\dfrac{z_1^2+z_2^2}{4}} \frac{\partial}{\partial z_2}\left(\frac{z_2}{\sqrt{z_1^2+z_2^2}}\right) + \frac{z_2}{\sqrt{z_1^2+z_2^2}}\frac{\partial}{\partial z_2}\left(e^{-\dfrac{z_1^2+z_2^2}{4}}\right) \\ &= e^{-\dfrac{z_1^2+z_2^2}{4}} \frac{1}{2(z_1^2+z_2^2)^{3/2}}\left[2z_1^2 - z_2^2(z_1^2+z_2^2)\right]\end{aligned}$$

By (1) we have the Jacobian determinant

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2}\end{vmatrix} \\ &= \frac{e^{-\dfrac{z_1^2+z_2^2}{2}}}{4(z_1^2+z_2^2)^3}\begin{vmatrix} 2z_2^2 - z_1^2(z_1^2+z_2^2) & -z_1z_2(2+z_1^2+z_2^2) \\ -z_1z_2(2+z_1^2+z_2^2) & 2z_1^2 - z_2^2(z_1^2+z_2^2)\end{vmatrix} \\ &= \frac{e^{-\dfrac{z_1^2+z_2^2}{2}}}{4(z_1^2+z_2^2)^3}\left\{4z_1^2z_2^2 - 2z_1^4(z_1^2+z_2^2) - 2z_2^4(z_1^2+z_2^2)\right. \\ &\quad \left. + z_1^2z_2^2(z_1^2+z_2^2)^2 - z_1^2z_2^2(4+4(z_1^2+z_2^2)+(z_1^2+z_2^2)^2)\right\} \\ &= \frac{e^{-\dfrac{z_1^2+z_2^2}{2}}}{4(z_1^2+z_2^2)^2}(-2z_1^4 - 2z_2^4 - 4z_1^2z_2^2) \\ &= \frac{e^{-\dfrac{z_1^2+z_2^2}{2}}}{4(z_1^2+z_2^2)^2}(-2)(z_1^2+z_2^2)^2 \\ &= -\frac{1}{2}e^{-\dfrac{z_1^2+z_2^2}{2}}\end{aligned}$$

Thus, the absolute value of the Jacobian determinant J is

$$|J| = \frac{1}{2}e^{-\dfrac{z_1^2+z_2^2}{2}} \tag{10}$$

Therefore, by (2), (5), and (10), the joint probability density function of Z_1 and Z_2 is

$$\begin{aligned} f_{Z_1,Z_2}(z_1,z_2) &= f_{X_1,X_2}(x_1,x_2)|J| \\ &= \frac{1}{\pi}\frac{1}{2}e^{-\dfrac{z_1^2+z_2^2}{2}}\end{aligned}$$

$$= \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{z_1^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{z_2^2}{2}}\right)$$

By Theorem 1.1, Z_1 and Z_2 are independent.

Hence, (Z_1, Z_2) is a pair of independent standard normal random variables. An alternative proof is completed. □

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