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RISK-MINIMIZING HEDGING FOR A SPECIAL CONTINGENTS†

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ABSTRACT. In this paper, we consider a risk-minimization hedging problem for a special European contingent claims. The existence and uniqueness of strategy are given constructively. Firstly, a non-standard European contingent is demonstrated as stochastic payment streams. Then the existence of the risk minimization strategy and also the uniqueness are proved under two kinds market information by using Galtchouk-Kunita-Watanabe decomposition and constructing a 0-achieving strategy risk-minimizing strategies in full information. And further, we have proven risk-minimizing strategies exists and is unique under restrict information by constructing a weakly mean-selffinancing strategy..

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1. Introduction

Taking conditional Mean square error process of the cost of the investment portfolio as a risk measurement is introduced by Föllmer,Sondermann [1]in 1986,. Since then, the risk-minimizing and local risk-minimizing became one of the most popular standard for the pricing and hedging. [2]-[6] studied risk-minimizing and local risk-minimizing for a T-contingent claim. The risk-minimizing for reinsurance contracts in diffusion approximation and equity-indexed annuity under Markov regime switching model are considered respectively in [7] and [8]. However after carefully checking the existing literature, most of them are for hedging standard European contingent under complete information. Few of them involve incomplete information or the non-standard European contingent. In fact in the financial and insurance market, non-standard European contingent claims and

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incomplete information can be seen everywhere. Life insurance contract holders or the investors who have several European contingent claims with different expiration date be sure to face hedging a non-standard European contingent. In actual financial markets there do exist some investors who can only know partial information for its own conditions limitations(for example the remote investors can not get and understand the national investment policy, some construction planning and related information in time while the general investor do know; Some of the listed company's financial disclosure information once several months,for example three months. It make investors's market information lag, and cause market information acquisition not congruent). They may just know partial market information (such as only prices of risky assets information). This has caused investors incomplete information. The previous two kind of problem has caused the attention of researchers. In complete information, [4],[5],[9] study respectively risk and local minimal risk of strategy for the insurance compensation contingent claims. [10] discuss dynamic hedging of counter-party risk for a portfolio of credit derivatives by the local risk-minimization approach and recover a closed-form representation for the locally risk minimizing strategy in terms of classical solutions to nonlinear recursive systems of Cauchy problems. [2]and [11] study how to hedge European with incomplete information. This paper also studies risk-minimizing hedging strategy. Compare with [2]and [11], we discuss hedging contingent claims with stochastic payment stream under incomplete information. Relative to the standard European contingent claims, with random pay flow payoff of the contingent claims can happen at any point in $[0, T]$. And thus hedging the European contingent claims is more complex. In the paper we first assume that there are two investors with different market information (complete information and incomplete information). By Galtchouk-Kunita-Watanabe decomposition and projection theorem in L^2 space, we show that risk minimization hedging strategies for investors with incomplete information exists and is unique. Furthermore, the constructing methods for optimal strategies is given.

2. Problem Formulation

Consider a financial markets with only two assets. One is risk-free asset with price B_t. The other is risky asset, denoted by (X_t). We demonstrate the market by probability space $\{\Omega, \mathbb{F}, P\}$ with a filter $\mathcal{F}$. $\mathcal{F}_t$ is the valid market information up to t. Assume $\mathcal{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfies common hypothesis. By the invariable law of Numeraire change ,we can assume $B_t \equiv 1$ for simplicity, i.e. (X_t) is the discounted price process. Moreover, we suppose that (X_t) is a local square-integrable $\mathcal{F}$-local martingale. We suppose that there is some investors who can not capture full valid market information for themselves limited conditions and can get less market information. $\mathcal{G}$: $\mathcal{G} \subset \mathcal{F}$. Assume that the investors hold some financial contracts,such as European options with different maturity, insurance contract etc. The investor then faces payment at any time during

$[0,T]$, in other words , there is a payment stream. We demonstrate the payment by $L^2(\mathcal{G},P)-$ process $H=(H_t)_{0\le t\le T}$. With the stochastic payment stream, the investor will face a random loss. For avoiding or reducing the future stochastic losses, hedgers will as far as possible use the market existing financial assets to construct a investment strategies. Because financial markets are often not complete, contingent claims is sometimes not replicated and hedging strategies do not completely eliminate contingent claims brought by random risk. To determine the optimal strategy, all kinds of risk standard are put forwarded.

Here, we select the optimal strategy from the point of the minimizing conditional mean square error of the cost process. Our goal is to find the risk-minimizing hedging strategies. Because $\mathcal{G}-$ risk-minimizing strategies is to be considered, we give the following assumptions.

$$X_T \text{ is } \mathcal{G}_T - \text{measureable}, H \text{ is } \mathcal{G}\text{-adaptable}.$$

Remark 2.1. Since an investor is sure to know himself instant stochastic payments at time t, we suppose that $H$is $\mathcal{G}-$ adaptable. The assumption X_T is $\mathcal{G}_T-$ measureable is also reasonable. Since every investor must know the total value of his portfolio and only hold a risk assets X_T from beginning to end, the terminal wealth X_T must be $\mathcal{G}_T-$ measurable.

Definition 2.1. Denote by $\Theta(\mathcal{F})$ the set of $\mathcal{F}-$ predictable processes with $\mathrm{E}[\int_0^T \vartheta_s^2 d\langle X\rangle_s]<+\infty$, i.e.

$$\Theta(\mathcal{F})=\left\{\vartheta \;\middle|\; E\left[\int_0^T \vartheta_s^2 d\langle X\rangle_s\right]<+\infty,\ \text{moreover } \vartheta \text{ is } \mathcal{F}\text{- predictable}\right\}.$$

Similarly, we can define $\Theta(\mathcal{G})$.

Definition 2.2. $\varphi=(\vartheta,\eta)$ is called a $\mathcal{F}$-strategy,if $\vartheta,\eta\in\Theta(\mathcal{F})$.

Similarly, we can define a $\mathcal{G}$-strategy.

Definition 2.3. For payments stream $H=(H_t)_{0\le t\le T}$, the accumulative cost process of $\varphi=(\vartheta,\eta)$ is defined as

$$C_t^H(\varphi)\hat{=}H_t+V_t(\varphi)-\int_0^t \vartheta_s \mathrm{d}X_s,$$

where $V_t(\varphi)\hat{=}\vartheta_t\cdot X_t+\eta_t$ is also known as the value process of $\varphi=(\vartheta,\eta)$.

Definition 2.4. For payments stream $H=(H_t)_{0\le t\le T}$, the risk process of a $\mathcal{F}$-strategy $\varphi=(\vartheta,\eta)$

$$R_s^{H,\mathcal{F}}(\varphi)=\mathrm{E}[(C_T^H(\varphi)-C_s^H(\varphi))^2|\mathcal{F}_s].$$

Definition 2.5. A strategy $\varphi=(\vartheta,\eta)$ is called risk minimization strategy, if for any $\hat{\varphi}=(\hat{\vartheta},\hat{\eta})$ satisfies the conditions as follows: for $t\in[0,T]$,

$$\begin{cases} V_T(\hat{\varphi})=V_T(\varphi), & \\ \quad\hat{\eta}_s=\eta_s, & s<t,\\ \quad\hat{\vartheta}_s=\vartheta_s, & s\ge t, \end{cases} \tag{1}$$

$R_t(\hat{\varphi}) \geq R_t(\varphi)$ holds.

Theorem 2.6. *Let $H = (H_t)_{0\leq t\leq T}$ be a given payments stream. For $\mathcal{F}$-strategy $\varphi = (\vartheta, \eta)$ and any $t \in [0, T]$, there is another $\mathcal{F}$-strategy $\hat{\varphi} = (\hat{\vartheta}, \hat{\eta})$ such that*

$$\begin{cases} V_T(\hat{\varphi}) = V_T(\varphi), \\ C_s^H(\hat{\varphi}) = \mathrm{E}[C_T^H(\hat{\varphi})|\mathcal{F}_s], \\ R_s^{H,\mathcal{F}}(\hat{\varphi}) \leq R_s^{H,\mathcal{F}}(\varphi). \end{cases}$$

Proof. Let $\hat{\vartheta} = \vartheta$,

$$\hat{\eta}_s = \begin{cases} \eta_s, & s < t, \\ \mathrm{E}[V_T(\varphi) + H_T - \int_0^T \vartheta_u \mathrm{d}X_u|\mathcal{F}_s] + \int_0^t \vartheta_u \mathrm{d}X_u - \vartheta_s \cdot X_s - H_s, & s \geq t. \end{cases}$$

Then

$$V_s(\hat{\varphi}) = \begin{cases} V_s(\varphi), & s < t, \\ \mathrm{E}[V_T(\varphi) + H_T - \int_0^T \vartheta_u \mathrm{d}X_u|\mathcal{F}_s] + \int_0^t \vartheta_u \mathrm{d}X_u - H_s, & s \geq t. \end{cases}$$

Thus

$$C_T^H(\hat{\varphi}) \quad = V_T(\hat{\varphi}) + H_T - \int_0^T \hat{\vartheta}_u \mathrm{d}X_u = V_T(\varphi) + H_T - \int_0^T \vartheta_u \mathrm{d}X_u = C_T^H(\varphi).$$

Hence, in case $s \geq t$, we have

$$\begin{aligned} &C_s^H(\hat{\varphi}) \\ &= V_s(\hat{\varphi}) + H_s - \int_0^s \hat{\vartheta}_u \mathrm{d}X_u = \mathrm{E}[V_T(\varphi) + H_T - \int_0^T \vartheta_u \mathrm{d}X_u|\mathcal{F}_s] \\ &+ \int_0^t \vartheta_u \mathrm{d}X_u - \int_0^t \hat{\vartheta}_u \mathrm{d}X_u \\ &= \mathrm{E}[V_T(\varphi) + H_T - \int_0^T \vartheta_u \mathrm{d}X_u|\mathcal{F}_s] = \mathrm{E}[C_T^H(\varphi)|\mathcal{F}_s]. \end{aligned}$$

Therefore, in case $s \geq t$,

$$\begin{aligned} R_s^{H,\mathcal{F}}(\hat{\varphi}) \quad &= \mathrm{E}[(C_T^H(\hat{\varphi}) - C_s^H(\hat{\varphi}))^2|\mathcal{F}_s] \\ &= \mathrm{E}[(C_T^H(\varphi) - C_s^H(\varphi) + C_s^H(\varphi) - C_s^H(\hat{\varphi}))^2|\mathcal{F}_s] \\ &= \mathrm{E}[(C_T^H(\varphi) - C_s(\varphi)^2|\mathcal{F}_s] \\ +\mathrm{E}[(C_s^H(\varphi) - C_s^H(\hat{\varphi}))^2|\mathcal{F}_s] \\ &\quad +\mathrm{E}[2(C_T^H(\varphi) - C_s^H(\varphi))(C_s^H(\varphi) - C_s^H(\hat{\varphi}))|\mathcal{F}_s] \\ &= R_s^{H,\mathcal{F}}(\varphi) - (C_s^H(\varphi) - C_s^H(\hat{\varphi}))^2 \leq R_s^{H,\mathcal{F}}(\varphi). \end{aligned}$$

□

Remark 2.2. We know by theorem 2.6 a risk minimization strategy must be mean self-financing (i.e. $C_s^H(\hat{\varphi}) = \mathrm{E}[C_T^H(\hat{\varphi})|\mathcal{F}_s]$).

Definition 2.7. we call a strategy $\varphi = (\vartheta, \eta)$ 0-achieving, if

$$V_T(\varphi) = \eta_T + \int_0^T \vartheta_s \mathrm{d}X_s = 0.$$

Since hedging is for the special contingent claims with stochastic payment stream, we modifies general risk-minimizing hedging problems which are for general European claims and define the following problems

(HF) risk minimization hedging problems with complete market information($\mathcal{F}$-risk-minimizing hedging problems): Find a 0-achieving $\mathcal{F}$-risk-minimizing strategy;

(HR) risk-minimizing hedging problems with restricted information ($\mathcal{G}$-risk-minimizing hedging problems): Find a 0-achieving $\mathcal{G}$-risk-minimizing strategy.

3. Risk-minimizing Hedging Strategies for Full Information

Here we assume investors can hold all the market information in time, and determine their own investment strategy based on the information.

we suppose that investors can know market information in time and based on it select theirs portfolio strategies according to their own information. As the preceding $\mathcal{F}_t$ represents all market valid information up to the time of t, So solving the minimum set of risk strategy problems under complete information is essentially looking for a 0-achieving $\mathcal{F}$ makes the market risk R_φ^H minimum.

Lemma 3.1 (Galtchouk-Kunita-Watanabe decomposition [2]). *For any $Y \in L^2(\mathcal{F}, P)$, it an only be written as*

$$Y = \mathrm{E}[H|\mathcal{F}_0] + \int_0^T \vartheta^Y \mathrm{d}X_s + L_Y, \tag{2}$$

where ϑ^Y is $\mathcal{F}$-adapted, $L^Y = (L_t^Y)$ is a squared integrable martingale with $Y_0 = 0$ and strongly orthogonal to X

Lemma 3.2. *For a given payment stream $H = (H_T)_{0 \le T \le T}$, If a strategy of $\varphi = (\vartheta, \eta)$ is average self-funded and 0-achieving, Then $\varphi = (\vartheta, \eta)$ is uniquely determined by ϑ.*

Proof. By the theorem conditions, we know

$$\begin{aligned} C_s^H(\varphi) &= \mathrm{E}[C_T^H(\varphi)|\mathcal{F}_s], V_T(\varphi) = 0. \qquad (3) \\ \eta_s &= C_s^H - H_s + \int_0^s \vartheta_t \mathrm{d}X_t - \vartheta_s \cdot X_s \\ &= \mathrm{E}[C_T^H(\varphi)|\mathcal{F}_s] - H_s + \int_0^s \vartheta_t \mathrm{d}X_t - \vartheta_s \cdot X_s \\ &= \mathrm{E}[H_T - \int_0^T \vartheta_t \mathrm{d}X_t|\mathcal{F}_s] - H_s + \int_0^s \vartheta_t \mathrm{d}X_t - \vartheta_s \cdot X_s \\ &= \mathrm{E}[H_T - H_s - \int_s^T \vartheta_t \mathrm{d}X_t|\mathcal{F}_s] - \vartheta_s \cdot X_s. \end{aligned}$$

The Lemma 3.2 is proved. □

Theorem 3.3. *For a given payment stream $H = (H_t)_{0 \le t \le T}$, the unique solution of question A) is $\varphi = (\vartheta, \eta)$.*

$$\vartheta_t = \vartheta_t^{H_T}, \quad \eta_t = V_t^{H_T} - \vartheta_t^{H_T} \cdot X_t,$$

where $V_t^{H_T} = \mathrm{E}[H_T|\mathcal{F}_0] + \int_0^t \vartheta_s^{H_T} \mathrm{d}X_s + L_t^{H_T} - H_t,\ 0 \leq t \leq T, \vartheta^{H_T}, L^{H_T}$ *is determined by* (2).

Further the cost process of the risk minimum strategy is $C_t^H(\varphi) = \mathrm{E}[H_T|\mathcal{F}_0] + L_t^{H_T},\ 0 \leq t \leq T$, *the risk-minimizing process is* $R_s^{H,\mathcal{F}}(\varphi) = \mathrm{E}[(L_T^{H_T} - L_s^{H_T})^2|\mathcal{F}_s]$.

Proof. Obviously, $V_T(\varphi) = \eta_T + \vartheta_T^{H_T} \cdot X_T = V_T^{H_T} = H_T - H_T = 0$, so φ is 0-achieving. Since $\vartheta = \vartheta^{H_T}, \eta = V^{H_T} - \vartheta^{H_T} \cdot X$, we have

$$C_t^H(\varphi) = V_t^H + H_t - \int_0^t \vartheta_s \mathrm{d}X_s = \mathrm{E}[H_T|\mathcal{F}_0] + L_t^{H_T},$$
$$R_s^H(\varphi) = \mathrm{E}[(C_T^H(\varphi) - C_t^H(\varphi))^2|\mathcal{F}_s] = \mathrm{E}[(L_T^{H_T} - L_s^{H_T})^2|\mathcal{F}_s].$$

Assume that there is another 0-achieving strategy $\hat{\varphi}$ which is the risk-minimizing, it is known from the remark of Theorem 2.6,

$$C_s^H(\hat{\varphi}) = \mathrm{E}[C_T^H(\hat{\varphi})|\mathcal{F}_s].$$

Because $\hat{\varphi}$ is 0-achieving strategy and $\int_0^\cdot \hat{\vartheta}_t \mathrm{d}X_t$ is a martingale, we have

$$C_T^H(\hat{\varphi}) - C_s^H(\hat{\varphi}) = H_T + \int_0^T \hat{\vartheta}_t \mathrm{d}X_t - \mathrm{E}\left[H_T + \int_0^T \hat{\vartheta}_t \mathrm{d}X_t|\mathcal{F}_s\right]$$
$$= \int_0^s \vartheta^{H_T} \mathrm{d}X_t + L_T^{H_T} + \int_0^T \hat{\vartheta}_t \mathrm{d}X_s - \mathrm{E}\left[\int_0^T \vartheta^{H_T} \mathrm{d}X_t + L_T^{H_T} + \int_0^T \hat{\vartheta}_t \mathrm{d}X_t|\mathcal{F}_s\right]$$
$$= L_T^{H_T} - L_s^{H_T} + \int_s^T (\vartheta^{H_T} - \hat{\vartheta}) \mathrm{d}X_t.$$

Then

$$\begin{aligned} R_s^{H,\mathcal{F}}(\hat{\varphi}) &= \mathrm{E}[(C_T^H(\varphi) - C_s^H(\varphi))^2|\mathcal{F}_s] \\ &= \mathrm{E}[(L_T^{H_T} - L_s^{H_T} + \int_s^T (\vartheta^{H_T} - \hat{\vartheta}) \mathrm{d}X_t)^2|\mathcal{F}_s] \\ &= \mathrm{E}[(L_T^{H_T} - L_s^{H_T})^2|\mathcal{F}_s] + \mathrm{E}[(\int_s^T (\vartheta_t^{H_T} - \hat{\vartheta}_t) \mathrm{d}X_t)^2|\mathcal{F}_s] \\ &= R_s^{H,\mathcal{F}}(\varphi) + \mathrm{E}[(\int_s^T (\vartheta_t^{H_T} - \hat{\vartheta}_t) \mathrm{d}X_t)^2|\mathcal{F}_s] \geq R_s^{H,\mathcal{F}}(\varphi). \end{aligned}$$

Therefore, the available strategy of $\varphi = (\vartheta, \eta)$ is risk-minimizing.

Next, we prove the the solution of question (HF) is unique. Assume that there is also another 0-achieving $\mathcal{F}$-strategy $\tilde{\varphi}$ with Minimum risk $R_s^H(\tilde{\varphi}) = \mathrm{E}[(L_T^{H_T} - L_s^{H_T})^2|\mathcal{F}_s]$, Tt is known from the above formula $\tilde{\vartheta}_t = \vartheta_t^{H_T}$. Since the risk-minimizing strategy must be mean self-financing, it is known by Lemma 3.2 $\tilde{\varphi} = \varphi$. That is, the solution to problem (HF) is unique. □

4. Risk-minimizing Hedging Strategies for Restricted Information

This section assumes that hedgers can not be informed of all market information $\mathcal{F}$ in a timely manner due to condition constraints, and only partial market information $\mathcal{G} \subset \mathcal{F}$ is available for them. Therefore, the hedger can only build his hedging strategy based on his own information set of $\mathcal{G}$. In mathematical terms, his hedging strategy should be $\mathcal{G}$-adapted. Since X is no longer $\mathcal{G}$-adapted, so for filtering $\mathcal{G}$, H_T no longer exists Kunita–Kunita-Watanabe decomposition. That is to say, ϑ^Y in (2) is not $\mathcal{G}$-adapted. Therefore, to solve the problem (HR), we must take a different approach from the problem (HF). Follow the idea of Schweizer[2], first of all, we prove a useful Lemma.

Lemma 4.1. *Let $H = (H_t)_{0 \le t \le T}$ is a given payment stream. For any strategy $\varphi = (\vartheta, \eta)$ and $t \in [0, T]$, There's another strategy $\hat{\varphi} = (\hat{\vartheta}, \hat{\eta})$ such that*

$$\begin{cases} V_T(\hat{\varphi}) & = V_T(\varphi), \qquad P\text{-}a.s., \\ \tilde{\eta}_s & = \eta_s, \qquad s < t, \\ 0 & = \mathrm{E}[C_T(\hat{\varphi}) - C_s(\hat{\varphi}) | \mathcal{G}_s], \\ R_s^{H,\mathcal{G}}(\hat{\varphi}) & \le R_s^{H,\mathcal{G}}(\varphi). \end{cases}$$

Proof. Set $\hat{\vartheta}_s = \vartheta_s, s \in [0, T]$. Let J is the $\mathcal{G}-$ optional projection of $V + H$, $K_t = \mathrm{E}[V_T + H_T | \mathcal{G}_t]$(Without loss of generality, we can take its right continuous form). Set

$$\hat{\eta}_s \hat{=} V_s(\varphi) + (K_s - J_s) I_{[t,T]}(s) - \vartheta_s \cdot X_s.$$

From this we can see $\hat{\varphi} = (\hat{\vartheta}, \hat{\eta})$ is an investment strategy and we have when $s < t$, $\hat{\eta}_s = \eta_s$, and when $s \ge t$

$$\begin{aligned} \hat{\eta}_s &= \mathrm{E}(\hat{\eta}_s | \mathcal{G}_s) = \mathrm{E}(V_s(\varphi) + K_s - J_s - \vartheta_s \cdot X_s | \mathcal{G}_s) \\ &= \mathrm{E}(V_T(\varphi) + H_T - H_s - \vartheta_s \cdot X_s | \mathcal{G}_s). \end{aligned} \tag{4}$$

Since H_T, X_T is $\mathcal{G}$-measurable, we have $V_T(\hat{\varphi}) = V_T(\varphi)$ P-a.s. From the definition of $\hat{\varphi}$, we have when $s \ge t$,

$$C_T^H(\hat{\varphi}) - C_s^H(\hat{\varphi}) = V_T(\hat{\varphi}) + H_T - V_s(\hat{\varphi}) - H_s - \int_s^T \hat{\vartheta}_t \mathrm{d}X_t \tag{5}$$

$$= V_T(\varphi) + H_T - V_s(\varphi) - H_s - \int_s^T \hat{\vartheta}_t \mathrm{d}X_t + J_s - K_s \tag{6}$$

$$= C_T^H(\hat{\varphi}) - C_s^H(\hat{\varphi}) + J_s - K_s. \tag{7}$$

and that when $s \ge t$, $\mathrm{E}[C_T(\hat{\varphi}) - C_s(\hat{\varphi}) | \mathcal{G}_s] = 0$. We also have from (7)that

$$R_s^{H,\mathcal{G}}(\varphi) = R_s(\hat{\varphi})^{H,\mathcal{G}} + \mathrm{E}[(J_s - K_s)^2 | \mathcal{G}_s] \le R_s^{H,\mathcal{G}}(\varphi). \tag{8}$$

□

Definition 4.2. For any strategy φ, the $\mathcal{G}-$ optional projection of the cost process $C^H(\varphi)$ is denoted by $C^{H,O}(\varphi)$. φ is weakly $\mathcal{G}$-mean-selffinancing if $C^{H,O}(\varphi)$ is a $\mathcal{G}-$martingale.

Remark 4.1. Known by the definition of an optional projection, weakly $\mathcal{G}$-mean-selffinancing implies

$$\mathrm{E}[C_T^H(\hat{\varphi}) - C_s^H(\hat{\varphi})|\mathcal{G}_s] = 0, \qquad s \in [0,T], \tag{9}$$

and

$$\hat{\eta}_s = \mathrm{E}(V_T(\varphi) + H_T - H_s - \vartheta_s \cdot X_s|\mathcal{G}_s), \qquad s \in [0,T]. \tag{10}$$

Lemma 4.3. *If φ is $\mathcal{G}$-risk-minimizing strategy, it is also weakly $\mathcal{G}$-mean-selffinancing.*

Take $t = 0$,we construct φ as Lemma 4.1. Note that φ is $\mathcal{G}$-risk-minimizing, we know from (8) J is a modification of K. Note that J and K are right continuous, then J and K are indistinguishable(see [12] for detail). Then we have

$$C(\varphi) = V(\varphi) + H - \int \vartheta \mathrm{d}X = V(\varphi) + H - J + K - \int \vartheta \mathrm{d}X.$$

Note that K and $\int \vartheta \mathrm{d}X$ are $\mathcal{G}$-martingale and $\mathcal{F}$-martingale respectively and that J is the optional projectin of $V + H$, we know $C^{H,O}(\varphi)$ is a $\mathcal{G}$-martingale.

Theorem 4.4. *For a given $H = (H_t)_{0 \le t \le T}$, a 0-achieving strategy φ is $\mathcal{G}$-risk-minimizing if and only if φ is weakly $\mathcal{G}$-mean-selffinancing and ϑ is the solution to the optimization problem*

$$\min_{\gamma \in \ \Theta(\mathcal{G})} \left[\left(H_T - \int_0^T \gamma_s \mathrm{d}X_s \right)^2 \right]. \tag{11}$$

Further, $\mathcal{G}$-risk-minimizing strategy exists and is unique.

Proof. Known from the space projection theorem on L^2, a process ξ is the solution to the optimization problem (11) if and only if

$$\mathrm{E}\left[\left(H_T - \int_0^T \xi_s \mathrm{d}X_t \right) \int_0^T \gamma_t \mathrm{d}X_t \right] = 0, \qquad \forall\, \gamma \in \Theta(\mathcal{G}). \tag{12}$$

From the G-K-W decompositon of H_T and that L is orthogonal to X, we know the above equality is equivalent to

$$\mathrm{E}\left[\left(\int_0^T (\vartheta_t^{H_T} - \xi_t) \mathrm{d}X_t \right) \int_0^T \gamma_t \mathrm{d}X_t \right] = 0, \qquad \forall\, \gamma \in \Theta(\mathcal{G}), \tag{13}$$

and then equivalent to

$$\mathrm{E}\left[\left(\int_s^T (\vartheta_t^{H_T} - \xi_t) \mathrm{d}X_t \right) \int_s^T \gamma_t \mathrm{d}X_t |\mathcal{G}_s \right] = 0, \qquad \forall\, \gamma \in \Theta(\mathcal{G}), s \in [0,T]. \tag{14}$$

Further it is equivalent to

$$\mathrm{E}\left[\left(\int_s^T \gamma_s \mathrm{d}X_s \right)^2 - 2\int_s^T (\vartheta_t^{H_T} - \xi_t) \mathrm{d}X_t \int_s^T \gamma_t \mathrm{d}X_t |\mathcal{G}_s \right] \ge 0, \forall\, \gamma \in \Theta(\mathcal{G}), s \in [0,T]. \tag{15}$$

1) The proof of the necessity. If $\varphi = (\vartheta, \eta)$ is 0-achieving and $\mathcal{G}$-risk-minimizing, the $\varphi = (\vartheta, \eta)$ is $\mathcal{G}$-mean selffinancing. For $s \in [0, T]$, we consider another strategy $\delta = (\varsigma, \tau)$. Construct a strategy $\hat{\delta} = (\hat{\varsigma}, \hat{\tau})$ as lemma 4.1, we know that $\hat{\delta} = (\hat{\varsigma}, \hat{\tau})$ satisfies(1). Note that (1), (4)and (10), we have $\hat{\tau} = \eta$. Known from (1) we have $V_t(\delta) = V_t(\varphi)$ and

$$C_T^H(\varphi) - C_s^H(\varphi) = V_T(\varphi) + H_T + \int_0^T \hat{\vartheta}_t \mathrm{d}X_t - V_s(\varphi) - H_s - \int_0^s \hat{\vartheta}_t \mathrm{d}X_t \tag{16}$$

$$= C_T^H(\hat{\delta}) - C_s^H(\hat{\delta}) + \int_s^T (\delta_t - \vartheta_t) \mathrm{d}X_t. \tag{17}$$

Since $\varphi = (\vartheta, \eta)$ is $\mathcal{G}$-risk-minimizing, then

$$\begin{aligned} 0 \quad &\leq R_s^{H,\mathcal{G}(\hat{\delta})} - R_s^{H,\mathcal{G}}(\varphi) \\ &= \mathrm{E}\left[\left(\int_s^T (\delta_t - \vartheta_t)\mathrm{d}X_t\right)^2 - 2\left(C_T^H(\varphi) - C_s^H(\varphi)\right)\int_s^T (\delta_t - \vartheta_t)\mathrm{d}X_t|\mathcal{G}_s\right] \\ &= \mathrm{E}\left[\left(\int_s^T (\delta_t - \vartheta_t)\mathrm{d}X_t\right)^2 - 2\int_s^T (\vartheta_t^{H_T} - \vartheta_t)\mathrm{d}X_t \int_s^T (\delta_t - \vartheta_t)\mathrm{d}X_t|\mathcal{G}_s\right], \end{aligned}$$

Take $\delta = \vartheta + \gamma I_{(s,T]}$ we have (15), and then ϑ satisfies the optimization problem(11).

2) The proof of the sufficiency. Suppose ϑ satisfies the optimization problem(11)and $\varphi = (\vartheta, \eta)$ is weakly $\mathcal{G}$-mean-selffinancing. For a $s \in [0, T]$,we consider another strategy $\delta = (\varsigma, \tau)$:

$$\begin{cases} V_T(\delta) &= V_T(\varphi), \\ \tau_s &= \eta_s, \qquad s < t, \\ \varsigma_s &= \vartheta_s, \qquad s \geq t, \end{cases}$$

As the proof of the necessity, we construct $\hat{\delta} = (\hat{\varsigma}, \hat{\tau})$. Then we have (17), and then

$$\begin{aligned} &R_s^{H,\mathcal{G}}(\hat{\delta}) - R_s^{H,\mathcal{G}}(\varphi) \\ &= \mathrm{E}\left[\left(\int_s^T (\delta_t - \vartheta_t)\mathrm{d}X_t\right)^2 - 2\int_s^T (\vartheta_t^{H_T} - \vartheta_t)\mathrm{d}X_t \int_s^T (\delta_t - \vartheta_t)\mathrm{d}X_t|\mathcal{G}_s\right]. \end{aligned} \tag{18}$$

Since ϑ the solution to the optimization problem(11), then we have (15). If we take $\gamma = \delta - \vartheta$ in (15), then

$$0 \leq R_s^{H,\mathcal{G}}(\hat{\delta}) - R_s^{H,\mathcal{G}}(\varphi) \leq R_s^{H,\mathcal{G}}(\delta) - R_s^{H,\mathcal{G}}(\varphi)$$

From the definition of the risk minimization strategy, we know $\varphi = (\vartheta, \eta)$ is risk-minimizing. We know from [2]that $\Theta(\mathcal{G})$ is a closed subspace of $L^2(\mathcal{F})$, the the projection exists and is unique, $\mathcal{G}$-risk-minimizing strategy existence and uniqueness.

For any finite variation process A, denote by $A^{P,\mathcal{G}}$ the predictable dual projection of process A.

□

Theorem 4.5. *For a given stream of payments $H = (H_t)_{0 \leq t \leq T}$, there is a unique 0-achieving $\mathcal{G}$-risk-minimizing strategy $\hat{\varphi}^H = (\hat{\vartheta}^H, \hat{\eta}^H)$:*

$$\hat{\vartheta}_s^H = \frac{d(\int_0^\cdot \vartheta_u^{H_T} d\langle X\rangle_u)_s^{P,\mathcal{G}}}{d\langle X\rangle_s^{P,\mathcal{G}}}, \tag{19}$$

$$\hat{\eta}_s^H = \mathrm{E}\left[H_T - H_s - \hat{\vartheta}_s^H \cdot X_s | \mathcal{G}_s\right]. \tag{20}$$

Proof. We know from theorem 4.4 that risk-minimizing strategy $\hat{\varphi}^H = (\hat{\vartheta}^H, \hat{\eta}^H)$ exists and is unique. Furthermore we know from (13) $\hat{\vartheta}^H$ satisfies

$$\mathrm{E}\left[\left(\int_0^T (\vartheta_s^{H_T} - \hat{\vartheta}_s^H)\mathrm{d}X_s\right)\int_0^T \gamma_s \mathrm{d}X_s\right] = 0, \qquad \forall \gamma \in \Theta(\mathcal{G}).$$

Therefore, $\forall\ \gamma \in \Theta(\mathcal{G})$

$$\begin{aligned} 0 &= \mathrm{E}\left[\left(\int_0^T (\vartheta_s^{H_T}\gamma_s)\mathrm{d}\langle X\rangle_s\right) - \int_0^T \hat{\vartheta}_s^H \gamma_s \mathrm{d}\langle X\rangle_s\right] \\ &= \mathrm{E}\left[\int_0^T \gamma_s \mathrm{d}\left(\int_0^\cdot \vartheta_u^{H_T}\mathrm{d}\langle X\rangle_u\right)_s^{P,\mathcal{G}}\right] - \mathrm{E}\left[\int_0^T \hat{\vartheta}_s^H \gamma_s \mathrm{d}\langle X\rangle_s^{P,\mathcal{G}}\right]. \end{aligned}$$

From the arbitrary of γ, we have

$$\hat{\vartheta}_s^H = \frac{\mathrm{d}(\int_0^\cdot \vartheta_u^{H_T}\mathrm{d}\langle X\rangle_u)_s^{P,\mathcal{G}}}{\mathrm{d}\langle X\rangle_s^{P,\mathcal{G}}}.$$

In addition, since the risk-minimizing strategy must be 0-achieving and mean-selffinancing, then (20) holds from(10). □

5. Conclusions

Considering the widespread existence of undetermined interests with random payment streams(such as insurance payments, etc.) in the real financial and insurance markets, this paper proposes and resolves the risk-minimizing hedging problem of this kind of undetermined rights and the existence and uniqueness of strategy are given by constructiveness. Due to the consideration of non-standard European undecided rights, this paper introduces a 0 -achieving strategy and modify the usual risk minimization strategy definition. Under different information conditions, the existence and uniqueness of risk-minimizing strategy are proven by Galtchouk-Kunita decomposition and L^2 spatial projection theorem , and the construction method of the specific strategy are given.

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