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EXISTENCE AND UNIQUENESS RESULT FOR RANDOM IMPULSIVE STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FINITE DELAYS

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ABSTRACT. This manuscript addressed, the existence and uniqueness result for random impulsive stochastic functional differential equations with finite time delays. The study of random impulsive stochastic system is a new area of research. We interpret the meaning of a stochastic derivative and how it differs from the classical derivative. We prove the existence and uniqueness of mild solutions to the equations by using the successive approximation method. We conclude the article with some interesting future extension. This work extends the work of [18, 12, 20]. Finally, an example is given to illustrate the theoretical result.

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1. Introduction

The existence, uniqueness and stability of solutions to random impulsive differential equations have been studied extensively by several researchers, pl. see the reference list. Random impulsive stochastic differential equations are widely used in the fields of medicine, biology, economy, finance and so on. For example, the classical stock price model [17] in which T. Wang and S. Wu discussed the random impulsive model for stock prices and its application for insurers as below:

$$\begin{aligned} d(S(t)) &= uS(t)dt + \sigma S(t)dB_t,\ t \geq 0,\ t \neq \tau_k, \\ S(\tau_k) &= a_k S(\tau_k^-), k = 1, 2, ..., \\ S(0) &= S_0, \end{aligned}$$

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This model describes the impulsive stochastic differential equation. Here B_t is a Brownian motion or Wiener process, $S(t)$ represents the price of the stock at time t, and τ_k represents the release time of the important information relating to the stock. $S_0 \in R$. In reality, τ_k is a sequence of random variables, which satisfies $0 < \tau_2 < \tau_3 < ....$

Inspired by above stock price model, in this article, we discuss the existence and uniqueness result for random impulsive stochastic functional differential equations with finite delays:

$$d\left[x(t)\right] = f(t, x_t)dt + \sigma(t, x_t)dw(t), \ t \neq \zeta_k \ t \in [-r, T], \tag{1}$$

$$x(\zeta_k) = b_k(r_k)x(\zeta_k^-), \ k = 1, 2, ..., \tag{2}$$

$$x_{t_0} = \varphi, \ -r \leq \theta \leq 0, \tag{3}$$

where r_k is a random variable define from a nonempty set Ω to $\mathcal{D}_k \overset{def.}{=} (0, d_k)$ with $0 < d_k < \infty$ for $k = 1, 2, ...$ Suppose that r_i and r_j are independent of each other as $i \neq j$ for $i, j = 1, 2, ...$ Here, $f : [t_0, T] \times \mathscr{C} \to \mathbb{R}^n$, $\sigma : [t_0, T] \times \mathscr{C} \to \mathbb{R}^{n \times m}$ and $b_k : \mathcal{D}_k \to \mathbb{R}^{n \times n}$, and x_t is $\mathbb{R}^n$-valued stochastic process such that $x_t \in \mathbb{R}^n$, $x_t = \left\{x(t + \theta) : -r \leq \theta \leq 0\right\}$. The impulsive moments ζ_k from a strictly increasing sequence, i.e. $\zeta_0 < \zeta_1 < \cdots < \zeta_k < \lim_{k\to\infty} \zeta_k$, and $x(\zeta_k^-) = \lim_{t\to\zeta_k - 0} x(t)$. We assume that $\zeta_0 = t_0$ and $\zeta_k = \zeta_{k-1} + r_k$ for $k = 1, 2, ...$ Obviously, $\{\zeta_k\}$ is a process with independent increment of the σ-algebra.

Stochastic differential equations (SDEs) have been investigated as mathematical models to describe the dynamical behavior of real life phenomena. It is essential to take into account the environmental disturbances as well as the time delay while constructing realistic models in the area of engineering, biology, mathematical finance, applied sciences and so on (see [10, 13, 6, 17] and the references therein). Existence of mild solutions for stochastic functional differential equations and impulsive SDEs has been extensively studied in the literature. Anguraj et. al [4] have investigated the impulsive stochastic semilinear functional differential equations with infinite delay. Very recently, Anguraj et. al [2] impulsive stochastic functional integrodifferential equations with Poisson jumps via successive approximation technique. Moreover, the study was conducted on stability through the continuous dependence on the initial values by means of Bihari's inequality. For more details see refer [13, 6, 18, 5].

Many evolution processes from fields as diverse as physics, population dynamics, telecommunications, economics and engineering are characterized by the fact that they undergo abrupt change of state at certain moments of time between intervals of continuous evolution. The duration of these changes are often negligible compared to the total duration of process act instantaneously in the form of impulses. The impulses may be deterministic or random. There are a lot of papers which investigate the qualitative properties of deterministic impulses see [1, 3, 8, 10] and the references therein. When the impulses exist at random points, the solutions of the differential systems are stochastic processes. It is very different from deterministic impulsive differential systems and also it is different from stochastic differential equations. Thus, the random impulsive systems give more realistic than deterministic impulsive systems. The study of random impulsive differential equations is a new research area. There are few publications in this field, Shujin Wu [19] first brought forward random impulsive ordinary differential equations and investigated boundedness of solutions to these models by Liapunov's direct function. Shujin Wu et. al [19, 18, 20, 17], studied some qualitative properties of random impulses. Anguraj et. al [5], studied the existence and uniqueness of random impulsive differential system by relaxing the linear growth conditions, sufficient conditions for stability through continuous dependence on initial conditions and exponential stability via fixed point theory.

On the other hand, SDEs is influenced by various disturbance factors from random inputs. By the interaction of stochastic process and mathematical models, the real-world system can be interpreted. Several systems are modeled using stochastic functional differential equations with impulses. In general, impulses appears at random time points, i.e., the impulse time and the impulsive functions are random variables. Random impulsive stochastic differential equations are widely used in the fields of biology, medicine, economics and so on. For instance, the classical stock price model [17] is described using a random impulsive stochastic differential equation. Recently, many monographs have been focusing their attention towards the theory and applications of SDEs with random impulses. To be more precise, existence, uniqueness and Hyers-Ulam stability results on SDEs with random impulses can be found in [12, 21]. Very recently, S. Li et. al [11] investigated existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays using Krasnoselskii's fixed point.

To the best of authors knowledge, up to now, no work has been reported to prove the existence and uniqueness results for random impulsive stochastic functional differential equations with finite delays using successive approximation. This work extends the work of [18, 12, 20]. The main contributions are summarized as follows:

1 Random impulsive stochastic functional differential equations with finite delays is formulated.
2 Existence and uniqueness of the mild solution for random impulsive stochastic functional differential equations with finite delays via successive approximation method are studied under some sufficient conditions.

2. Preliminaries

In this section, we collect basic concepts, definitions and lemmas which will be used in the sequel to obtain the main results.

Let us denote $\{N(t), t \geq 0\}$ is the simple counting process generated by $\{\zeta_k\}$, and $\{w(t) : t \geq 0\}$ is a given m-dimensional Wiener process. We denote $\Im_t^{(1)}$ the σ-algebra generated by $\{N(t), t \geq 0\}$, and denote $\Im_t^{(2)}$ σ-algebra generated by $\{w(s), s \leq t\}$. We assume that $\Im_\infty^{(1)}, \Im_\infty^{(2)}$ are mutually independent. Let $(\Omega, \Im, \mathbb{P})$ with the filtration $\{\Im_t\}_{t\geq 0}$ satisfying $\Im_t = \Im_t^{(1)} \vee \Im_t^{(2)}$. Let $\mathcal{L}^2(\Omega, \mathbb{R}^n)$ be the collection of all strongly measurable, $\mathbb{R}^n$-valued random variable x with norm

$$\|x\|_{\mathcal{L}^2} = \left(\mathbf{E}\,\|x\|^2\right)^{1/2}.$$

Let $r > 0$ and denote the Banach space of all piecewise continuous $\mathbb{R}^n$-valued stochastic process $\{\varphi : t \in [-r, 0]\}$ by $\mathscr{C}([-r, 0], \mathcal{L}^2(\Omega, \mathbb{R}^n))$ equipped with the norm

$$\|\varphi\|_{\mathscr{C}} = \sup_{t\in[-r,0]} \left(\mathbf{E}\,\|\varphi(t)\|^2\right)^{1/2},$$

where $\varphi(t) \in \mathscr{C}$, φ is a function from $[-r, 0]$ to $\mathbb{R}^n$.

Let $\mathscr{B}$ be the space $\mathscr{B} = \mathscr{C}([-r, 0], \mathcal{L}^2(\Omega, \mathbb{R}^n))$ endowed with the norm

$$\|x\|^2_{\mathscr{B}} = \sup_{t\in[t_0,T]} \|x\|^2_{\mathscr{C}}, \text{ where } \|x\|^2_{\mathscr{C}} = \sup_{t-r\leq s\leq t} \mathbf{E}\,\|x(s)\|^2.$$

Definition 2.1. For a given $T \in (t_0, \infty)$, a $\mathbb{R}^n$-valued stochastic process $x(t)$ on $t_0 - r \leq t \leq T$, is called a solution to (1)-(3) if for every $t_0 \leq t \leq T$,

(i) $x(t)$ is $\Im_t$-adapted for $t \geq t_0$,

(ii) $x(t_0 + s) = \varphi(s)$ when $s \in [-r, 0]$, and

$$\begin{aligned} x(t) &= \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} b_i(r_i)\varphi(0) + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(r_j) \int_{\varphi_{i-1}}^{\varphi_i} f(s, x_s)ds + \int_{\varphi_k}^{t} f(s, x_s)ds \\ &+ \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(r_j) \int_{\varphi_{i-1}}^{\varphi_i} \sigma(s, x_s)dw(s) \\ &+ \int_{\varphi_k}^{t} \sigma(s, x_s)dw(s)\Bigg] I_{(\zeta_k,\zeta_{k+1}]}(t), \text{ a.s.} \end{aligned} \tag{4}$$

where $\prod_{j=i}^{k} b_j(r_j) = b_k(r_k)b_{k-1}(r_{k-1})\cdots b_i(r_i)$, and $I_A(\cdot)$ is the index function, i.e.

$$I_A(t) = \begin{cases} i, & \text{if } t \in A, \\ 0, & \text{if } t \neq A. \end{cases}$$

3. Existence and uniqueness

In this section, we discuss the existence and uniqueness of solution of system (1)-(3). Before giving the main results, we introduce several assumptions:

(H1) There exists a function $H : [t_0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t, x_t)\|^2 + \|\sigma(t, x_t)\|^2 \leq H\left(t, \|x_t\|^2_{\mathscr{C}}\right),$$

hold for any stochastic process $t \in [0, T]$ and $x_t : [-r, 0] \to \mathbb{R}$.

(H2) The function $H(u,v)$ is locally integrable in u for each fixed $v \in \mathbb{R}^+$ and is continuous, concave and monotone non-decreasing in v for each fixed $u \in [t_0, T]$.

(H3) There exists a function $Z : [t_0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(t,x_t) - f(t,y_t)\|^2 + \|\sigma(t,x_t) - \sigma(t,y_t)\|^2 \quad \leq \quad Z\left(t, \|x_t - y_t\|^2_{\mathscr{C}}\right),$$

hold for any two functions $x_t, y_t : [-r, 0] \to \mathbb{R}$.

(H4) The function $Z(t,v)$ is locally integrable in t for each fixed $v \in \mathbb{R}^+$ and is continuous, non-decreasing and convex in v for each fixed $t \in [t_0, T]$, and the inequality

$$v(t) \quad = \quad K \int_{t_0}^{t} Z(t, v(s))ds$$

has a nonzero, non-decreasing and non-negative solution, where $K > 0$.

(H5) For any constant $D > 0$, the differential equation

$$\begin{aligned} \frac{du}{dt} &= DH(t,u), \\ u(t_0) &= u_0, \ t \in [0,T], \end{aligned} \tag{5}$$

has a global solution.

(H6) $\mathbf{E}\left[\max_{i,k}\left\{\prod_{j=i}^{k} \|b_j(r_j)\|^2\right\}\right] < \infty$.

Let us now introduce the successive approximations to the equation (4) as follows:

$$x^n(t_0 + s) \quad = \quad \varphi(s), \ n = 0, 1, ... \text{ for all } s \in [-r, 0],$$

with arbitrary non-negative initial value $\varphi \in \mathscr{B}$, $t \in [-r, 0]$,

$$x^0(t) \quad = \quad \sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k} b_i(r_i)\varphi(0)\right] I_{[\zeta_k, \zeta_{k+1})}(t), \ \text{ for } t \in [t_0, T],$$

and

$$\begin{aligned} x^{n+1}(t) \quad = \quad & \sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k} b_i(r_i)\varphi(0) + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(r_j)\int_{\zeta_{i-1}}^{\zeta_i} f(s, x_s^{(n)})ds\right. \\ & + \int_{\zeta_k}^{t} f(s, x_s^{(n)})ds + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(r_j)\int_{\zeta_{i-1}}^{\zeta_i} \sigma(s, x_s)dw(s) \\ & \left. + \int_{\zeta_k}^{t} \sigma(s, x_s)dw(s)\right] I_{(\zeta_k, \zeta_{k+1}]}(t), \ n = 0, 1, 2, ..., \end{aligned} \tag{6}$$

where $\prod_{j=i}^{k} b_j(r_j) = b_k(r_k)b_{k-1}(r_{k-1})\cdots b_i(r_i)$, and $I_A(\cdot)$ is the index function, i.e.

$$I_A(t) = \begin{cases} i, \text{ if } t \in A, \\ 0, \text{ if } t \neq A. \end{cases}$$

In addition to the successive approximations to (5), we also need the following successive approximations:

$$v_{n+1}(t) \quad = \quad K\int_{t_0}^{t} Z(t, v_n(s))ds, \;\; t_0 \le t \le T, \;\; n = 1, 2, ... \tag{7}$$

with arbitrary non-negative initial approximation $v_0 \in \mathcal{C}_T(0)$.

Lemma 3.1. [16] *Suppose that the function $H(u, v) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone non-decreasing with respect to $v \in \mathbb{R}^+$ for each $u \in \mathbb{R}^+$ and is locally integrable with respect to $u \in \mathbb{R}^+$ for each fixed $v \in \mathbb{R}^+$. If two continuous functions $a(t)$ and $b(t)$ defined on $[s, \theta)$ ($s \ge 0$ and θ may be ∞) satisfy the inequality*

$$a(t) - \int_s^t H(u, a(u))du < b(t) - \int_s^t H(u, b(u))du$$

for all $t \in (s, \theta)$ and $a(s) < b(s)$, then $a(t) < b(t)$ for all $t \in [s, \theta)$.

Lemma 3.2. [15] *Let the assumption* (**H4**) *holds. Then for every $\eta > 0$, there exists $T_0 \in [t_0, T]$ such that for $0 \le v_0(t) \le \eta$ the successive approximations (6) tends uniformly to zero.*

Remark 3.1. The number T_0 is chosen by the condition $K \int_{t_0}^{t} Z(t, \eta)ds \le \eta$. Using boundedness of the function $Z(t, \eta)$ on $[t_0, T]$ one can show that for each fixed $\eta > 0$ there exists $T_0 > t_0$ such that the successive approximations

$$v_{n+1}(t) \quad = \quad K\int_{t_0}^{t} Z(t, v_n(s))ds \;\; \to 0 \text{ as } n \to \infty,$$

and for every $\theta \in [t_0, T]$, $v_0(t) \le \eta$, $t \in [\theta, \min \theta + T_0, T]$, $v(s) = 0$, and $s \ge \theta$.

Remark 3.2. How does one interpret the meaning of a stochastic derivative? Can we still think of it as an instantaneous rate of change?
A continuous-time stochastic process, which is formally a random variable that takes values in a function space, interpreted as a function of time. One can certainly speak about differentiating the particular function that is the value of the process in one experiment, which will give you another function-valued random variable.

I think we need to disclose more context for truly satisfying explanation. If we have given a random function F (formally a map from the sample space into a space of functions), then the derivative of that function is another random function G, such that for each particular point in the sample space the function G at that point in the sample space is the derivative of the function F at that point in the sample space. Let's talk about one-dimensional Brownian motion, which is a stochastic process: It is a family of random variables indexed by a continuous parameter, which is usually called "time" and is written as B_t.
Another point of view is that Brownian motion is a probability measure on a suitable set of functions. Since it can be shown that Brownian motion has

continuous sample paths with probability one, we can think of it as probability measure on the set $C[0,T]$, the set of of continuous functions

$$f : [0,T] \mapsto R.$$

In addition, one can prove that Brownian motion has with probability one sample paths that are not differentiable and not even of bounded variation. This means it is not possible to define a Riemann-Stieltjes integral with respect to the sample paths. This is why one needs to develop a new concept of an integral with respect to Brownian motion, for example the Ito or the Stratonovich integral. It is possible to give precise meaning to the expressions like this one:

$$X_T = \int_0^T f(t,x)dB_t$$

and prove (with appropriate assumptions for f) that there is a unique stochastic process X_T satisfying this relation. These integral equations are usually abbreviated, with an abuse of notation, as

$$dX_t = f\ dB_t$$

but one has to keep in mind that the symbol dB_t is actually undefined. Only the integral with respect to Brownian motion is defined in the Ito- or the Stratonovich calculus. This means that there is no "stochastic derivative", and that the notion of "velocity" is undefined for Brownian motion. There just is no room for the interpretation of a "velocity" in physical terms in the theory.

There is some ambiguity associated with the term "stochastic derivative". It can mean "Nelson derivative" and "Malliavin derivative", for example.

A real-valued stochastic process $\{X(t)\}$ is said to be stochastically differentiable (differentiable with probability one, differentiable in the L_p sense) at a point t_0 if there exists a random variable eta such that, for $t \mapsto t_0$,

$$\frac{X(t_0) - X(t)}{t_0 - t} \mapsto \eta$$

in probability (with probability one, or in L_p respectively). The random variable η is called the stochastic derivative of the process at the point t_0, and is denoted $X^{'}(t_0)$.

In other words, even though $\{X(t)\}$ may not be differentiable, there is an η from which we can make the approximation

$$X(t) = X(t_0) + (t_o - t)\eta + r(t)$$

where $r(t)$ is a remainder small enough in probability or in a L_p sense.

For differentiability in probability, we have

$$r(t) = op(\big|t_0 - t\big|)$$

which means that for any $\epsilon > 0$,

$$lim_{t\mapsto t_0} P\Big(\Big|\frac{r(t)}{t_0 - t}\Big| \geq \epsilon\Big) = 0.$$

Theorem 3.3. *Let the assumptions* $(\mathbf{H1}) - (\mathbf{H6})$ *hold, then there exists a* $\mathcal{C} > 0$ *such that*

$$\mathbf{E}\left[\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(r_j)\|^2\right\}\right] \quad \leq \quad \mathcal{C},$$

then equation (1)-(3) has a unique global solution $x(t)$ *in* $\mathscr{B}$ *and*

$$\|x_n(t) - x(t)\|^2_{\mathscr{B}} \to 0 \text{ as } n \to \infty,$$

where $\{x_n\}_{n\geq 1}$ *are the successive approximations of (1)-(3).*

Proof. **Step 1:** Let T be an arbitrary positive number $t_0 < T < \infty$ and

$$x^{(0)} \in \mathscr{B}$$

be a fixed initial approximation to (1)-(3). For $n = 0$, we have

$$\begin{aligned} x^{(1)}(t) &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(r_i)\varphi(0) + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(r_j)\int_{\zeta_{i-1}}^{\zeta_i} f(s, x_s^{(0)})ds \\ &+ \int_{\zeta_k}^{t} f(s, x_s^{(0)})ds + \sum_{i=1}^{k}\prod_{j=1}^{k} b_j(r_j)\int_{\zeta_{i-1}}^{\zeta_i}\sigma(s, x_s^{(0)})dw(s) \\ &+ \int_{\zeta_k}^{t}\sigma(s, x_s^{(0)})dw(s)\Big] I_{(\zeta_k,\zeta_{k+1}]}(t), \end{aligned}$$

so

$$\begin{aligned} \left\|x^{(1)}(t)\right\|^2 &\leq \Bigg[\sum_{k=0}^{+\infty}\Big[\Big\|\prod_{i=1}^{k} b_i(r_i)\Big\|\,\|\varphi(0)\| \\ &+ \sum_{i=1}^{k}\Big\|\prod_{j=i}^{k} b_j(r_j)\Big\|\int_{\zeta_{i-1}}^{\zeta_i}\left\|f(s, x_s^{(0)})\right\|ds + \int_{\zeta_k}^{t}\left\|f(s, x_s^{(0)})\right\|ds \\ &+ \sum_{i=1}^{k}\Big\|\prod_{j=i}^{k} b_j(r_j)\Big\|\int_{\zeta_{i-1}}^{\zeta_i}\left\|\sigma(s, x_s^{(0)})\right\|dw(s) \\ &+ \int_{\zeta_k}^{t}\left\|\sigma(s, x_s^{(0)})\right\|dw(s)\Big] I[\zeta_k, \zeta_{k+1})(t)\Bigg]^2 \\ &\leq 3\Bigg[\sum_{k=0}^{+\infty}\Big[\Big\|\prod_{i=1}^{k} b_i(r_i)\Big\|^2\|\varphi(0)\|^2 I_{(\zeta_k,\zeta_{k+1}]}(t)\Big] \end{aligned}$$

$$
\begin{aligned}
&+ \quad \Big[\sum_{k=0}^{+\infty}\Big(\sum_{i=1}^{k}\Big\|\prod_{j=i}^{k} b_j(r_j)\Big\|\int_{\zeta_{i-1}}^{\zeta_i}\Big\|f(s,x_s^{(0)})\Big\|ds \\
&+ \quad \int_{\zeta_k}^{t}\Big\|f(s,x_s^{(0)})\Big\|ds + \sum_{i=1}^{k}\Big\|\prod_{j=i}^{k} b_j(r_j)\Big\|\int_{\zeta_{i-1}}^{\zeta_i}\Big\|\sigma(s,x_s^{(0)})\Big\|dw(s) \\
&+ \quad \int_{\zeta_k}^{t}\Big\|\sigma(s,x_s^{(0)})\Big\|dw(s)\Big)I_{(\zeta_k,\zeta_{k+1}]}(t)\Big]^2\Big] \\
&\leq \quad 3\max_k\left\{\prod_{i=1}^{k}\|b_i(r_i)\|^2\right\}\mathbf{E}\,\|\varphi(0)\|^2 \\
&+ \quad 3\mathbf{E}\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\|b_j(r_j)\|\right\}\right]^2\mathbf{E}\Big(\sum_{k=0}^{+\infty}\int_{t_0}^{t}\Big\|f(s,x_s^{(0)})\Big\|dsI_{(\zeta_k,\zeta_{k+1}]}(t) \\
&+ \quad \sum_{k=0}^{+\infty}\int_{t_0}^{t}\Big\|\sigma(s,x_s^{(0)})\Big\|dw(s)I_{(\zeta_k,\zeta_{k+1}]}(t)\Big)^2 \\
&\leq \quad 3\mathcal{C}\mathbf{E}\,\|\varphi(0)\|^2 + 3\max\{1,\mathcal{C}\}\,\mathbf{E}\left(\int_{t_0}^{t}\Big\|f(s,x_s^{(0)})\Big\|ds\right)^2 \\
&+ \quad 3\max\{1,\mathcal{C}\}\,\mathcal{C}_p\mathbf{E}\left(\int_{t_0}^{t}\Big\|\sigma(s,x_s^{(0)})\Big\|ds\right)^2 \\
&\leq \quad 3\mathcal{C}\mathbf{E}\,\|\varphi(0)\|^2 \\
&+ \quad 3\max\{1,\mathcal{C}\}\,(1+\mathcal{C}_p)\,(T-t_0)\Big[\int_{t_0}^{t}\mathbf{E}\Big\|f(s,x_s^{(0)})\Big\|^2 ds \\
&+ \quad \int_{t_0}^{t}\mathbf{E}\Big\|\sigma(s,x_s^{(0)})\Big\|^2 ds\Big].
\end{aligned}
$$

By assumption (**H1**),

$$
\mathbf{E}\Big\|x^{(1)}(t)\Big\|^2 \leq 3\mathcal{C}\mathbf{E}\,\|\varphi(0)\|^2 + 3\max\{1,\mathcal{C}\}\,(1+\mathcal{C}_p)\,(T-t_0)\int_{t_0}^{t}H(s,\mathbf{E}\Big\|x^{(0)}\Big\|^2)ds.
$$

Let $u(t)$ be a global solution to equation (5) with $D = 3\max\{1,\mathcal{C}\}\,(1+\mathcal{C}_p)\,(T-t_0)$, then

$$
u(t) \quad = \quad u_0 + 3\max\{1,\mathcal{C}\}\,(1+\mathcal{C}_p)\,(T-t_0)\int_{t_0}^{t}H(s,u(s))ds.
$$

Choosing $u_0 > 3\mathcal{C}\mathbf{E}\,\|\varphi(0)\|^2$, it follows that

$$
\mathbf{E}\Big\|x^{(1)}(t)\Big\|^2 - 3\max\{1,\mathcal{C}\}\,(1+\mathcal{C}_p)\,(T-t_0)\int_{t_0}^{t}H(s,\mathbf{E}\Big\|x^{(0)}\Big\|^2)ds.
$$

$$< u(t) - 3\max\{1,\mathcal{C}\}(1+\mathcal{C}_p)(T-t_0)\int_{t_0}^{t} H(s,u(s))ds.$$

By Lemma 3.1, we get

$$\mathbf{E}\left\|x^{(1)}(t)\right\|^2 < u(t), \text{ for all } t\in[t_0,T].$$

Continuing the proof by mathematical induction, one can show that

$$\mathbf{E}\left\|x^{(n)}(t)\right\|^2 < u(t), \text{ for all } t\in[t_0,T],\ n=1,2,... \tag{8}$$

Since $u(t)$ is continuous on $[t_0,T]$, there exists an $R>0$ such that $\mathbf{E}\left\|x^{(n)}(t)\right\|^2 < R$ for all $t\in[t_0,T]$, thus we proved that for every $n=1,2,...$, the solution is uniformly bounded in $\mathscr{B}$.

Step 2: Next, in order to show that $\{x^{(n)}\}$ is Cauchy sequence in $\mathscr{B}$, Consider,

$$\begin{aligned}
&\left\|x^{(n)}-x^{(0)}\right\|_t^2\\
&\leq 3\mathbf{E}\left\|\sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k} b_i(r_i)\varphi(0)I_{[\zeta_k,\zeta_{k+1})}(t) - x^{(0)}(s)\right]\right\|^2\\
&\quad + 3\mathbf{E}\Bigg\|\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{i=1}^{k} b_i(r_i)\int_{\zeta_{i-1}}^{\zeta_i} f(s,x_s^{(n)})ds + \int_{\zeta_k}^{t} f(s,x_s^{(n)})ds\\
&\quad + \sum_{i=1}^{k}\prod_{i=1}^{k} b_i(r_i)\int_{\zeta_{i-1}}^{\zeta_i}\sigma(s,x_s^{(n)})dw(s) + \int_{\zeta_k}^{t}\sigma(s,x_s^{(n)})dw(s)\Bigg]I_{[\zeta_k,\zeta_{k+1})}(t)\Bigg\|^2\\
&\leq 3\left(\mathcal{C}\mathbf{E}\left\|\varphi(0)\right\|^2 + \mathbf{E}\left\|x^{(0)}\right\|_t^2\right) + 3\max\{1,\mathcal{C}\}(1+\mathcal{C}_p)(T-t_0)\int_{t_0}^{t} H(s,u(s))ds\\
&\leq 3\left(\mathcal{C}\mathbf{E}\left\|\varphi(0)\right\|^2 + \mathbf{E}\left\|x^{(0)}\right\|_t^2\right) + 3\max\{1,\mathcal{C}\}(1+\mathcal{C}_p)(T-t_0)\int_{t_0}^{t} H(s,q_0)ds,
\end{aligned}$$

where $q_0 = \max\{u(t), t\in[t_0,T]\} < \infty$ because $u(t)$ is continuous on $[t_0,T]$. By assumption (**H2**), there exists a time $T_1\in[t_0,T]$ such that

$$\left\|x^{(n)}-x^{(0)}\right\|_t^2 \leq \mathcal{C}_0,$$

where $\mathcal{C}_0 \equiv \mathcal{C}_0(\varphi_0, x_0, T_1-t_0, \mathcal{C}, 1+\mathcal{C}_p, q_0)$ is a constant.

We assume $T_1 < T_0$, where T_0 is defined in the Lemma 3.2 and Remark 3.1, by the above argument

$$\mathbf{E}\sup_{T_1\leq s\leq 2T_1}\left\|x^{(n)}(s)-x^{(0)}(s)\right\|^2 \quad \leq \quad Q_0, \tag{9}$$

here $Q_0 \equiv Q_0(\varphi_0, x_0, T_1 - t_0, \mathcal{C}, 1 + \mathcal{C}_p, q_0)$ is some constant. By repeating this argument on successive intervals $[2T_1, 3T_1]$, $[3T_1, 4T_1]$ and so on, it follows that

$$\left\| x^{(n)} - x^{(0)} \right\|_{\mathscr{B}}^2 \quad \leq \quad H_0, \tag{10}$$

where H_0 is constant. Next choose the initial approximation to (5) as $v_0(t) = H_0$. We next prove by mathematical induction that for every n and m, $t \in [t_0, T]$,

$$\left\| x^{(m+n)} - x^{(m)} \right\|_{\mathscr{B}}^2 \quad \leq \quad v_m(t),$$

Indeed, for every n,

$$\left\| x^{(n)} - x^{(0)} \right\|_{\mathscr{B}}^2 \leq v_0(t), \; n = 1, 2, ..., \; t_0 \leq t \leq T,$$

Suppose that (10) takes place for some m, then

$$\begin{aligned}
&\mathbf{E}\left\| x^{(m+n+1) - x^{(m)}} \right\|_t^2 \\
&= 2\mathbf{E}\Bigg[\Bigg\| \sum_{k=0}^{+\infty} \Bigg[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(r_j) \int_{\zeta_{i-1}}^{\zeta_i} [f(s, x_s^{(m+n)}) - f(s, x_s^{(m)})] ds \\
&\quad + \int_{\zeta_k}^{t} [f(s, x_s^{(m+n)}) - f(s, x_s^{(m)})] ds \\
&\quad + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(r_j) \int_{\zeta_{i-1}}^{\zeta_i} [\sigma(s, x_s^{(m+n)}) - \sigma(s, x_s^{(m)})] dw(s) \\
&\quad + \int_{\zeta_k}^{t} [\sigma(s, x_s^{(m+n)}) - \sigma(s, x_s^{(m)})] dw(s) \Bigg] I_{[\zeta_k, \zeta_{k+1})}(t) \Bigg\| \Bigg]^2 \\
&\leq 2\mathbf{E}\left[\max_{i,k} \left\{ 1, \prod_{j=i}^{k} \|b_j(r_j)\| \right\} \right]^2 \\
&\qquad \times \mathbf{E}\left(\sum_{k=0}^{+\infty} \int_{t_0}^{t} \left\| f(s, x_s^{(m+n)}) - f(s, x_s^{(m)}) \right\| ds I_{[\zeta_k, \zeta_{k+1})}(t) \right)^2 \\
&\quad + 2\mathbf{E}\left[\max_{i,k} \left\{ 1, \prod_{j=i}^{k} \|b_j(r_j)\| \right\} \right]^2 \\
&\qquad \times \mathbf{E}\left(\sum_{k=0}^{+\infty} \int_{t_0}^{t} \left\| \sigma(s, x_s^{(m+n)}) - \sigma(s, x_s^{(m)}) \right\| dw(s) I_{[\zeta_k, \zeta_{k+1})}(t) \right)^2 \\
&\leq 2 \max\{1, \mathcal{C}\} (1 + \mathcal{C}_p)(T - t_0) \int_{t_0}^{t} Z(s, \mathbf{E}\left\| x^{(m+n) - x^{(m)}} \right\|_s^2) ds
\end{aligned}$$

$$\leq 2\max\{1,\mathcal{C}\}(1+\mathcal{C}_p)(T-t_0)\int_{t_0}^{t} Z(s,v_m(s))ds = v_{m+1}(t).$$

By Lemma 3.2,

$$\left\|x^{(m+n+1)-x^{(m)}}\right\|_{\mathscr{B}}^2 \to 0 \text{ as } m\to\infty.$$

Using Remark 3.1 and equation (8), we have

$$\left\|x^{(m+n+1)-x^{(m)}}\right\|_{\mathscr{B}}^2 \to 0 \text{ as } m\to\infty.$$

and for every n. Hence $\left\{x^{(n)}\right\}_{n\geq 1}$ is a Cauchy sequence in $\mathscr{B}$. Let $x^* - \lim_{n\to\infty} x^{(n)}$. We now claim that x^* is a solution of the problem (1)-(3).

$$\begin{aligned}
&\mathbf{E}\left\|x^{(n)-x^*}\right\|_t^2\\
&\leq \max\{1,\mathcal{C}\}(1+\mathcal{C}_p)(T-t_0)\int_{t_0}^{t} Z(s,\mathbf{E}\left\|x^{(n)}-x^*\right\|_s^2)ds \to 0 \text{ as } n\to\infty,
\end{aligned}$$

Proving the claim.

Step 3 Uniqueness: Let $x(t)$ and $y(t)$ be two solutions existing with initial values φ. Then, we obtain

$$\mathbf{E}\left\|x-y\right\|_t^2 \quad\leq\quad \max\{1,\mathcal{C}\}(1+\mathcal{C}_p)(T-t_0)\int_{t_0}^{t} Z(s,\mathbf{E}\left\|x-y\right\|_s^2)ds,$$

for all $t\in[t_0,T]$, $t_0<T<\infty$. From assumptions (**H4**) and (**H5**),

$$\mathbf{E}\left\|x-y\right\|_t^2 \quad = 0$$

for all $t\in[t_0,T]$, $t_0<T<\infty$. This proves the uniqueness of the solution. This completes the proof. □

4. An application

Consider the random impulsive stochastic functional differential equations with finite delays of the form

$$\begin{aligned}
d[x(t)] &= \left[\int_{-r}^{0} u_1(\theta)x(t+\theta)d\theta\right]dt\\
&+ \left[\int_{-r}^{0} u_2(\theta)x(t+\theta)d\theta\right]dw(t),\ t\geq 0,\ t\neq\zeta_k,\\
x(\zeta_k) &= b(k)r_k x(\zeta_k^-),\ k=1,2,...,\\
x_0 &= \varphi,\ -r\leq\theta\leq 0. \qquad (11)
\end{aligned}$$

Let $r>0$, x is $\mathbb{R}$-valued stochastic process, $\zeta\in\mathscr{C}([-r,0],\mathcal{L}^2(\Omega,\mathbb{R}))$. τ_k is defined from Ω to $\mathcal{D}_k \stackrel{def}{=} (0,d_k)$ for all $k=1,2,...$, Suppose that r_k following the distribution and r_i and r_j are independent of each other as $i\neq j$ for $i,j = 1,2,...t_0=\zeta_0<\zeta_1<\zeta_2<\cdots<\zeta_k<\cdots$, and $\zeta_k=\zeta_{k-1}+r_k$ for $k=1,2,...$

Let $w(t) \in \mathbb{R}$ is a one-dimensional Brownian motions, where b is a function of k. $u_1, u_2 : [-r, 0] \to \mathbb{R}$ are continuous functions. Let

$$f(t, x_t) = \int_{-r}^{0} u_1(\theta)x(t+\theta)d\theta, \ \ \sigma(t, x_t) = \int_{-r}^{0} u_2(\theta)x(t+\theta)d\theta.$$

For $x(t+\theta) \in \mathscr{C}([-r, 0], \mathcal{L}^2(\Omega, \mathbb{R}))$, we suppose that the following conditions hold:

$$(i) \ \max_{i,k} \left\{ \prod_{j=i}^{k} \mathbf{E} \left\| b(j)(r_j) \right\|^2 \right\} < \infty,$$

$$(ii) \ \int_{-r}^{0} u_1(\theta)^2 d\theta, \int_{-r}^{0} u_2(\theta)^2 d\theta < \infty.$$

Suppose the state (i) and (ii) get satisfied. Condition (i) says that the assumption (**H6**) holds. Condition (ii) is useful to prove the assumptions (**H1**) – (**H5**). Thus system (11) has a unique mild solution.

5. Conclusion

This paper addressed the problem of existence and uniqueness result for random impulsive stochastic functional differential equations with finite delays. Successive approach and stochastic analysis techniques are employed for achieving the existence and uniqueness of mild solutions of the aforementioned random impulsive dynamical systems. For future research, it is interesting to investigate the random impulsive stochastic differential equations with Levy noise, fractional Brownian motion with time and state varying finite/infinite delays.

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