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# COEFFICIENT INEQUALITIES FOR ANALYTIC FUNCTIONS CONNECTED WITH $k$-FIBONACCI NUMBERS

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**Abstract.** In this paper, we introduce a new class $\mathcal{R}_{\lambda}^{k}$ ($\lambda \geq 1$, $k$ is any positive real number) of univalent complex functions, which consists of functions $f$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (|z| < 1)$ satisfying the subordination condition

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \frac{1+\tau_k^2 z^2}{1-k\tau_k z - \tau_k^2 z^2}, \quad \tau_k = \frac{k-\sqrt{k^2+4}}{2},$$

and investigate the Fekete-Szegö problem for the coefficients of $f \in \mathcal{R}_{\lambda}^{k}$ which are connected with $k$-Fibonacci numbers

$$F_{k,n} = \frac{(k-\tau_k)^n - \tau_k^n}{\sqrt{k^2+4}} \qquad (n \in \mathbb{N} \cup \{0\}).$$

We obtain sharp upper bound for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ when $\mu \in \mathbb{R}$. We also generalize our result for $\mu \in \mathbb{C}$.

## 1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers.

Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Let the class $\mathcal{P}(\beta)$ be defined by

$$\mathcal{P}(\beta) = \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > \beta, \ z \in \mathbb{U}\}.$$

In particular, we set $\mathcal{P}(0) = \mathcal{P}$.

For two functions $f, g \in \mathcal{H}$, we say that the function $f$ is subordinate to $g$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

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if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ (z \in \mathbb{U})\},$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalence

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions $f$ normalized by

$$f(0) = f'(0) - 1 = 0.$$

Each function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1}$$

We also denote by $\mathcal{S}$ the class of univalent functions in $\mathcal{A}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem, see [3], ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function $F = f^{-1}$ is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots =: w + \sum_{n=2}^{\infty} A_n w^n. \tag{2}$$

For a function $f \in \mathcal{S}$, the logarithmic coefficients $\delta_n$ $(n \in \mathbb{N})$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \delta_n z^n \qquad (z \in \mathbb{U}), \tag{3}$$

and play a central role in the theory of univalent functions. The idea of studying the logarithmic coefficients helped Kayumov [11] to solve Brennan's conjecture for conformal mappings. If $f \in \mathcal{S}$, then it is known that

$$|\delta_1| \leq 1$$

and

$$|\delta_2| \leq \frac{1}{2}\left(1 + 2e^{-2}\right) \approx 0,635\ldots$$

(see [3]). The problem of the best upper bounds for $|\delta_n|$ of univalent functions for $n \geq 3$ is still open.

For $f \in \mathcal{S}$ given by (1), Fekete and Szegö [8] proved a noticeable result that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu & , \quad \mu \leq 0, \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) & , \quad 0 \leq \mu \leq 1, \\ 4\mu - 3 & , \quad \mu \geq 1 \end{cases} \tag{4}$$

holds. The result is sharp in the sense that for each $\mu$ there is a function in the class under consideration for which equality holds. The coefficient functional

$$\phi_\mu(f) = a_3 - \mu a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\mu}{2}\left(f''(0)\right)^2\right)$$

on $f \in \mathcal{A}$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$\phi_\mu\left(e^{-i\theta} f\left(e^{i\theta} z\right)\right) = e^{2i\theta}\phi_\mu(f) \qquad (\theta \in \mathbb{R}).$$

By means of the principle of subordination, we introduce the following class for functions $f \in \mathcal{S}$:

**Definition 1.1.** *Let $k$ be any positive real number and $\lambda \geq 1$. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{R}_\lambda^k$ if it satisfies the condition that*

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec \tilde{p}_k(z) \qquad (z \in \mathbb{U}), \tag{5}$$

*where*

$$\tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = \frac{1 + \tau_k^2 z^2}{1 - \left(\tau_k^2 - 1\right) z - \tau_k^2 z^2} \tag{6}$$

*with*

$$\tau_k = \frac{k - \sqrt{k^2+4}}{2}. \tag{7}$$

Yılmaz Özgür and Sokół [13] showed that the function $\tilde{p}_k$ given by (6) belongs to the class $\mathcal{P}\left(\frac{k\sqrt{k^2+4}}{2(k^2+4)}\right)$.

On the other hand, the subordination (5) may be written as a linear differential equation

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = \tilde{p}_k(w(z)) \qquad (z \in \mathbb{U}), \tag{8}$$

for some $w \in \Omega$. Therefore, the solution of (8)

$$f(z) = \frac{1}{\lambda} z^{\frac{\lambda-1}{\lambda}} \int_0^z t^{\frac{1-\lambda}{\lambda}}\, \tilde{p}_k(w(t))\, \mathrm{d}t$$

gives one-to-one correspondence between classes $\Omega$ and $\mathcal{R}_\lambda^k$.

For $k=1$, the class $\mathcal{R}_{\lambda}^{k}$ reduces to the class $\mathcal{R}_{\lambda}$ which consists of functions $f\in\mathcal{A}$ defined by (1) satisfying

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\prec\tilde{p}(z),$$

where

$$\tilde{p}(z):=\tilde{p}_1(z)=\frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2},\qquad \tau:=\tau_1=\frac{1-\sqrt{5}}{2}.$$

For more details please refer to [4, 5, 6, 9, 10, 14, 15, 16, 17, 18].

For $\lambda=1$, the class $\mathcal{R}_{\lambda}^{k}$ reduces to the class $\mathcal{R}(\tilde{p}_k)$ which consists of functions $f\in\mathcal{A}$ satisfying

$$f'(z)\prec\tilde{p}_k(z)\qquad(z\in\mathbb{U}),$$

where $\tilde{p}_k$ is given by (6). In particular, we get the class $\mathcal{R}_1^1=\mathcal{R}(\tilde{p})$. The classes $\mathcal{R}(\tilde{p}_k)$ and $\mathcal{R}(\tilde{p})$ are introduced by Sumalatha et al. [19].

**Definition 1.2.** [7] *For any positive real number $k$, the $k$-Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ is defined recurrently by*

$$F_{k,n+1}=kF_{k,n}+F_{k,n-1}\qquad(n\in\mathbb{N})$$

*with initial conditions*

$$F_{k,0}=0,\qquad F_{k,1}=1.$$

*Furthermore $n^{th}$ $k$-Fibonacci number is given by*

$$F_{k,n}=\frac{(k-\tau_k)^n-\tau_k^n}{\sqrt{k^2+4}}, \tag{9}$$

*where $\tau_k$ is given by* (7).

For $k=1$, we obtain the classic Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}_0}$:

$$F_0=0,\qquad F_1=1,\qquad\text{and}\quad F_{n+1}=F_n+F_{n-1}\quad(n\in\mathbb{N}).$$

Yılmaz Özgür and Sokół [13] showed that the coefficients of the function $\tilde{p}_k(z)$ defined by (6) are connected with $k$-Fibonacci numbers. This connection is pointed out in the following theorem.

**Theorem 1.3.** [13] *Let $\{F_{k,n}\}_{n\in\mathbb{N}_0}$ be the sequence of $k$-Fibonacci numbers defined in Definition* 1.2. *If*

$$\tilde{p}_k(z)=\frac{1+\tau_k^2z^2}{1-k\tau_kz-\tau_k^2z^2}:=1+\sum_{n=1}^{\infty}\tilde{p}_{k,n}z^n, \tag{10}$$

*then we have*

$$\tilde{p}_{k,1}=k\tau_k,\quad \tilde{p}_{k,2}=\left(k^2+2\right)\tau_k^2,\quad \tilde{p}_{k,n}=\left(F_{k,n-1}+F_{k,n+1}\right)\tau_k^n\qquad(n\in\mathbb{N}). \tag{11}$$

The main purpose of this paper is to obtain Fekete-Szegö inequalities for functions belonging to the class $\mathcal{R}_{\lambda}^{k}$. For this purpose, we need the following lemmas:

**Lemma 1.4.** [3] *Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in \mathcal{P}$. Then*

$$|c_n| \leq 2 \qquad (n \in \mathbb{N}).$$

**Lemma 1.5.** [12] *If $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2 & , \quad \nu \leq 0, \\ 2 & , \quad 0 \leq \nu \leq 1, \\ 4\nu - 2 & , \quad \nu \geq 1. \end{cases}$$

*When $\nu < 0$ or $\nu > 1$, equality holds true if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then equality holds true if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, then the equality holds true if and only if*

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \qquad (0 \leq \eta \leq 1)$$

*or one of its rotations. If $\nu = 1$, then the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in the case when $\nu = 0$.*

*Although the above upper bound is sharp, in the case when $0 < \nu < 1$, it can be further improved as follows:*

$$\left|c_2 - \nu c_1^2\right| + \nu\left|c_1\right|^2 \leq 2 \qquad \left(0 < \nu \leq \frac{1}{2}\right)$$

*and*

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} < \nu \leq 1\right).$$

**Lemma 1.6.** [1] *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and*

$$p(z) \prec \tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2}, \qquad \tau_k = \frac{k - \sqrt{k^2+4}}{2},$$

*then we have*

$$\left|p_2 - \gamma p_1^2\right| \leq k\left|\tau_k\right| \max\left\{1, \ \left|k^2 + 2 - \gamma k^2\right| \frac{|\tau_k|}{k}\right\} \qquad \text{for all } \gamma \in \mathbb{C}.$$

*The above estimates are sharp.*

**Lemma 1.7.** [2] *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and*

$$p(z) \prec \tilde{p}_k(z) = \frac{1 + \tau_k^2 z^2}{1 - k\tau_k z - \tau_k^2 z^2} = 1 + \sum_{n=1}^{\infty}(F_{k,n-1} + F_{k,n+1})\tau_k^n z^n, \tag{12}$$

*then we have*

$$|p_n| \leq (F_{k,n-1} + F_{k,n+1})\left|\tau_k\right|^n \qquad (n \in \mathbb{N}). \tag{13}$$

*The result is sharp.*

## 2. Main results

In this section, we firstly give the upper bound of the Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ of functions $f\in\mathcal{R}_\lambda^k$ given by (1) when $\mu\in\mathbb{R}$.

**Theorem 2.1.** *If the function $f$ given by (1) is in the class $\mathcal{R}_\lambda^k$, then we have*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{(1+\lambda)^2\left(k^2+2\right)-\mu(1+2\lambda)k^2}{(1+\lambda)^2(1+2\lambda)}\tau_k^2 &, \quad \mu\leq \frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k+k\right]}{(1+2\lambda)k^2\tau_k}, \\ \frac{k\left|\tau_k\right|}{1+2\lambda} &, \quad \frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k+k\right]}{(1+2\lambda)k^2\tau_k}\leq\mu\leq\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k-k\right]}{(1+2\lambda)k^2\tau_k}, \\ \frac{\mu(1+2\lambda)k^2-(1+\lambda)^2\left(k^2+2\right)}{(1+\lambda)^2(1+2\lambda)}\tau_k^2 &, \quad \mu\geq\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k-k\right]}{(1+2\lambda)k^2\tau_k}. \end{cases}$$

*If* $\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k+k\right]}{(1+2\lambda)k^2\tau_k}\leq\mu\leq\frac{(1+\lambda)^2\left(k^2+2\right)}{(1+2\lambda)k^2}$, *then*

$$\left|a_3-\mu a_2^2\right|+\left(\mu-\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k+k\right]}{(1+2\lambda)k^2\tau_k}\right)\left|a_2\right|^2\leq\frac{k\left|\tau_k\right|}{1+2\lambda}.$$

*Furthermore, if* $\frac{(1+\lambda)^2\left(k^2+2\right)}{(1+2\lambda)k^2}\leq\mu\leq\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k-k\right]}{(1+2\lambda)k^2\tau_k}$, *then*

$$\left|a_3-\mu a_2^2\right|+\left(\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k-k\right]}{(1+2\lambda)k^2\tau_k}-\mu\right)\left|a_2\right|^2\leq\frac{k\left|\tau_k\right|}{1+2\lambda}.$$

*Each of these results is sharp.*

*Proof.* If $f\in\mathcal{R}_\lambda^k$, then by the principle of subordination, there exists a Schwarz function $\omega\in\Omega$ such that

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)=\tilde{p}_k(\omega(z)) \quad (z\in\mathbb{U}), \tag{14}$$

where the function $\tilde{p}_k$ is given by (10). Therefore, the function

$$g(z):=\frac{1+\omega(z)}{1-\omega(z)}=1+c_1z+c_2z^2+\cdots \quad (z\in\mathbb{U}) \tag{15}$$

is in the class $\mathcal{P}$. Now, defining the function $p(z)$ by

$$p(z)=(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)=1+p_1z+p_2z^2+\cdots, \tag{16}$$

it follows from (14) and (15) that

$$p(z)=\tilde{p}_k\left(\frac{g(z)-1}{g(z)+1}\right). \tag{17}$$

Note that

$$\omega(z)=\frac{c_1}{2}z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots$$

and so

$$\tilde{p}_k(\omega(z)) = 1 + \frac{\tilde{p}_{k,1}c_1}{2}z + \left[\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tilde{p}_{k,1} + \frac{1}{4}c_1^2\tilde{p}_{k,2}\right]z^2 + \cdots . \tag{18}$$

Thus, by using (15) in (17), and considering the values $\tilde{p}_{k,j}$ $(j = 1, 2)$ given in (11), we obtain

$$p_1 = \frac{k\tau_k}{2}c_1 \qquad \text{and} \qquad p_2 = \frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{\left(k^2+2\right)\tau_k^2}{4}c_1^2. \tag{19}$$

On the other hand, a simple calculation shows that

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = 1 + (1+\lambda)a_2 z + (1+2\lambda)a_3 z^2 + \cdots ,$$

which, in view of (16), yields

$$p_1 = (1+\lambda)a_2 \qquad \text{and} \qquad p_2 = (1+2\lambda)a_3. \tag{20}$$

Thus, we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{1+2\lambda}\left[p_2 - \mu\frac{(1+2\lambda)}{(1+\lambda)^2}p_1^2\right] \\ &= \frac{1}{1+2\lambda}\left[\frac{k\tau_k}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{\left(k^2+2\right)\tau_k^2}{4}c_1^2 - \mu\frac{(1+2\lambda)k^2\tau_k^2}{4(1+\lambda)^2}c_1^2\right] \\ &= \frac{k\tau_k}{2(1+2\lambda)}\left(c_2 - \nu c_1^2\right), \end{aligned}$$

where

$$\nu = \frac{1}{2} - \frac{(1+\lambda)^2\left(k^2+2\right) - \mu(1+2\lambda)k^2}{2(1+\lambda)^2}\frac{\tau_k}{k}.$$

The assertion of Theorem 2.1 now follows by an application of Lemma 1.5.

To show that the bounds asserted by Theorem 2.1 are sharp, we define the following functions:

$$K_{\tilde{p}_{k,n}}(z) \qquad (n \in \mathbb{N}\setminus\{1\}),$$

with

$$K_{\tilde{p}_{k,n}}(0) = 0 = K'_{\tilde{p}_{k,n}}(0) - 1,$$

by

$$(1-\lambda)\frac{K_{\tilde{p}_{k,n}}(z)}{z} + \lambda K'_{\tilde{p}_{k,n}}(z) = \tilde{p}_k\left(z^{n-1}\right), \tag{21}$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ $(0 \leq \eta \leq 1)$, with

$$F_\eta(0) = 0 = F'_\eta(0) - 1 \quad \text{and} \quad G_\eta(0) = 0 = G'_\eta(0) - 1,$$

by

$$(1-\lambda)\frac{F_\eta(z)}{z} + \lambda F'_\eta(z) = \tilde{p}_k\left(\frac{z(z+\eta)}{1+\eta z}\right)$$

and

$$(1-\lambda)\frac{G_\eta(z)}{z}+\lambda G'_\eta(z)=\tilde{p}_k\left(-\frac{z(z+\eta)}{1+\eta z}\right),$$

respectively. Then, clearly, the functions $K_{\tilde{p}_{k,n}}, F_\eta, G_\eta \in \mathcal{R}_\lambda^k$. We also write

$$K_{\tilde{p}_k}=K_{\tilde{p}_{k,2}}.$$

If $\mu<\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k+k\right]}{(1+2\lambda)k^2\tau_k}$ or $\mu>\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k-k\right]}{(1+2\lambda)k^2\tau_k}$, then the equality in Theorem 2.1 holds if and only if $f$ is $K_{\tilde{p}_k}$ or one of its rotations. When

$$\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k+k\right]}{(1+2\lambda)k^2\tau_k}<\mu<\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k-k\right]}{(1+2\lambda)k^2\tau_k},$$

the equality holds if and only if $f$ is $K_{\tilde{p}_{k,3}}$ or one of its rotations. If $\mu=\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k+k\right]}{(1+2\lambda)k^2\tau_k}$, then the equality holds if and only if $f$ is $F_\eta$ or one of its rotations. If $\mu=\frac{(1+\lambda)^2\left[\left(k^2+2\right)\tau_k-k\right]}{(1+2\lambda)k^2\tau_k}$, then the equality holds if and only if $f$ is $G_\eta$ or one of its rotations. □

If the function $f$ given by (1) is in the class $\mathcal{R}_\lambda^k$, then from (19), (20) and Lemma 1.4, we have

$$|a_2|\le\frac{k|\tau_k|}{1+\lambda} \tag{22}$$

and using the bound for $\left|a_3-\mu a_2^2\right|$ with $\mu=0$ we obtain

$$|a_3|\le\frac{(k^2+2)\tau_k^2}{1+2\lambda}. \tag{23}$$

For the general case, if we consider (16) , then we get

$$a_n=\frac{p_{n-1}}{1+(n-1)\lambda}.$$

Therefore using Lemma 1.7, we get following result.

**Theorem 2.2.** *If the function $f$ given by* (1) *is in the class $\mathcal{R}_\lambda^k$, then we have*

$$|a_n|\le\frac{F_{k,n-2}+F_{k,n}}{1+(n-1)\lambda}|\tau_k|^{n-1}\qquad(n\ge2). \tag{24}$$

*Equality holds in* (24) *for the function*

$$\widetilde{f}_{k,\lambda}(z)=\frac{1}{\lambda}z^{\frac{\lambda-1}{\lambda}}\int_0^z t^{\frac{1-\lambda}{\lambda}}\tilde{p}_k(t)\,\mathrm{d}t,$$

*where the function $\tilde{p}_k$ is given by* (10) .

Now, we give the upper bound of the Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ of functions $f\in\mathcal{R}_\lambda^k$ given by (1) when $\mu\in\mathbb{C}$.

**Theorem 2.3.** *If the function $f$ given by* (1) *is in the class $\mathcal{R}_{\lambda}^{k}$, then we have*

$$\left|a_3-\mu a_2^2\right| \leq \frac{k\left|\tau_k\right|}{1+2\lambda} \max\left\{1, \left|k^2+2-\mu\frac{1+2\lambda}{(1+\lambda)^2}k^2\right| \frac{\left|\tau_k\right|}{k}\right\}$$

*for all $\mu \in \mathbb{C}$. The result is sharp.*

*Proof.* Let the function $f \in \mathcal{A}$ given by (1) be in the class $\mathcal{R}_{\lambda}^{k}$. Define the function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ by

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) = p(z),$$

where $p(z) \prec \tilde{p}_k(z)$ and $\tilde{p}_k(z)$ is defined by $(6)$. Considering the equalities in $(20)$, for any $\mu \in \mathbb{C}$, we have

$$\left|a_3-\mu a_2^2\right| = \frac{1}{1+2\lambda}\left|p_2-\mu\frac{(1+2\lambda)}{(1+\lambda)^2}p_1^2\right|.$$

Now, by Lemma 1.6, this equality implies that

$$\left|a_3-\mu a_2^2\right| \leq \frac{k\left|\tau_k\right|}{1+2\lambda} \max\left\{1, \left|k^2+2-\mu\frac{(1+2\lambda)}{(1+\lambda)^2}k^2\right| \frac{\left|\tau_k\right|}{k}\right\}.$$

This evidently completes the proof of theorem. □

**Theorem 2.4.** *Let $f \in \mathcal{R}_{\lambda}^{k}$ be given by* (1) *be univalent and its inverse $f^{-1}$ has the coefficients of the form* (2). *Then we have*

$$\left|A_2\right| \leq \frac{k\left|\tau_k\right|}{1+\lambda}$$

*and*

$$\left|A_3\right| \leq \frac{k\left|\tau_k\right|}{1+2\lambda} \max\left\{1, \left|k^2+2-\frac{2(1+2\lambda)}{(1+\lambda)^2}k^2\right| \frac{\left|\tau_k\right|}{k}\right\}.$$

*Proof.* Let the function $f \in \mathcal{R}_{\lambda}^{k}$ and of the form (1). Then for the initial coefficients $A_2$ and $A_3$ of the inverse function $f^{-1}$ given by (2) we get

$$A_2 = -a_2 \qquad \text{and} \qquad A_3 = 2a_2^2 - a_3. \tag{25}$$

The upper bound for $A_2$ is obtained by using the equalities (19) and (20). Also the upper bound for $A_3$ is easily obtained from Theorem 2.3. □

**Theorem 2.5.** *Let $f \in \mathcal{R}_{\lambda}^{k}$ be given by* (1) *be univalent and its inverse $f^{-1}$ has the coefficients of the form* (2). *Then we have*

$$\left|A_3-\mu A_2^2\right| \leq \frac{k\left|\tau_k\right|}{1+2\lambda} \max\left\{1, \left|k^2+2-\frac{(1+2\lambda)(2-\mu)}{(1+\lambda)^2}k^2\right| \frac{\left|\tau_k\right|}{k}\right\}$$

*for all $\mu \in \mathbb{C}$.*

*Proof.* Let the function $f \in \mathcal{R}_{\lambda}^{k}$ and of the form (1). Then from (25) and (20), we get

$$\left|A_3 - \mu A_2^2\right| = \frac{1}{1+2\lambda}\left|p_2 - \frac{(1+2\lambda)(2-\mu)}{(1+\lambda)^2}p_1^2\right|.$$

Now using Lemma 1.6, we obtain the required result. □

**Theorem 2.6.** *Let $f \in \mathcal{R}_{\lambda}^{k}$ be given by (1) be univalent and the coefficients of $\log(f(z)/z)$ be given by (3). Then*

$$|\delta_1| \leq \frac{k}{2(1+\lambda)}|\tau_k|$$

*and*

$$|\delta_2| \leq \frac{k|\tau_k|}{2(1+2\lambda)} \max\left\{1, \left|k^2+2-\frac{1+2\lambda}{2(1+\lambda)^2}k^2\right|\frac{|\tau_k|}{k}\right\}.$$

*Each of these results is sharp.*

*Proof.* Let the function $f \in \mathcal{R}_{\lambda}^{k}$ and of the form (1). By differentiating (3) and equating coefficients, we have

$$\delta_1 = \frac{1}{2}a_2, \qquad \delta_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right).$$

Thus the desired results obtained from (19) and (20) for $|\delta_1|$, and from Theorem 2.3 for $|\delta_2|$. □

The following result is obtained from Theorem 2.3 (or Lemma 1.6).

**Theorem 2.7.** *Let $f \in \mathcal{R}_{\lambda}^{k}$ be given by (1) be univalent and the coefficients of $\log(f(z)/z)$ be given by (3). Then*

$$\left|\delta_2 - \mu\delta_1^2\right| \leq \frac{k|\tau_k|}{2(1+2\lambda)} \max\left\{1, \left|k^2+2-\frac{(1+2\lambda)(1+\mu)}{2(1+\lambda)^2}k^2\right|\frac{|\tau_k|}{k}\right\}.$$

## 3. Corollaries and Consequences

Setting $\lambda = 1$ in Theorem 2.1, we get the following consequence.

**Corollary 3.1.** *If the function $f$ given by (1) is in the class $\mathcal{R}(\tilde{p}_k)$, then we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{4\left(k^2+2\right)-3\mu k^2}{12}\tau_k^2 & , \quad \mu \leq \frac{4\left[\left(k^2+2\right)\tau_k+k\right]}{3k^2\tau_k}, \\ \frac{k|\tau_k|}{3} & , \quad \frac{4\left[\left(k^2+2\right)\tau_k+k\right]}{3k^2\tau_k} \leq \mu \leq \frac{4\left[\left(k^2+2\right)\tau_k-k\right]}{3k^2\tau_k}, \\ \frac{3\mu k^2-4\left(k^2+2\right)}{12}\tau_k^2 & , \quad \mu \geq \frac{4\left[\left(k^2+2\right)\tau_k-k\right]}{3k^2\tau_k}. \end{cases}$$

*If* $\frac{4\left[\left(k^2+2\right)\tau_k+k\right]}{3k^2\tau_k} \leq \mu \leq \frac{4\left(k^2+2\right)}{3k^2}$, *then*

$$\left|a_3 - \mu a_2^2\right| + \left(\mu - \frac{4\left[\left(k^2+2\right)\tau_k+k\right]}{3k^2\tau_k}\right)\left|a_2\right|^2 \leq \frac{k\left|\tau_k\right|}{3}.$$

*Furthermore, if* $\frac{4\left(k^2+2\right)}{3k^2} \leq \mu \leq \frac{4\left[\left(k^2+2\right)\tau_k-k\right]}{3k^2\tau_k}$, *then*

$$\left|a_3 - \mu a_2^2\right| + \left(\frac{4\left[\left(k^2+2\right)\tau_k-k\right]}{3k^2\tau_k} - \mu\right)\left|a_2\right|^2 \leq \frac{k\left|\tau_k\right|}{3}.$$

*Each of these results is sharp.*

**Remark 3.2.** *Note that Corollary* 3.1 *gives a worthy improvement of* [19, Theorem 3.3] *for the real values of* $\mu$.

Setting $k = 1$ in Theorem 2.1, we get the following consequence.

**Corollary 3.3.** *If the function* $f$ *given by* (1) *is in the class* $\mathcal{R}_\lambda$, *then we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{3(1+\lambda)^2-\mu(1+2\lambda)}{(1+\lambda)^2(1+2\lambda)}\tau^2 & , \quad \mu \leq \frac{(1+\lambda)^2(3\tau+1)}{(1+2\lambda)\tau}, \\ \frac{|\tau|}{1+2\lambda} & , \quad \frac{(1+\lambda)^2(3\tau+1)}{(1+2\lambda)\tau} \leq \mu \leq \frac{(1+\lambda)^2(3\tau-1)}{(1+2\lambda)\tau}, \\ \frac{\mu(1+2\lambda)-3(1+\lambda)^2}{(1+\lambda)^2(1+2\lambda)}\tau^2 & , \quad \mu \geq \frac{(1+\lambda)^2(3\tau-1)}{(1+2\lambda)\tau}. \end{cases}$$

*If* $\frac{(1+\lambda)^2(3\tau+1)}{(1+2\lambda)\tau} \leq \mu \leq \frac{3(1+\lambda)^2}{1+2\lambda}$, *then*

$$\left|a_3 - \mu a_2^2\right| + \left(\mu - \frac{(1+\lambda)^2(3\tau+1)}{(1+2\lambda)\tau}\right)\left|a_2\right|^2 \leq \frac{|\tau|}{1+2\lambda}.$$

*Furthermore, if* $\frac{3(1+\lambda)^2}{1+2\lambda} \leq \mu \leq \frac{(1+\lambda)^2(3\tau-1)}{(1+2\lambda)\tau}$, *then*

$$\left|a_3 - \mu a_2^2\right| + \left(\frac{(1+\lambda)^2(3\tau-1)}{(1+2\lambda)\tau} - \mu\right)\left|a_2\right|^2 \leq \frac{|\tau|}{1+2\lambda}.$$

*Each of these results is sharp.*

Setting $k = 1$ and $\lambda = 1$ in Theorem 2.1, we get the following consequence.

**Corollary 3.4.** *If the function* $f$ *given by* (1) *is in the class* $\mathcal{R}\left(\tilde{p}\right)$, *then we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{4-\mu}{4}\tau^2 & , \quad \mu \leq \frac{4(3\tau+1)}{3\tau}, \\ \frac{|\tau|}{3} & , \quad \frac{4(3\tau+1)}{3\tau} \leq \mu \leq \frac{4(3\tau-1)}{3\tau}, \\ \frac{\mu-4}{4}\tau^2 & , \quad \mu \geq \frac{4(3\tau-1)}{3\tau}. \end{cases}$$

*If* $\frac{4(3\tau+1)}{3\tau} \leq \mu \leq 4$, *then*

$$\left|a_3 - \mu a_2^2\right| + \left(\mu - \frac{4\left(3\tau+1\right)}{3\tau}\right)\left|a_2\right|^2 \leq \frac{|\tau|}{3}.$$

*Furthermore, if* $4 \leq \mu \leq \frac{4(3\tau-1)}{3\tau}$, *then*

$$\left|a_3 - \mu a_2^2\right| + \left(\frac{4\left(3\tau-1\right)}{3\tau} - \mu\right)\left|a_2\right|^2 \leq \frac{|\tau|}{3}.$$

*Each of these results is sharp.*

Setting $\lambda = 1$ in Theorem 2.3, we get the following consequence.

**Corollary 3.5.** *If the function* $f$ *given by* (1) *is in the class* $\mathcal{R}\left(\tilde{p}_k\right)$, *then we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{k\left|\tau_k\right|}{3}\max\left\{1,\ \left|k^2 + 2 - \frac{3\mu}{4}k^2\right|\frac{\left|\tau_k\right|}{k}\right\}$$

*for all* $\mu \in \mathbb{C}$. *The result is sharp.*

Setting $\lambda = 1$ in Theorem 2.4, we get the following consequence.

**Corollary 3.6.** *Let* $f \in \mathcal{R}\left(\tilde{p}_k\right)$ *be given by* (1) *be univalent and its inverse* $f^{-1}$ *has the coefficients of the form* (2). *Then we have*

$$\left|A_2\right| \leq \frac{k}{2}\left|\tau_k\right|$$

*and*

$$\left|A_3\right| \leq \frac{k\left|\tau_k\right|}{3}\max\left\{1,\ \frac{\left|4 - k^2\right|}{2k}\left|\tau_k\right|\right\}.$$

Setting $\lambda = 1$ in Theorem 2.5, we get the following consequence.

**Corollary 3.7.** *Let* $f \in \mathcal{R}\left(\tilde{p}_k\right)$ *be given by* (1) *be univalent and its inverse* $f^{-1}$ *has the coefficients of the form* (2). *Then we have*

$$\left|A_3 - \mu A_2^2\right| \leq \frac{k\left|\tau_k\right|}{3}\max\left\{1,\ \frac{\left|8 - \left(2 - 3\mu\right)k^2\right|}{4k}\left|\tau_k\right|\right\}$$

*for all* $\mu \in \mathbb{C}$.

Setting $\lambda = 1$ in Theorem 2.6, we get the following consequence.

**Corollary 3.8.** *Let* $f \in \mathcal{R}\left(\tilde{p}_k\right)$ *be given by* (1) *be univalent and the coefficients of* $\log\left(f(z)/z\right)$ *be given by* (3). *Then*

$$\left|\delta_1\right| \leq \frac{k}{4}\left|\tau_k\right|$$

*and*

$$\left|\delta_2\right| \leq \frac{k\left|\tau_k\right|}{6}\max\left\{1,\ \frac{\left|5k^2 + 16\right|}{8k}\left|\tau_k\right|\right\}.$$

*Each of these results is sharp.*

Setting $\lambda = 1$ in Theorem 2.7, we get the following consequence.

**Corollary 3.9.** *Let $f \in \mathcal{R}(\tilde{p}_k)$ be given by* (1) *be univalent and the coefficients of* $\log(f(z)/z)$ *be given by* (3). *Then*

$$\left|\delta_2 - \mu\delta_1^2\right| \leq \frac{k\,|\tau_k|}{6} \max\left\{1,\ \frac{\left|(5-3\mu)\,k^2+16\right|}{8k}\,|\tau_k|\right\}.$$

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