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SUMS OF DUAL TOEPLITZ PRODUCTS ON THE WEIGHTED SPACES

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Abstract. On the setting of the orthogonal complement of the weighted Bergman space, we study the problem of when a finite sum of dual Toeplitz products is compact or zero. Our results extend several known results on the unweighted space to the weighted spaces.

1. Introduction

For a fixed integer $n \geq 1$, we let B be the unit ball in the n-dimensional complex space $\mathbb{C}^n$ and $S := \partial B$ be the unit sphere. Also, let σ denote the rotation-invariant positive Borel measure on S normalized to have total mass 1. Let ρ denote a positive probability Borel measure on $[0, 1)$ and μ be the product measure of ρ and σ. So, μ is a regular Borel probability measure on B such that

$$\int_B f(z)\, d\mu(z) = \int_0^1 \int_S f(r\zeta)\, d\sigma(\zeta)\, d\rho(r)$$

holds for every $f \in L^1$. Here, $L^p := L^p(B, \mu)$ for $0 < p \leq \infty$.

The weighted Bergman space $A^2(\mu)$ is the subspace of L^2 consisting of all holomorphic functions on B. In order to ensure the completeness of $A^2(\mu)$, we are concerned with measures μ whose support is not entirely contained in a compact subset of B. In other words, we consider measures μ such that $\mu(\{z \in B : |z| \geq r\}) > 0$ for all $0 < r < 1$, Equivalently, we assume throughout the paper that the measure ρ satisfies

$$\rho([r, 1)) > 0 \quad \text{for every} \quad r \in [0, 1). \tag{1}$$

Then, by using (1), one can see that for any compact subset K of B, there exists a constant C, depending only on K and n, for which

$$\sup_{z \in K} |f(z)| \leq C \left(\int_B |f|^2\, d\mu \right)^{\frac{1}{2}} \tag{2}$$

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for every $f \in A^2(\mu)$. Then, by a routine argument, one can see that (2) implies that $A^2(\mu)$ is a closed subspace of L^2 and hence $A^2(\mu)$ is a Hilbert space. Let $P : L^2 \to A^2(\mu)$ be the Hilbert space orthogonal projection and write $A^2(\mu)^\perp$ for the orthogonal complement of $A^2(\mu)$ as usual.

Given $\varphi \in L^\infty$, we define an operator $S_\varphi : A^2(\mu)^\perp \to A^2(\mu)^\perp$ by

$$S_\varphi f = (I - P)(\varphi f)$$

for functions $f \in A^2(\mu)^\perp$. We call S_φ the *dual Toeplitz operator* with symbol φ. Clearly, S_φ is a bounded linear operator on $A^2(\mu)^\perp$.

For the case of $d\rho(r) = 2nr^{2n-1}dr$, the corresponding space $A^2(\mu)$ is the well known classical Bergman space on B. On the setting of the orthogonal complement of the classical Bergman space of the unit disk, Stroethoff and Zheng [9] characterized (essentially) commuting dual Toeplitz operators and obtained a characterization of when a product of two dual Toeplitz operators is another dual Toeplitz operator. Later, their results have been extended to the unit ball as in [6], [7] and references therein. Recently, Kong and Lu [4] recovered the known results above concerning the commutativity or product problem by characterizing zero sums of products of two dual Toeplitz operators. Also, the corresponding problems on the Hardy space, Dirichlet space and Fock space have been studied in [3], [5] and [1] respectively.

Motivated by the results mentioned above, in this paper, we consider operators which are finite sums of products of several dual Toeplitz operators on the orthogonal complement of the weighted Bergman space $A^2(\mu)$ under consideration. We first show that the compactness for operators which are finite sums of products of several dual Toeplitz operators with continuous symbol implies that the corresponding sum of products of symbols equals 0; see Theorem 5.

Using this result on the compactness, we next consider finite sums of products of two dual Toeplitz operators with pluriharmonic symbol and obtain a characterization of when such an operator equals 0; see Theorem 8. As applications, we obtain several consequences concerning commuting and product problem for dual Toeplitz operators. Also, we characterize normal dual Toeplitz operators. See Corollaries 9 and 10. Our results extend several known results on the orthogonal complement of the classical Bergman space to that of our weighted Bergman space.

2. Preliminaries

Given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ where each α_k is a nonnegative integer, we use

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!,$$

and write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $z = (z_1, \cdots, z_n) \in B$. Also, we will use the inner product notation

$$\langle f, g \rangle = \int_B f\overline{g}\, d\mu$$

and let $||f|| = \sqrt{\langle f, f \rangle}$ for functions $f, g \in L^2$. Given a multi-index α, since

$$\int_S |\zeta^\alpha|^2\, d\sigma(\zeta) = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!},$$

we have

$$\hat{\rho}(\alpha) := \int_B |z^\alpha|^2\, d\mu(z) = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} \int_0^1 r^{2|\alpha|}\, d\rho(r);$$

see Proposition 1.4.10 of [8]. By (2), we see that each point evaluation is a bounded linear functional on $A^2(\mu)$. Hence, for each $z \in B$, there exists a unique function $K_z \in A^2(\mu)$ which has the following reproducing property

$$f(z) = \langle f, K_z \rangle$$

for every $f \in A^2(\mu)$. Since the set $\{z^\alpha : |\alpha| \geq 0\}$ spans a dense subset of $A^2(\mu)$, it can be easily seen that K_z can be written as

$$K_z(w) = \sum_{|\alpha| \geq 0} \frac{1}{\hat{\rho}(\alpha)} w^\alpha \overline{z^\alpha}, \qquad w \in B. \tag{3}$$

Thus the projection P can be represented as the following integral formula

$$P\psi(z) = \int_B \psi(w) \overline{K_z(w)}\, d\mu(w), \qquad z \in B \tag{4}$$

for every $\psi \in L^2$.

3. Compact sums of dual Toeplitz products

In this section, we study the compactness for operators being finite sums of products of several dual Toeplitz operators. We will use some ideas as in [1] and [9] in our proofs.

Given $a \in B$ and $0 < \ell < 1 - |a|$, we define

$$G_{a,\ell}(z) = \overline{(z_1 - a_1)} \chi_{B(a;\ell)}(z), \qquad z \in B$$

where $B(w; r)$ is the euclidean ball centered at w with radius r and χ_E is the usual characteristic function for a set E. Also, w_j denotes the j-th component of a point $w \in B$. Put

$$g_{a,\ell} := \frac{G_{a,\ell}}{||G_{a,\ell}||}$$

for simplicity. Then, by an application of the mean value property for holomorphic functions, one sees that each $g_{a,\ell}$ belongs to $A^2(\mu)^\perp$. Also, using the Cauchy-Schwarz inequality, we have

$$|\langle \varphi, g_{a,\ell}\rangle|^2 \leq ||g_{a,\ell}||^2 \int_{B(a;\ell)} |\varphi|^2 \, d\mu = \int_{B(a;\ell)} |\varphi|^2 \, d\mu$$

for all a, ℓ and $\varphi \in L^2$, which shows that $g_{a,\ell}$ converges weakly to 0 in $A^2(\mu)^\perp$ as $\ell \to 0$.

The following proposition will be useful in our characterization of the compactness. The notation $C(B)$ stands for the set of all continuous functions on B.

Proposition 1. *For $u \in C(B) \cap L^\infty$, we have*

$$\lim_{\ell \to 0} ||S_u g_{a,\ell}|| = |u(a)|$$

for all $a \in B$.

Proof. Fix $a \in B$. Since u is continuous at a, we can see that

$$\lim_{\ell \to 0} ||(u - u(a))g_{a,\ell}|| = 0$$

and hence

$$\lim_{\ell \to 0} ||S_{u-u(a)} g_{a,\ell}|| \leq \lim_{\ell \to 0} ||(u - u(a))g_{a,\ell}|| = 0.$$

Since S_1 is the identity on $A^2(\mu)^\perp$ and each $g_{a,\ell}$ has norm 1, the above gives the desired result. The proof is complete. □

As a simple application of Proposition 1, we characterize compact dual Toeplitz operators with continuous symbol.

Corollary 2. *Let $u \in C(B) \cap L^\infty$. Then S_u is compact if and only if $u = 0$ on B.*

Given $u \in L^\infty$, the *Hankel operator* $H_u : A^2(\mu) \to A^2(\mu)^\perp$ with symbol u is the bounded linear operator defined by

$$H_u f = (I - P)(uf)$$

for $f \in A^2(\mu)$. It is not hard to see that the adjoint operator H_u^* of H_u is given by $H_u^* f = P(\bar{u} f)$ for functions $f \in A^2(\mu)^\perp$.

Lemma 3. *Let $u, v \in L^\infty$. Then we have*

$$\lim_{\ell \to 0} ||H_u H_v^* g_{a,\ell}|| = 0$$

for all $a \in B$.

Proof. Fix $a \in B$ and let $0 < \ell < \frac{1-|a|}{2}$. Note from (4) that

$$\begin{aligned} |H_v^* g_{a,\ell}(z)|^2 &= |P(\bar{v} g_{a,\ell})(z)|^2 \\ &= |\int_{B(a;\ell)} \bar{v} g_{a,\ell} \overline{K_z}\, d\mu|^2 \\ &\le ||g_{a,\ell}||^2 ||v||_\infty^2 \int_{B(a;\ell)} |K_z|^2\, d\mu \\ &= ||v||_\infty^2 \int_{B(a;\ell)} |K_z|^2\, d\mu \end{aligned}$$

for all $z \in B$. Also, by (2) and the reproducing property, one can see that there exists a constant C, depending only on a and n, such that

$$K_w(w) \le C||K_w|| = C\sqrt{\langle K_w, K_w\rangle} = C\sqrt{K_w(w)}$$

for every $w \in B(a; \frac{1-|a|}{2})$ and hence

$$\sup_{w \in B(a;\ell)} K_w(w) \le \sup_{w \in B(a; \frac{1-|a|}{2})} K_w(w) \le C^2.$$

Since $|K_z(w)| = |K_w(z)|$ for all $z, w \in B$ by (3), it follows from the reproducing property that

$$\begin{aligned} ||H_u H_v^* g_{a,\ell}||^2 &= ||(I-P)(u H_v^* g_{a,\ell})||^2 \\ &\le ||u||_\infty^2 \int_B |H_v^* g_{a,\ell}|^2\, d\mu \\ &\le ||u||_\infty^2 ||v||_\infty^2 \int_B \int_{B(a;\ell)} |K_z(w)|^2\, d\mu(w)\, d\mu(z) \\ &= ||u||_\infty^2 ||v||_\infty^2 \int_{B(a;\ell)} \int_B |K_w(z)|^2\, d\mu(z)\, d\mu(w) \\ &= ||u||_\infty^2 ||v||_\infty^2 \int_{B(a;\ell)} K_w(w) d\mu(w) \\ &\le C^2 ||u||_\infty^2 ||v||_\infty^2 \int_{B(a;\ell)} d\mu \end{aligned}$$

for each ℓ, which implies the desired result. The proof is complete. □

We let $\mathscr{H}$ be the set of all operators of the form $\sum_{j=1}^M L_j H_u H_v^*$ where $M \ge 1$ is an integer, L_j is a bounded linear operator on $A^2(\mu)^\perp$ and $u, v \in L^\infty$. By Lemma 3, one can see that

$$\lim_{\ell \to 0} ||K g_{a,\ell}|| = 0 \tag{5}$$

for every $a \in B$ and $K \in \mathscr{H}$. Since

$$S_u S_v = S_{uv} - H_u H_{\bar{v}}^* \tag{6}$$

for any $u, v \in L^\infty$, the product $S_u S_v$ can be written as a sum of an operator in $\mathscr{H}$ and a single dual Toeplitz operator with symbol uv. The following lemma shows that same is true for operators which are products of several dual Toeplitz operators.

Lemma 4. *Let $u_j \in L^\infty$ for $j = 1, \cdots, N$. Then*

$$S_{u_1} \cdots S_{u_N} = S_{u_1 u_2 \cdots u_N} + K$$

for some $K \in \mathscr{H}$.

Proof. As mentioned above, the result is true for $N = 2$. Now, suppose the result holds for $N - 1$ and then by (6)

$$\begin{aligned} S_{u_1} \cdots S_{u_N} &= S_{u_1}[S_{u_2} \cdots S_{u_N}] \\ &= S_{u_1}[S_{u_2 \cdots u_N} + K] \\ &= S_{u_1 u_2 \cdots u_N} - H_{u_1} H^*_{\overline{u_2 \cdots u_N}} + S_{u_1} K \end{aligned}$$

for some $K \in \mathscr{H}$. Since $-H_{u_1} H^*_{\overline{u_2 \cdots u_N}} + S_{u_1} K \in \mathscr{H}$, the above shows the desired result. The proof is complete. □

The following theorem is the our main result of this section and extends Theorem 5.1 of [4] to our weighted cases for continuous symbols. The case $M_i = 2$ will be used in our characterization of zero sums of dual Toeplitz products in the next section.

Theorem 5. *Let $u_{ij} \in C(B) \cap L^\infty$. If $\sum_{i=1}^N \prod_{j=1}^{M_i} S_{u_{ij}}$ is compact on $A^2(\mu)^\perp$, then $\sum_{i=1}^N \prod_{j=1}^{M_i} u_{ij} = 0$ on B.*

Proof. Put

$$S := \sum_{i=1}^N \prod_{j=1}^{M_i} S_{u_{ij}}, \qquad U := \sum_{i=1}^N \prod_{j=1}^{M_i} u_{ij}$$

for simplicity. By Lemma 4, we see that

$$S = \sum_{i=1}^N \left[S_{\prod_{j=1}^{M_i} u_{ij}} + K_i \right] = S_U + \sum_{i=1}^N K_i$$

for some $K_1, \cdots, K_N \in \mathscr{H}$. For each $a \in B$, recall $g_{a,\ell}$ converges weakly to 0 in $A^2(\mu)^\perp$ as $\ell \to 0$. Since S is compact by the assumption, (5) shows that

$$\lim_{\ell \to 0} ||S_U g_{a,\ell}|| = 0$$

for all $a \in B$. Now, the result follows from Proposition 1, as desired. □

With the same notations S and U above, Theorem 5 shows that S is a compact perturbation of a dual Toeplitz operator with continuous symbol if and only if S is a compact perturbation of S_U. Also, the following application of Theorem 5 shows that a product of several dual Toeplitz operators with harmonic symbol can only be zero or compact in a trivial case, which extends Corollary 8.8 of [9] to our weighted cases.

Corollary 6. *Let $u_1, \cdots, u_N \in L^\infty$ be harmonic. Then the following statements are equivalent.*

(a) $S_{u_1} \cdots S_{u_N}$ *is compact.*
(b) $S_{u_1} \cdots S_{u_N} = 0$
(c) $u_j = 0$ *for some* j.

4. Zero sums of dual Toeplitz products

In this section, we characterize zero sums of products of two dual Toeplitz operators with pluriharmonic symbol. Recall that a complex valued function on B is said to be *pluriharmonic* if its restriction to an arbitrary complex line is harmonic as a function of one complex variable. As is well known, each pluriharmonic function u can be decomposed as $u = f + \bar{g}$ for some $f, g \in H(B)$, the set of all holomorphic functions on B. Note that $u \in L^\infty$ implies $f, g \in A^2(\mu)$. Also, by the explicit formulas (3) and (4) for K_z and P respectively, we can see

$$Pu = f + \overline{g(0)}. \tag{7}$$

The following proposition will be useful in our characterization. We remark that the converse of (b) below does not hold in general. Consider $G_{a,\ell}$ introduced at Section 3 for example.

Proposition 7. *For $u \in L^\infty$, the following statements hold.*

(a) *The following conditions are equivalent.*
  (a1) $H_{\bar{u}} = 0$.
  (a2) $P(u\varphi) = 0$ *for every* $\varphi \in A^2(\mu)^\perp$.
  (a3) $\bar{u} \in A^2(\mu)$.
(b) *One of conditions in (a) above implies that Pu is constant.*
(c) *In addition, if u is pluriharmonic, the converse of (b) is also true.*

Proof. Since $H^*_{\bar{u}}\varphi = P(u\varphi)$ for every $\varphi \in A^2(\mu)^\perp$, we see (a1) and (a2) are equivalent. If (a1) holds, then $0 = H_{\bar{u}}1 = \bar{u} - P(\bar{u})$ and hence (a3) holds. Since $(a3) \Longrightarrow (a1)$ holds clearly, we see (a1) and (a3) are equivalent. If (a3) holds, then $\bar{u}K_z \in A^2(\mu)$ for all $z \in B$. Thus, by an application of the mean value property for holomorphic functions, we see

$$Pu(z) = \langle u, K_z \rangle = \langle 1, \bar{u}K_z \rangle = \overline{u(0)}K_z(0) = \overline{u(0)}, \qquad z \in B,$$

so (b) holds. Finally, (7) shows that (c) holds. The proof is compete. □

Now we are ready to characterize zero sums of products of two dual Toeplitz operators. Given an integer $N \geq 1$ and $a = (a_1, \cdots, a_N), b = (b_1, \cdots, b_N) \in \mathbb{C}^N$, we use the notation $a \cdot \overline{b} = a_1\overline{b_1} + \cdots + a_N\overline{b_N}$. Also, we let

$$\mathcal{R}f(z) = \sum_{i=1}^{n} z_i \frac{\partial f}{\partial z_i}(z), \qquad \widetilde{\mathcal{R}}f(z) = \sum_{i=1}^{n} \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(z)$$

for $z = (z_1, \dots, z_n) \in B$. Given $u, v \in L^\infty$, we note that

$$S_u S_v f = uvf - uP(vf) - P(uvf) + P[uP(vf)] \tag{8}$$

for functions $f \in A^2(\mu)^\perp$.

Theorem 8. *Let $u_j, v_j \in L^\infty$ be pluriharmonic for $j = 1, \cdots, N$ and $h \in L^\infty \cap C(B)$. Then $S_h = \sum_{j=1}^N S_{u_j} S_{v_j}$ if and only if $h = \sum_{j=1}^N u_j v_j$ on B and one of the following equivalent conditions holds.*

(a) $\sum_{j=1}^N \overline{P(\overline{u_j})} P(v_j \varphi) \in H(B)$ *for all* $\varphi \in A^2(\mu)^\perp$.

(b) $\sum_{j=1}^N \overline{\mathcal{R}P(\overline{u_j})} P(v_j \varphi) = 0$ *for all* $\varphi \in A^2(\mu)^\perp$.

(c) *There exist $\epsilon_j, \tau_j \in \mathbb{C}^N$ for $j = 1, \cdots, N$ such that $\epsilon_i \cdot \overline{\tau_j} = 0$ for all i, j and*

$$(P\overline{U_1}, \cdots, P\overline{U_N}) = \sum_{j=1}^N (P\overline{U_j}) \epsilon_j,$$

$$(PV_1, \cdots, PV_N) = \sum_{j=1}^N (PV_j) \tau_j$$

where $U_j = u_j - u_j(0)$ and $V_j = v_j - v_j(0)$ for each j.

Proof. Write $u_j = f_j + \overline{g_j}$ for some $f_j, g_j \in A^2(\mu)$. Using (8), we have

$$\begin{aligned}\Big[S_h - \sum_{j=1}^N S_{u_j} S_{v_j}\Big]\varphi &= \varphi\Big(h - \sum_{j=1}^N u_j v_j\Big) + \sum_{j=1}^N u_j P(v_j \varphi) \\ &\quad - P\Big[\varphi\Big(h - \sum_{j=1}^N u_j v_j\Big)\Big] - P\Big[\sum_{j=1}^N u_j P(v_j \varphi)\Big]\end{aligned} \tag{9}$$

for all $\varphi \in A^2(\mu)^\perp$. By (7), we note that $\overline{g_j} = \overline{P\overline{u_j}} - f_j(0)$ for each j. Then, by Theorem 5, we see that $S_h = \sum_{j=1}^N S_{u_j} S_{v_j}$ if and only if $h = \sum_{j=1}^N u_j v_j$ and (a) holds. Thus, in order to complete the proof, it is sufficient to show that (a), (b) and (c) are all equivalent.

First, by taking $\widetilde{\mathcal{R}}$ in (a), we see (a) $\Rightarrow$ (b) holds. Next, by Theorem 3.2 of [2], one can see that (b) holds if and only if there exist $\epsilon_j, \tau_j \in \mathbb{C}^N$ for $j = 1, \cdots, N$ such that $\epsilon_i \cdot \overline{\tau_j} = 0$ for all i, j and

$$\begin{aligned}\Big(\mathcal{R}P(\overline{u_1}), \cdots, \mathcal{R}P(\overline{u_N})\Big) &= \sum_{k=1}^N [\mathcal{R}P(\overline{u_k})]\epsilon_k, \\ \Big(P(v_1\varphi), \cdots, P(v_N\varphi)\Big) &= \sum_{k=1}^N [P(v_k\varphi)]\tau_k\end{aligned} \tag{10}$$

for all $\varphi \in A^2(\mu)^\perp$. Writing $\epsilon_k = (\epsilon_k^1, \cdots, \epsilon_k^N)$ and $\tau_k = (\tau_k^1, \cdots, \tau_k^N)$ for each k, we note that (10) is equivalent to that

$$\mathcal{R}\left[P\Big(\overline{u_j} - \sum_{k=1}^N \epsilon_k^j \overline{u_k}\Big)\right] = \mathcal{R}P(\overline{u_j}) - \sum_{k=1}^N \epsilon_k^j \mathcal{R}P(\overline{u_k}) = 0,$$
$$P\left[\Big(v_j - \sum_{k=1}^N \tau_k^j v_k\Big)\varphi\right] = P(v_j\varphi) - \sum_{k=1}^N \tau_k^j P(v_k\varphi) = 0$$

for each j and all $\varphi \in A^2(\mu)^\perp$. Then, by Proposition 7, the above is equivalent to that

$$\begin{aligned} P\Big(\overline{u_j} - \sum_{k=1}^N \epsilon_k^j \overline{u_k}\Big) &= P\Big(\overline{u_j} - \sum_{k=1}^N \epsilon_k^j \overline{u_k}\Big)(0), \\ P\Big(v_j - \sum_{k=1}^N \tau_k^j v_k\Big) &= P\Big(v_j - \sum_{k=1}^N \tau_k^j v_k\Big)(0) \end{aligned} \tag{11}$$

for each j. Noting $(P\psi)(0) = P(\psi(0))$ for every pluriharmonic $\psi \in L^\infty$ by (7), we see that (11) is equivalent to (c), thus (b) $\Longleftrightarrow$ (c) holds. Finally, if (c) holds, by a direct computation using Proposition 7 again and condition $\epsilon_i \cdot \overline{\tau_j} = 0$ for all i, j, we see that (a) holds. Thus (a), (b) and (c) are all equivalent and the proof is complete. □

The special case of $N = 2$ in Theorem 8 gives a more concrete description in the following corollary. Because the proof is similar to that of Corollary 9 of [1], we omit the details.

Corollary 9. *Let $u, v, x, y \in L^\infty$ be pluriharmonic and $h \in L^\infty \cap C(B)$. Then $S_h = S_uS_v + S_xS_y$ if and only if $h = uv + xy$ on B and one of the following conditions holds.*

(a) $u, x \in H(B)$.
(b) $\bar{v}, \bar{y} \in H(B)$.
(c) $u, \bar{y} \in H(B)$.
(d) $\bar{v}, x \in H(B)$.
(e) $u + \beta x \in H(B)$ *and* $\bar{y} - \overline{\beta v} \in H(B)$ *for some constant* $\beta \neq 0$.

As immediate applications of Corollary 9, the case of $x = v$, $y = -u$ and $h = 0$ characterizes commuting dual Toeplitz operators. Also, the case $x = y = 0$ solves the product problem. More specially, the case $h = uv$ in Corollary 10(b) below characterizes semi-commuting dual Toeplitz operators.

Corollary 10. *Let $u, v \in L^\infty$ be pluriharmonic and $h \in L^\infty \cap C(B)$.*

(a) *S_u and S_v are commuting if and only if $u, v \in H(B)$, or $\bar{u}, \bar{v} \in H(B)$, or a nontrivial linear combination of u, v is constant.*
(b) *$S_uS_v = S_h$ if and only if $h = uv$ on B, and $u \in H(B)$ or $\bar{v} \in H(B)$.*
(c) *$S_uS_v = S_{uv}$ if and only if $u \in H(B)$ or $\bar{v} \in H(B)$.*

As an application of Corollary 10(a), we characterize normal dual Toeplitz operators with pluriharmonic symbol. Note $S_u^* = S_{\bar{u}}$ for all $u \in L^\infty$.

Corollary 11. *Let $u \in L^\infty$ be pluriharmonic. Then S_u is normal if and only if $u(B)$ is contained in a line in $\mathbb{C}$.*

Proof. Without loss of generality, we may assume $u(0) = 0$. Note that S_u is normal if and only if S_u and $S_{\bar{u}}$ are commuting. Also, one can easily see that $u(B)$ is contained in a line if and only if $u = \alpha\bar{u}$ for some constant α. Now the result follows from Corollary 10(a). The proof is complete. □

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