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A FEW CLASSES OF INFINITE SERIES IDENTITIES FROM A MODULAR TRANSFORMATION FORMULA

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Abstract. The author proved a modular transformation formula for a function related to generalized non-holomorphic Eisenstein series and, using this formula, established a lot of infinite series identities. In this paper, we find more generalized series relations which contain the author's previous work.

1. Introduction

B. C. Berndt established a lot of infinite series identities from a modular transformation formula for a function coming from generalized Eisenstein series([2, 3]). Especially some of identities which he found are generalizations of Ramanujan's formulae in [7]. Following his work, the author considered a more general function which stems from generalized non-holomorphic Eisenstein series([4]) and, using a modular transformation formula for this function, established a lot of infinite series identities([5, 6]). In this paper, we find more generalized infinite series identities.

We shall introduce the following notations. Let $\mathbb{C}$ be the set of complex numbers. The branch of the argument for $z \in \mathbb{C}$ is defined by $-\pi \leq \arg z < \pi$. For any non-negative integer n, the rising factorial $(x)_n$ is defined by

$$(x)_n = x(x+1)\cdots(x+n-1) \text{ for } n > 0,\ (x)_0 = 1.$$

Let $\Gamma(s)$ be the Gamma function. The confluent hypergeometric function of the first kind ${}_1F_1(\alpha;\beta;z)$ is defined by

$${}_1F_1(\alpha;\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n n!} z^n$$

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and the confluent hypergeometric function of the second kind $U(\alpha,\beta,z)$ is defined to be

$$U(\alpha;\beta;z)=\frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\ {}_1F_1(\alpha;\beta;z)+\frac{\Gamma(\beta-1)}{\Gamma(\alpha)}z^{1-\beta}\ {}_1F_1(1+\alpha-\beta;2-\beta;z).$$

Let ${}_2F_1(\alpha,\beta;\gamma;z)$ be a hypergeometric function defined by

$${}_2F_1(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n,$$

which has the following integral representation, for $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0$ and $z\in\mathbb{C}\backslash[1,\infty)$([1], p. 65),

$${}_2F_1(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}\,dt.$$

Let $\mathbb{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}(\tau)>0\}$ be the upper half-plane. For real r_k and $h_k(k=1,2)$, let $\mathbf{r}=(r_1,r_2)$ and $\mathbf{h}=(h_1,h_2)$. For $\tau\in\mathbb{H}$ and $s_1,\ s_2\in\mathbb{C}$ with $s=s_1+s_2$, define

$$\mathcal{A}(\tau,s_1,s_2;\mathbf{r},\mathbf{h})=\sum_{m+r_1>0}\sum_{n-h_2>0}\frac{e^{2\pi i(mh_1+((m+r_1)\tau+r_2)(n-h_2))}}{(n-h_2)^{1-s}}\times U(s_2;s;4\pi(m+r_1)(n-h_2)\mathrm{Im}(\tau))$$

and

$$\bar{\mathcal{A}}(\tau,s_1,s_2;\mathbf{r},\mathbf{h})=\sum_{m+r_1>0}\sum_{n+h_2>0}\frac{e^{2\pi i(mh_1-((m+r_1)\bar{\tau}+r_2)(n+h_2))}}{(n+h_2)^{1-s}}\times U(s_1;s;4\pi(m+r_1)(n+h_2)\mathrm{Im}(\tau)).$$

Let

$$\mathcal{H}(\tau,s_1,s_2;\mathbf{r},\mathbf{h})=\mathcal{A}(\tau,s_1,s_2;\mathbf{r},\mathbf{h})+e^{\pi is}\mathcal{A}(\tau,s_1,s_2;-\mathbf{r},-\mathbf{h})$$

and

$$\bar{\mathcal{H}}(\tau,s_1,s_2;\mathbf{r},\mathbf{h})=\bar{\mathcal{A}}(\tau,s_1,s_2;\mathbf{r},\mathbf{h})+e^{\pi is}\bar{\mathcal{A}}(\tau,s_1,s_2;-\mathbf{r},-\mathbf{h}).$$

Let

$$\mathbf{H}(\tau,\bar{\tau},s_1,s_2;\mathbf{r},\mathbf{h})=\frac{1}{\Gamma(s_1)}\mathcal{H}(\tau,s_1,s_2;\mathbf{r},\mathbf{h})+\frac{1}{\Gamma(s_2)}\bar{\mathcal{H}}(\tau,s_1,s_2;\mathbf{r},\mathbf{h}).$$

For real x, y and $t\in\mathbb{C}$ with $\mathrm{Re}\ t>1$, let

$$\psi(x,y,t)=\sum_{n+y>0}\frac{e^{2\pi inx}}{(n+y)^t}$$

and

$$\Psi(x,y,t)=\psi(x,y,t)+e^{\pi it}\psi(-x,-y,t),$$

$$\Psi_{-1}(x,y,t) = \psi(x,y,t-1) + e^{\pi i t}\psi(-x,-y,t-1).$$

The characteristic function of the integers is denoted by λ, i.e.,

$$\lambda(x) = \begin{cases} 1, & \text{if } x \text{ is an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} = x - [x]$. Let

$$V\tau = \frac{a\tau + b}{c\tau + d}$$

denote a modular transformation with $c > 0$ for $\tau \in \mathbb{C}$. Let

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

The following theorem is the principal theorem which we shall use to obtain main results.

THEOREM 1.1. [4]. *Let $Q = \{\tau \in \mathbb{H} \mid \text{Re } \tau > -d/c\}$ and $\varrho = c\{R_2\} - d\{R_1\}$. Let $s_1, s_2 \in \mathbb{C}$ with $s = s_1 + s_2$ and assume that s is not an integer less than or equal to 1. Then, for $\tau \in Q$,*

$$\begin{aligned} z^{-s_1}\bar{z}^{-s_2}\mathbf{H}(V\tau, V\bar{\tau}, s_1, s_2; \mathbf{r}, \mathbf{h}) &= \mathbf{H}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &+ \lambda(R_1)(2\pi i)^{-s} e^{-\pi i(2R_1H_1 + s_2)}\Psi(-H_2, -R_2, s) \\ &- \lambda(r_1)(2\pi i)^{-s} e^{-\pi i(2r_1h_1 - s_1)} z^{-s_1}\bar{z}^{-s_2}\Psi(h_2, r_2, s) \\ &+ \lambda(H_2)(4\pi \text{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)}\Psi_{-1}(H_1, R_1, s) \\ &- \lambda(h_2)(4\pi \text{Im}(\tau))^{1-s}\frac{\Gamma(s-1)}{\Gamma(s_1)\Gamma(s_2)} z^{s_2-1}\bar{z}^{s_1-1}\Psi_{-1}(h_1, r_1, s) \\ &+ \frac{(2\pi i)^{-s}e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)}\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H), \end{aligned}$$

where $z = c\tau + d$ and

$$\begin{aligned} &\mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &= \sum_{j=1}^{c} e^{-2\pi i(H_1(j + [R_1] - c) + H_2([R_2] + 1 - [(jd + \varrho)/c] + d))} \\ &\quad \times \int_0^1 v^{s_1-1}(1-v)^{s_2-1}\int_C u^{s-1}\frac{e^{-(zv + \bar{z}(1-v))(j - \{R_1\})u/c}}{e^{-(zv + \bar{z}(1-v))u} - e^{2\pi i(cH_1 + dH_2)}} \\ &\qquad \times \frac{e^{\{(jd + \varrho)/c\}u}}{e^u - e^{-2\pi i H_2}}\, du dv, \end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$(e^{-(zv+\bar{z}(1-v))u} - e^{2\pi i(cH_1+dH_2)})(e^u - e^{-2\pi iH_2})$$

lying inside the loop and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

Let $s = s_1 + s_2$ be an integer. Suppose $H_1 = H_2 = 0$. The contour integral in Theorem 1.1 can be evaluated by using the residue theorem, i.e., we find that

$$\begin{aligned}&\int_C u^{s-1} \frac{e^{-(zv+\bar{z}(1-v))(j-\{R_1\})u/c}}{e^{-(zv+\bar{z}(1-v))u} - 1} \frac{e^{\{(\varrho+jd)/c\}u}}{e^u - 1}\, dudv\\ &= 2\pi i \sum_{k=0}^{-s+2} \frac{B_k((j-\{R_1\})/c)\bar{B}_{-s+2-k}((\varrho+jd)/c)}{k!(-s+2-k)!}(-z)^{k-1},\end{aligned}$$

where $B_n(x)$ is the n-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ (|t| < 2\pi).$$

2. A few classes of infinite series identities

In this section, we obtain new infinite series identities using Theorem 1.1. The Eulerian number $E(n, j)$ is defined to be the number of permutations of numbers from 1 to n such that exactly j numbers are greater than the previous elements. Note that $E(n, j) = E(n, n-j-1)$. For any integer n, the polylogarithm function $\mathrm{Li}_n(z)$ is defined by

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n},$$

where $z \in \mathbb{C}$ and $|z| < 1$. For $n > 0$, we see that

$$\mathrm{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{j=0}^{n-1} E(n, j) z^{n-j}.$$

Here, we set

$$V\tau = \frac{\tau - 1}{c\tau - c + 1},$$

where c is a positive integer. For any integer $k \geq 1$, let

$$\mu_k = \begin{cases} \frac{1}{2}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Put $\mathbf{r} = (r, 0)$, $0 < r < 1$, $\mathbf{h} = (0, 0)$ and $c\tau - c + 1 = \frac{\pi}{\alpha} i$. Let $s_1 = A \geq 1$, $s_2 = -B \leq 0$ and $s = s_1 + s_2 = 2N \geq 2$. Then, employing

$$\begin{aligned} &U(-B; 2N; 4\pi(m+r)n\mathrm{Im}(V\tau)) \\ &= (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-4\pi(m+r)n\mathrm{Im}(V\tau))^k}{(2N-1+k)!}, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{A}(V\tau, A, -B; \mathbf{r}, \mathbf{h}) &= \sum_{m+r>0} \sum_{n=1}^{\infty} \frac{e^{2\pi i(m+r)nV\tau}}{n^{1-2N}} \\ &\qquad \times U(-B; 2N; 4\pi(m+r)n\mathrm{Im}(V\tau)) \\ &= (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-4\alpha/c)^k}{(2N+k-1)!} \\ &\qquad \times \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-2(m+r)n(\alpha - i\pi)/c}}{(m+r)^{-k} n^{1-2N-k}} \\ &= (-1)^B (A-1)! \sum_{k=0}^{B} \binom{B}{k} \frac{(-4\alpha/c)^k}{(2N+k-1)!} \\ &\qquad \times \sum_{m=0}^{\infty} \frac{\mathrm{Li}_{(1-2N-k)}(e^{-2(m+r)(\alpha - i\pi)/c})}{(m+r)^{-k}}. \end{aligned}$$

Put $w = (m+r)(\alpha - i\pi)/c$. Applying $E(n, j) = E(n, n-j-1)$, we have

$$\begin{aligned} \mathrm{Li}_{(1-2N-k)}(e^{-2w}) &= \frac{1}{(1-e^{-2w})^{2N+k}} \sum_{j=0}^{2N+k-2} \frac{E(2N+k-1, j)}{e^{2(2N+k-j-1)w}} \\ &= \frac{2^{1-2N-k}}{\sinh^{2N+k} w} \sideset{}{'}\sum_{j=0}^{N+[k/2]-1} \cosh(2w(j+\mu_k)) \\ &\qquad \times E(2N+k-1, N+[k/2]-j-1), \end{aligned}$$

where $'$ means that if k is odd, then the term with $j = 0$ is multiplied by $\frac{1}{2}$. Thus

$$\mathcal{A}(V\tau, A, -B; \mathbf{r}, \mathbf{h}) = \frac{(-1)^B (A-1)!}{2^{2N-1}} \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!}$$

$$\times \sum_{j=0}^{N+[k/2]-1}{}' E(2N+k-1, N+[k/2]-j-1)$$
$$\times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\alpha-i\pi)/c)}{(m+r)^{-k}\sinh^{2N+k}((m+r)(\alpha-i\pi)/c)}.$$

Similarly we have

$$\mathcal{A}(V\tau, A, -B; -\mathbf{r}, -\mathbf{h}) = \frac{(-1)^B(A-1)!}{2^{2N-1}} \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!}$$
$$\times \sum_{j=0}^{N+[k/2]-1}{}' E(2N+k-1, N+[k/2]-j-1)$$
$$\times \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\alpha-i\pi)/c)}{(m-r)^{-k}\sinh^{2N+k}((m-r)(\alpha-i\pi)/c)}.$$

Hence we find

$$\begin{aligned}\mathbf{H}(V\tau, A, -B; \mathbf{r}, \mathbf{h}) &= \frac{1}{(A-1)!}\left(\mathcal{A}(V\tau, A, -B; \mathbf{r}, \mathbf{h}) + \mathcal{A}(V\tau, A, -B; -\mathbf{r}, -\mathbf{h})\right) \\ &= \frac{(-1)^B}{2^{2N-1}} \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} \\ &\times \sum_{j=0}^{N+[k/2]-1}{}' E(2N+k-1, N+[k/2]-j-1) \\ &\times \left(\sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\alpha-i\pi)/c)}{(m+r)^{-k}\sinh^{2N+k}((m+r)(\alpha-i\pi)/c)} \right. \\ &\left. + \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\alpha-i\pi)/c)}{(m-r)^{-k}\sinh^{2N+k}((m-r)(\alpha-i\pi)/c)} \right).\end{aligned}$$

By the same way, we obtain that

$$\begin{aligned}\mathcal{A}(\tau, A, -B; R, H) &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{e^{-2(m+r)n(\beta+i\pi)/c}}{n^{1-2N}} U(-B; 2N; 4\beta(m+r)n/c) \\ &= \frac{(-1)^B(A-1)!}{2^{2N-1}} \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \\ &\times \sum_{j=0}^{N+[k/2]-1}{}' E(2N+k-1, N+[k/2]-j-1)\end{aligned}$$

$$\times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\beta+i\pi)/c)}{(m+r)^{-k}\sinh^{2N+k}((m+r)(\beta+i\pi)/c)},$$

$$\begin{aligned}\mathcal{A}(\tau, A, -B; -R, -H) &= \frac{(-1)^B (A-1)!}{2^{2N-1}} \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \\ &\quad \times \sideset{}{'}\sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ &\quad \times \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\beta+i\pi)/c)}{(m-r)^{-k}\sinh^{2N+k}((m-r)(\beta+i\pi)/c)}\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}(\tau, A, -B; R, H) &= \frac{(-1)^B}{2^{2N-1}} \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} \\ &\quad \times \sideset{}{'}\sum_{j=0}^{N+[k/2]-1} E(2N+k-1, N+[k/2]-j-1) \\ &\quad \times \left(\sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\beta+i\pi)/c)}{(m+r)^{-k}\sinh^{2N+k}((m+r)(\beta+i\pi)/c)} \right. \\ &\quad \left. + \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\beta+i\pi)/c)}{(m-r)^{-k}\sinh^{2N+k}((m-r)(\beta+i\pi)/c)} \right).\end{aligned}$$

Using the residue theorem and the integral representation of the hypergeometric function, we have

$$\begin{aligned}&\frac{(2\pi i)^{-s} e^{-\pi i s_2}}{\Gamma(s_1)\Gamma(s_2)} \mathbf{L}(\tau, \bar{\tau}, s_1, s_2; R, H) \\ &= \frac{(-1)^B (2\pi i)^{1-2N}}{\Gamma(2N)} \sum_{j=1}^{c} \sum_{k=0}^{2-2N} \frac{B_k((j-\{R_1\})/c)\bar{B}_{2-2N-k}((\varrho+j(1-c))/c)}{k!(2-2N-k)!} \\ &\qquad\qquad \times \left(-\frac{i\pi}{\alpha}\right)^{k-1} {}_2F_1(-B, 1-k, 2N; 2) \\ &= \begin{cases} \dfrac{c(1+(-1)^B)}{4\beta(B+1)}, & N=1, \\ 0, & N \geq 2. \end{cases}\end{aligned}$$

Finally, we obtain the following theorem.

Theorem 2.1. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < r < 1$. For any integers $B \geq 0$ and $N \geq 1$,*

$$\begin{aligned}
&(-1)^B \alpha^N \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\alpha/c)^k}{(2N+k-1)!} {\sum_{j=0}^{N+[k/2]-1}}' E(2N+k-1, N+[k/2]-j-1) \\
&\qquad \times \Bigg(\sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\alpha - i\pi)/c)}{(m+r)^{-k}\sinh^{2N+k}((m+r)(\alpha - i\pi)/c)} \\
&\qquad\qquad + \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\alpha - i\pi)/c)}{(m-r)^{-k}\sinh^{2N+k}((m-r)(\alpha - i\pi)/c)} \Bigg) \\
&= (-\beta)^N \sum_{k=0}^{B} \binom{B}{k} \frac{(-2\beta/c)^k}{(2N+k-1)!} {\sum_{j=0}^{N+[k/2]-1}}' E(2N+k-1, N+[k/2]-j-1) \\
&\qquad \times \Bigg(\sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\mu_k)(\beta + i\pi)/c)}{(m+r)^{-k}\sinh^{2N+k}((m+r)(\beta + i\pi)/c)} \\
&\qquad\qquad + \sum_{m=0}^{\infty} \frac{\cosh(2(m-r)(j+\mu_k)(\beta + i\pi)/c)}{(m-r)^{-k}\sinh^{2N+k}((m-r)(\beta + i\pi)/c)} \Bigg) \\
&\quad -\nu_N(B,c),
\end{aligned}$$

where

$$\nu_N(B,c) = \begin{cases} \dfrac{c(1+(-1)^B)}{2(B+1)}, & N = 1, \\ 0, & N \geq 2. \end{cases}$$

The case of $r = \frac{1}{2}$ in Theorem 2.1 is evaluated in [5]

Corollary 2.2. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. For any integer $N \geq 1$,*

$$\begin{aligned}
&\alpha^N \sum_{j=0}^{N-1} E(2N-1, N-j-1) \\
&\qquad \times \sum_{m \equiv 1,2 \pmod 3} \frac{\cosh(2mj(\alpha - i\pi)/3)}{\sinh^{2N}(m(\alpha - i\pi)/3)} \\
&= (-\beta)^N \sum_{j=0}^{N-1} E(2N-1, N-j-1) \\
&\qquad \times \sum_{m \equiv 1,2 \pmod 3} \frac{\cosh(2mj(\beta + i\pi)/3)}{\sinh^{2N}(m(\beta + i\pi)/3)} - \nu_N(0,1).
\end{aligned}$$

Proof. Put $r = \frac{1}{3}$, $B = 0$ and $c = 1$ in Theorem 2.1. □

Corollary 2.3. *We have*

$$\sum_{m=1}^{\infty}\operatorname{csch}^2(m\pi(1-i)/3)+\sum_{m=1}^{\infty}\operatorname{csch}^2(m\pi(1+i)/3)$$
$$=2\sum_{m=1}^{\infty}\operatorname{csch}^2(m\pi)-\frac{2}{\pi}.$$

Proof. Let $\alpha=\beta=\pi$ and $N=1$ in Corollary 2.2. Then

$$\sum_{m\equiv 1,2 \pmod 3}\operatorname{csch}^2(m\pi(1-i)/3)$$
$$=-\sum_{m\equiv 1,2 \pmod 3}\operatorname{csch}^2(m\pi(1+i)/3)-\frac{2}{\pi}.$$

Use $\sinh(m\pi(1\pm i))=(-1)^m\sinh(m\pi)$ to obtain the result. □

The identity in Corollary 2.3 can be written as a different form. By using

$$\operatorname{csch}(m\pi(1\pm i)/3)=\frac{\sinh(m\pi/3)\cos(m\pi/3)}{\sinh^2(m\pi/3)+\sin^2(m\pi/3)}\mp i\frac{\cosh(m\pi/3)\sin(m\pi/3)}{\sinh^2(m\pi/3)+\sin^2(m\pi/3)},$$

we see that

$$\operatorname{csch}^2(m\pi(1-i)/3)+\operatorname{csch}^2(m\pi(1+i)/3)$$
$$=2\frac{\sinh^2(m\pi/3)\cos^2(m\pi/3)-\cosh^2(m\pi/3)\sin^2(m\pi/3)}{(\sinh^2(m\pi/3)+\sin^2(m\pi/3))^2}$$
$$=2\frac{\sinh^2(m\pi/3)-\cosh(2m\pi/3)\sin^2(m\pi/3)}{(\sinh^2(m\pi/3)+\sin^2(m\pi/3))^2}.$$

Applying $\sin^2(\frac{m\pi}{3})=\frac{3}{4}$ for $m\equiv 1,\ 2 \pmod 3$, we have

$$\sum_{m\equiv 1 \pmod 3}\frac{4\sinh^2(m\pi/3)-3\cosh(2m\pi/3)}{(4\sinh^2(m\pi/3)+3)^2}$$
$$=-\sum_{m\equiv 2 \pmod 3}\frac{4\sinh^2(m\pi/3)-3\cosh(2m\pi/3)}{(4\sinh^2(m\pi/3)+3)^2}-\frac{1}{4\pi}.$$

Now we compute another infinite series identity by changing the values of $s_1,\ s_2$. Put $\mathbf{r}=(r,0),\ 0<r<1$ and $\mathbf{h}=(0,0)$. Let $s_1=A+1\geq 1$, $s_2=B+1\geq 1$ and $s=s_1+s_2=2N+2\geq 4$. In this case, for any integer $k \geq 1$, we put

$$\hat{k} = \left[\frac{k-1}{2}\right], \ \upsilon_k = \begin{cases} 0, & k \text{ odd}, \\ \dfrac{1}{2}, & k \text{ even}. \end{cases}$$

Let $z = c\tau - c + 1$ and put $z = \frac{\pi}{\alpha} i$. Then, using

$$e^{2\pi i((m+r)\tau - r)n} = e^{-2n(m+r)(\beta + i\pi)/c}$$

and

$$\begin{aligned} &U(B+1; 2N+2; 4\pi(m+r)n\mathrm{Im}(\tau)) \\ &= \frac{1}{B!}\sum_{k=0}^{A}\binom{A}{k}\frac{(2N-k)!}{(4\pi(m+r)n\beta/c)^{2N-k+1}}, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{A}(\tau, s_1, s_2; R, H) &= \frac{1}{B!}\sum_{k=0}^{A}\binom{A}{k}\frac{(2N-k)!}{(4\beta/c)^{2N-k+1}}\sum_{m=0}^{\infty}\frac{\mathrm{Li}_{-k}(e^{-2(m+r)(\beta+i\pi)/c})}{(m+r)^{2N-k+1}} \\ &= \frac{1}{B!}\sum_{k=1}^{A}\binom{A}{k}\frac{2^{-k}(2N-k)!}{(4\beta/c)^{2N-k+1}}\sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \\ &\quad \times \sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\beta+i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\beta+i\pi)/c)} \\ &\quad + \frac{(2N)!}{B!(4\beta/c)^{2N+1}}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta+i\pi)/c}-1}, \end{aligned}$$

where $'$ means that if k is odd, then the term with $j = 0$ is multiplied by $\frac{1}{2}$. By the same way, other functions can be evaluated, namely, we have

$$\begin{aligned} \mathcal{A}(\tau, s_1, s_2; -R, -H) &= \frac{1}{B!}\sum_{k=1}^{A}\binom{A}{k}\frac{2^{-k}(2N-k)!}{(4\beta/c)^{2N-k+1}}\sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \\ &\quad \times \sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\beta+i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\beta+i\pi)/c)} \\ &\quad + \frac{(2N)!}{B!(4\beta/c)^{2N+1}}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta+i\pi)/c}-1}, \end{aligned}$$

$$\bar{\mathcal{A}}(\tau, s_1, s_2; R, H) = \frac{1}{A!}\sum_{k=1}^{B}\binom{B}{k}\frac{2^{-k}(2N-k)!}{(4\beta/c)^{2N-k+1}}\sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j)$$

$$\times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\upsilon_k)(\beta - i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\beta - i\pi)/c)} + \frac{(2N)!}{A!(4\beta/c)^{2N+1}} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta-i\pi)/c} - 1},$$

$$\bar{\mathcal{A}}(\tau, s_1, s_2; -R, -H) = \frac{1}{A!} \sum_{k=1}^{B} \binom{B}{k} \frac{2^{-k}(2N-k)!}{(4\beta/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+\upsilon_k)(\beta - i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\beta - i\pi)/c)} + \frac{(2N)!}{A!(4\beta/c)^{2N+1}} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta-i\pi)/c} - 1},$$

$$\mathcal{A}(V\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \frac{1}{B!} \sum_{k=1}^{A} \binom{A}{k} \frac{2^{-k}(2N-k)!}{(4\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\upsilon_k)(\alpha - i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\alpha - i\pi)/c)} + \frac{(2N)!}{B!(4\alpha/c)^{2N+1}} \sum_{m=0}^{\infty} \frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha-i\pi)/c} - 1},$$

$$\mathcal{A}(V\tau, s_1, s_2; -\mathbf{r}, -\mathbf{h}) = \frac{1}{B!} \sum_{k=1}^{A} \binom{A}{k} \frac{2^{-k}(2N-k)!}{(4\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+\upsilon_k)(\alpha - i\pi)/c)}{(m-r)^{2N-k+1} \sinh^{k+1}((m-r)(\alpha - i\pi)/c)} + \frac{(2N)!}{B!(4\alpha/c)^{2N+1}} \sum_{m=1}^{\infty} \frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha-i\pi)/c} - 1},$$

$$\bar{\mathcal{A}}(V\tau, s_1, s_2; \mathbf{r}, \mathbf{h}) = \frac{1}{A!} \sum_{k=1}^{B} \binom{B}{k} \frac{2^{-k}(2N-k)!}{(4\alpha/c)^{2N-k+1}} \sum_{j=0}^{\hat{k}}{}' E(k, \hat{k}-j) \times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\upsilon_k)(\alpha + i\pi)/c)}{(m+r)^{2N-k+1} \sinh^{k+1}((m+r)(\alpha + i\pi)/c)}$$

$$+\frac{(2N)!}{A!(4\alpha/c)^{2N+1}}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha+i\pi)/c}-1},$$

$$\begin{aligned}\bar{\mathcal{A}}(V\tau,s_1,s_2;-\mathbf{r},-\mathbf{h}) &= \frac{1}{A!}\sum_{k=1}^{B}\binom{B}{k}\frac{2^{-k}(2N-k)!}{(4\alpha/c)^{2N-k+1}}\sideset{}{'}\sum_{j=0}^{\hat{k}}E(k,\hat{k}-j)\\ &\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\alpha+i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\alpha+i\pi)/c)}\\ &+\frac{(2N)!}{A!(4\alpha/c)^{2N+1}}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha+i\pi)/c}-1}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\Psi_{-1}(H_1,R_1,s) &= \psi(0,r,2N+1)+\psi(0,-r,2N+1)\\ &= \zeta(2N+1,r)+\zeta(2N+1,1-r),\end{aligned}$$

$$\begin{aligned}\Psi_{-1}(h_1,r_1,s) &= \psi(0,r,2N+1)+\psi(0,-r,2N+1)\\ &= \zeta(2N+1,r)+\zeta(2N+1,1-r),\end{aligned}$$

$$z^{-s_1}\bar{z}^{-s_2} = (-1)^{B+1}\alpha^{N+1}(-\beta)^{-N-1},$$

$$z^{s_2-1}\bar{z}^{s_1-1} = (-1)^{B}\alpha^{-N}(-\beta)^{N}.$$

Note that $s=2N+2\geq 4$. Then, by using the residue theorem, we have

$$\frac{1}{\Gamma(s_1)(s_2)}\mathbf{L}(\tau,\bar{\tau},s_1,s_2;R,H)=0.$$

We now obtain the following theorem.

Theorem 2.4. *Let $\alpha,\ \beta>0$ with $\alpha\beta=\pi^2$ and let $0<r<1$. For any integers $A\geq 0$, $B\geq 0$ and $N\geq 1$ with $A+B=2N$,*

$$\begin{aligned}(-1)^{B}\alpha^{-N}&\sum_{k=1}^{A}\binom{A}{k}\frac{(2N-k)!}{(2\alpha/c)^{-k}}\sideset{}{'}\sum_{j=0}^{\hat{k}}E(k,\hat{k}-j)\\ &\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\alpha-i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\alpha-i\pi)/c)}\\ &+(-1)^{B}(2N)!\alpha^{-N}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha-i\pi)/c}-1}\\ +(-1)^{B}\alpha^{-N}&\sum_{k=1}^{A}\binom{A}{k}\frac{(2N-k)!}{(2\alpha/c)^{-k}}\sideset{}{'}\sum_{j=0}^{\hat{k}}E(k,\hat{k}-j)\end{aligned}$$

$$\times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+\upsilon_k)(\alpha-i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\alpha-i\pi)/c)}$$

$$+(-1)^B(2N)!\alpha^{-N}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha-i\pi)/c}-1}$$

$$+(-1)^B\alpha^{-N}\sum_{k=1}^{B}\binom{B}{k}\frac{(2N-k)!}{(2\alpha/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)$$

$$\times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\upsilon_k)(\alpha+i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\alpha+i\pi)/c)}$$

$$+(-1)^B(2N)!\alpha^{-N}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha+i\pi)/c}-1}$$

$$+(-1)^B\alpha^{-N}\sum_{k=1}^{B}\binom{B}{k}\frac{(2N-k)!}{(2\alpha/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)$$

$$\times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+\upsilon_k)(\alpha+i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\alpha+i\pi)/c)}$$

$$+(-1)^B(2N)!\alpha^{-N}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha+i\pi)/c}-1}$$

$$+(-1)^B(2N)!\alpha^{-N}\left(\zeta(2N+1,r)+\zeta(2N+1,1-r)\right)$$

$$=(-1)^N\beta^{-N}\sum_{k=1}^{A}\binom{A}{k}\frac{(2N-k)!}{(2\beta/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)$$

$$\times \sum_{m=0}^{\infty} \frac{\cosh(2(m+r)(j+\upsilon_k)(\beta+i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\beta+i\pi)/c)}$$

$$+(-1)^N(2N)!\beta^{-N}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta+i\pi)/c}-1}$$

$$+(-1)^N\beta^{-N}\sum_{k=1}^{A}\binom{A}{k}\frac{(2N-k)!}{(2\beta/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)$$

$$\times \sum_{m=1}^{\infty} \frac{\cosh(2(m-r)(j+\upsilon_k)(\beta+i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\beta+i\pi)/c)}$$

$$+(-1)^N(2N)!\beta^{-N}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta+i\pi)/c}-1}$$

$$
\begin{aligned}
&+(-1)^N\beta^{-N}\sum_{k=1}^{B}\binom{B}{k}\frac{(2N-k)!}{(2\beta/c)^{-k}}{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\beta-i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\beta-i\pi)/c)}\\
&\qquad+(-1)^N(2N)!\beta^{-N}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta-i\pi)/c}-1}\\
&+(-1)^N\beta^{-N}\sum_{k=1}^{B}\binom{B}{k}\frac{(2N-k)!}{(2\beta/c)^{-k}}{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\beta-i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\beta-i\pi)/c)}\\
&\qquad+(-1)^N(2N)!\beta^{-N}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta-i\pi)/c}-1}\\
&+(-1)^N(2N)!\beta^{-N}\left(\zeta(2N+1,r)+\zeta(2N+1,1-r)\right).
\end{aligned}
$$

Note that the series in Theorem 2.4 contains sums of complex conjugates, which shows they are real. A specific case($r=\frac{1}{2}$) of Theorem 2.4 is established in [6]

Corollary 2.5. *Let $0<r<1$ and let $A\not\equiv B \pmod 4$. Then*

$$
\begin{aligned}
&\sum_{k=1}^{A}\binom{A}{k}\frac{(2N-k)!}{(2\pi/c)^{-k}}{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\pi-i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\pi-i\pi)/c)}\\
&+\sum_{k=1}^{A}\binom{A}{k}\frac{(2N-k)!}{(2\pi/c)^{-k}}{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\pi-i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\pi-i\pi)/c)}\\
&+\sum_{k=1}^{B}\binom{B}{k}\frac{(2N-k)!}{(2\pi/c)^{-k}}{\sum_{j=0}^{\hat{k}}}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\pi+i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\pi+i\pi)/c)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{B}\binom{B}{k}\frac{(2N-k)!}{(2\pi/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j) \\
&\qquad\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\pi+i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\pi+i\pi)/c)} \\
&+\sum_{m=0}^{\infty}\frac{(2N)!}{(m+r)^{2N+1}}\frac{\cos(2\pi(m+r)/c)-e^{-2\pi(m+r)/c}}{\cosh(2\pi(m+r)/c)-\cos(2\pi(m+r)/c)} \\
&+\sum_{m=1}^{\infty}\frac{(2N)!}{(m-r)^{2N+1}}\frac{\cos(2\pi(m-r)/c)-e^{-2\pi(m-r)/c}}{\cosh(2\pi(m-r)/c)-\cos(2\pi(m-r)/c)} \\
&=\frac{(2N)!}{2}\left(\zeta(2N+1,r)+\zeta(2N+1,1-r)\right).
\end{aligned}
$$

Proof. Put $\alpha=\beta=\pi$ in Theorem 2.4 and assume that B and N have the different parity. Apply

$$
\frac{1}{e^{x+iy}-1}+\frac{1}{e^{x-iy}-1}=\frac{\cos y-e^{-x}}{\cosh x-\cos y}
$$

to obtain the desired result. □

COROLLARY 2.6. *Let $0<r<1$. For any positive integer N,*

$$
\begin{aligned}
&\sum_{k=1}^{4N-2}\frac{(2\pi)^k}{k!}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j) \\
&\qquad\qquad\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\pi-i\pi))}{(m+r)^{4N-k-1}\sinh^{k+1}((m+r)(\pi-i\pi))} \\
&+\sum_{k=1}^{4N-2}\frac{(2\pi)^k}{k!}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j) \\
&\qquad\qquad\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\pi-i\pi))}{(m-r)^{4N-k-1}\sinh^{k+1}((m-r)(\pi-i\pi))} \\
&+\sum_{m=0}^{\infty}\frac{1}{(m+r)^{4N-1}}\frac{\cos(2\pi r)-e^{-2\pi(m+r)}}{\cosh(2\pi(m+r))-\cos(2\pi r)} \\
&+\sum_{m=1}^{\infty}\frac{1}{(m-r)^{4N-1}}\frac{\cos(2\pi r)-e^{-2\pi(m-r)}}{\cosh(2\pi(m-r))-\cos(2\pi r)} \\
&=\frac{1}{2}\left(\zeta(4N-1,r)+\zeta(4N-1,1-r)\right).
\end{aligned}
$$

Proof. Put $c=1$, $B=0$ and $\alpha=\beta=\pi$ in Theorem 2.4. Replace N by $2N-1$. □

The following two corollaries show elegant symmetric series relations with respect to α and β.

Corollary 2.7. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < r < 1$. For any integer $N \geq 1$,*

$$\begin{aligned}
&\alpha^{-N}\sum_{k=1}^{N}\binom{N}{k}\frac{(2N-k)!}{(2\alpha/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j) \\
&\quad\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\alpha-i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\alpha-i\pi)/c)} \\
&\quad+(2N)!\alpha^{-N}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha-i\pi)/c}-1} \\
&+\alpha^{-N}\sum_{k=1}^{N}\binom{N}{k}\frac{(2N-k)!}{(2\alpha/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j) \\
&\quad\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\alpha-i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\alpha-i\pi)/c)} \\
&\quad+(2N)!\alpha^{-N}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha-i\pi)/c}-1} \\
&+\alpha^{-N}\sum_{k=1}^{N}\binom{N}{k}\frac{(2N-k)!}{(2\alpha/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j) \\
&\quad\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\alpha+i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\alpha+i\pi)/c)} \\
&\quad+(2N)!\alpha^{-N}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\alpha+i\pi)/c}-1} \\
&+\alpha^{-N}\sum_{k=1}^{N}\binom{N}{k}\frac{(2N-k)!}{(2\alpha/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j) \\
&\quad\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\alpha+i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\alpha+i\pi)/c)} \\
&\quad+(2N)!\alpha^{-N}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\alpha+i\pi)/c}-1} \\
&+(2N)!\alpha^{-N}\left(\zeta(2N+1,r)+\zeta(2N+1,1-r)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \beta^{-N}\sum_{k=1}^{N}\binom{N}{k}\frac{(2N-k)!}{(2\beta/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\beta+i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\beta+i\pi)/c)}\\
&\qquad+(2N)!\beta^{-N}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta+i\pi)/c}-1}\\
&+\beta^{-N}\sum_{k=1}^{N}\binom{N}{k}\frac{(2N-k)!}{(2\beta/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\beta+i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\beta+i\pi)/c)}\\
&\qquad+(2N)!\beta^{-N}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta+i\pi)/c}-1}\\
&+\beta^{-N}\sum_{k=1}^{N}\binom{N}{k}\frac{(2N-k)!}{(2\beta/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=0}^{\infty}\frac{\cosh(2(m+r)(j+\upsilon_k)(\beta-i\pi)/c)}{(m+r)^{2N-k+1}\sinh^{k+1}((m+r)(\beta-i\pi)/c)}\\
&\qquad+(2N)!\beta^{-N}\sum_{m=0}^{\infty}\frac{(m+r)^{-2N-1}}{e^{2(m+r)(\beta-i\pi)/c}-1}\\
&+\beta^{-N}\sum_{k=1}^{N}\binom{N}{k}\frac{(2N-k)!}{(2\beta/c)^{-k}}\sum_{j=0}^{\hat{k}}{}' E(k,\hat{k}-j)\\
&\qquad\times\sum_{m=1}^{\infty}\frac{\cosh(2(m-r)(j+\upsilon_k)(\beta-i\pi)/c)}{(m-r)^{2N-k+1}\sinh^{k+1}((m-r)(\beta-i\pi)/c)}\\
&\qquad+(2N)!\beta^{-N}\sum_{m=1}^{\infty}\frac{(m-r)^{-2N-1}}{e^{2(m-r)(\beta-i\pi)/c}-1}\\
&+(2N)!\beta^{-N}\left(\zeta(2N+1,r)+\zeta(2N+1,1-r)\right).
\end{aligned}
$$

Proof. Let $A = B$ in Theorem 2.4. □

COROLLARY 2.8. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$ and let $0 < r < 1$. Then*

$$
\sum_{m=0}^{\infty}\frac{2\cos(2\pi r)\cosh(2(m+r)\alpha)}{(m+r)^2(\cosh(2(m+r)\alpha)-\cos(2\pi r))^2}
$$

$$
\begin{aligned}
&+\sum_{m=1}^{\infty}\frac{2\cos(2\pi r)\cosh(2(m-r)\alpha)}{(m-r)^2(\cosh(2(m-r)\alpha)-\cos(2\pi r))^2}\\
&+\alpha^{-1}\sum_{m=0}^{\infty}\frac{\cos(2\pi r)-e^{-2(m+r)\alpha}}{(m+r)^3(\cosh(2(m+r)\alpha)-\cos(2\pi r))}\\
&\alpha^{-1}\sum_{m=1}^{\infty}\frac{\cos(2\pi r)-e^{-2(m-r)\alpha}}{(m-r)^3(\cosh(2(m-r)\alpha)-\cos(2\pi r))}\\
&+\alpha^{-1}\left(\zeta(3,r)+\zeta(3,1-r)\right)\\
&=\sum_{m=0}^{\infty}\frac{2\cos(2\pi r)\cosh(2(m+r)\beta)}{(m+r)^2(\cosh(2(m+r)\beta)-\cos(2\pi r))^2}\\
&+\sum_{m=1}^{\infty}\frac{2\cos(2\pi r)\cosh(2(m-r)\beta)}{(m-r)^2(\cosh(2(m-r)\beta)-\cos(2\pi r))^2}\\
&+\beta^{-1}\sum_{m=0}^{\infty}\frac{\cos(2\pi r)-e^{-2(m+r)\beta}}{(m+r)^3(\cosh(2(m+r)\beta)-\cos(2\pi r))}\\
&\beta^{-1}\sum_{m=1}^{\infty}\frac{\cos(2\pi r)-e^{-2(m-r)\beta}}{(m-r)^3(\cosh(2(m-r)\beta)-\cos(2\pi r))}\\
&+\beta^{-1}\left(\zeta(3,r)+\zeta(3,1-r)\right).
\end{aligned}
$$

Proof. Put $A = B = 1$ and $c = 1$ in Corollary 2.7. Using

$$\sinh^2(x\pm iy)=\frac{1}{2}(\cosh(2x\pm i2y)-1)$$

and

$$\cosh(x\pm iy)=\cosh x\cos y\pm i\sinh x\sin y,$$

the proof is done. □

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