J. Chosun Natural Sci.
Vol. 14, No. 3 (2021) pp. 73 – 77
https://doi.org/10.13160/ricns.2021.14.3.73

Simultaneous Optimization Using Loss Functions in Multiple Response Robust Designs

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Abstract

Robust design is an approach to reduce the performance variation of mutiple responses in products and processes. In fact, in many experimental designs require the simultaneous optimization of multiple responses. In this paper, we propose how to simultaneously optimize multiple responses for robust design when data are collected from a combined array. The proposed method is based on the quadratic loss function. An example is illustrated to show the proposed method.

Keywords : Robust design, Simultaneous optimization, Multiple responses, Robust design, Combined array, Loss function

(Recceived August 1, 2021 Revised August 9, 2021 Accepted August 20, 2021)

1. Introduction

Robust design designs experiments with control factors and noise factors. Robust design is to find the optimal conditions for control factors that are less affected by noise factors that cause fluctuations in quality characteristics (response variables). The noise factor is different from the conventional random factor. The noise factor can be controlled in an experiment, but it is a difficult factor to control in an actual situation. The noise factor is mainly an environmental factor, so the control factor is called a design factor and the noise factor is called an environmental factor.

In many experimental designs, optimal conditions for design factors (design variables or independent variables) are sought when there is only one response variable (quality characteristic or independent variable), but in practice, in most cases, there are several response variables to be considered at the same time. That is, most of the cases are multiple responses.

The method of finding the optimal condition of design factors in multiple responses was made by Derringer and Suich[1], Khuri and Conlon[2], and Vining[3]. Derringer and Suich[1] proposed a multi-response simultaneous optimization method using an expectation function. Although this method is convenient to use, it does not consider variance-covariance between response variables. Khuri and Conlon[2] proposed a multi-response simultaneous optimization method using a distance function. Although this method considers variance-covariance between response variables, it only considers the distance from the target value and does not consider the economic aspects required in the actual field at all. Vining[3] proposed a multi-response simultaneous optimization method using a loss function. However, all of the methods presented above are simultaneous optimization methods that do not take into account the quality variation due to the influence of noise factors. In addition, studies on multiple responses in Taguchi parameter design are scarce.

The traditional experimental design method, which tends to find the optimal conditions, mainly focusing on improving the average of the response variables (quality characteristics). But, Taguchi's robust design (Tagu-

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chi[4,5], Taguchi and Wu[6], and Kackar[7]) aims to reduce as much as possible the variation as well as the mean of response variables. Product array using orthogonal array table in Taguchi parameter design has many disadvantages, such as too many experiments and not considering successive experiments by considering all interactions between control and noise factors.Therefore, not only can the number of experiments be reduced, but alternative methods using the well-established experimental design theory are being studied. Among them, the combined array approach was first proposed by Welch, Yu, Kang, and Sacks[8. Since then, it has been studied by Vining and Myers[9] and Box and Jones[10]. The combined array refers to an experimental method in which a noise factor is arranged in one experimental array as a control factor.

Vining[3] proposed a multi-response simultaneous optimization method using a loss function in an experimental design consisting of only control factors. However, in this paper, we propose a multi-response simultaneous optimization method for robust design using a loss function in a combined array. In Chapter 2, a simultaneous optimization method is presented, in Chapter 3, an example is illustrated to show the proposed method.

2. Simultaneous Optimization Using Loss Functions in Multiple Response Robust Designs

Suppose the response y, depends on control variables (or factors) and noise variables. Let a set of control variables be denoted by $\underline{x} = (z_1, z_2, \cdots, z_l)'$ and a set of noise variables by $\underline{z} = (z_1, z_2, \cdots, z_m)'$. Suppose that all response functions in a multiresponse system depend on the same set of $\underline{x}$ and $\underline{z}$ that they can be represented by second order models within a certain region of interest. Let N be the number of experimental runs and r be the number of response functions. The ith second order model is

$$y_i(\underline{x},\underline{z}) = \beta_{io} + \underline{x}'\underline{\beta}_i + \underline{x}'B_i\underline{x} + \underline{z}'R_i\underline{z} + \underline{z}'\underline{\gamma}_i + \underline{z}'D_i\underline{x} + \varepsilon_i$$
$$i = 1, 2, \cdots, r \qquad (1)$$

where $\underline{\beta}_i$ is $l \times 1$, $\underline{\gamma}_i$ is $m \times 1$, $B_i' = B_i$ is $l \times l$, $R_i' = R_i$ is $m \times m$, D_i is $l \times m$, which are vectors or matrices of unknown regression parameters, and ε_i is a random error associated with the ith response.

Equation (1) can be expressed in matrix notation as

$$\underline{y}_i = X\underline{\theta}_i + \underline{\varepsilon}_i$$
$$i = 1, 2, \cdots, r \qquad (2)$$

in which $\underline{y}_i$ is an $N \times 1$ vector of observations on the ith response, X is an $N \times P$ full column rank matrix of known constants, $\underline{\theta}_i$ is the $p \times 1$ column vector of unknown regression parameters, and $\underline{\varepsilon}_i$ is a vector of random errors associated with the ith response. We also assume that

$$E(\underline{\varepsilon}_i) = \underline{0}, \ Var(\underline{\varepsilon}_i) = \sigma_{ii}I_N, \ Cov(\underline{\varepsilon}_i, \underline{\varepsilon}_j) = \sigma_{ij}I_N$$
$$i,j = 1, 2, \cdots, r, \ i \neq j$$

The $r \times r$ matrix whose (i,j)th element is σ_{ij} will be denoted by Σ. An unbiased estimator of Σ is given by

$$\hat{\Sigma} = Y'[I_N - X(X'X)^{-1}X']Y/(N-p) \qquad (3)$$

where $Y = (\underline{y}_1, \underline{y}_2, \cdots, Y_r)$, and I_N is an identity matrix of order $N \times N$. The variance-covariance matrix of $\hat{\underline{\theta}}_1$ is as follows.

$$Var(\hat{\theta}) = (X'X)^{-1}\Sigma .$$

The estimation formula for the ith response variable(or quality characteristic) is as follows.

$$\hat{y}_i(\underline{x},\underline{z}) = g'(\underline{x},\underline{z})\hat{\underline{\theta}}_1 ,$$
$$i = 1, 2, \cdots, r \qquad (4)$$

where $g'(\underline{x}, \underline{z})$ is a vector of the same form as the row of matrix X computed at point $(\underline{x}, \underline{z})$. From Eq. (4), it becomes the following Equation.

$$\Sigma_{\hat{y}}(x,z) = Var[\underline{\hat{y}}(\underline{x},\underline{z})] = g'(\underline{x},\underline{z})(X'X)^{-1}g(\underline{x},\underline{z})\Sigma \qquad (5)$$

$\underline{\hat{y}}(\underline{x},\underline{z}) = (\hat{y}_1(\underline{x},\underline{z}), \hat{y}_2(\underline{x},\underline{z}), \ldots, \hat{y}_r(\underline{x},\underline{z}))'$ is the vector of predicted responses at the point $(\underline{x}, \underline{z})$. The unbiased estimator of Eq. (5) is as follows.

$$\hat{\Sigma}_{\hat{y}}(x,z) = \hat{Var}[\hat{y}(x,z)] = g'(\underline{x},\underline{z})(X'X)^{-1}g(\underline{x},\underline{z})\hat{\Sigma} \qquad (6)$$

From (2) and (3), ordinary least squares (OLS) estimation is the same as generalized least squares (GLS) estimation. Thus, the fitted ith second-order model can be written as

$$\hat{y}_i(\underline{x},\underline{z}) = b_{io} + \underline{x}'\underline{b}_i + \underline{x}'\hat{B}_i\underline{x} + \underline{z}'\hat{R}_i\underline{z} + \underline{z}'\underline{r}_i + \underline{z}'\hat{D}_i\underline{x}$$
$$i = 1, 2, \cdots, r$$

The noise variables $\underline{z}$ are not controllable and they are random variables. In the absence of other knowledge, $\underline{z}$ would be usually uniformly distributed over R_z $(-1 \le z \le 1)$.

Box and Jones[10] modeled the mean and variance seperately in a single response. But, we are interested in showing the estimated mean and variance response models in multiple responses.

Let $\hat{m}_i(x)$ ith estimated mean response at an $\underline{x}$ averaged over the noise variables

$$\hat{m}_i(\underline{x}) = \int_{R_z} \hat{y}_i(\underline{x},\underline{z})p(\underline{z})d_z\,,$$
$$i = 1, 2, \cdots, r$$

where $p(\underline{z})$ is a probability density function of $\underline{z}$, and $\underline{z}$ has a uniform distribution over R_z. Box and Jones[10] showed that the ith estimated mean becomes

$$\hat{m}_i(\underline{x}) = b_{io} + \underline{x}'\underline{b}_i + \underline{x}'\hat{B}_i\underline{x} + \frac{1}{3}tr\hat{R}_i\,,$$
$$i = 1, 2, \cdots, r \tag{7}$$

where $tr\hat{R}_i$ is the trace of the matrix $\hat{R}_i$.

When there is only one response variable, the estimation formula for $y(\underline{x}, \underline{z})$ is $\hat{y}(\underline{x}, \underline{z})$, and the loss function is as follows when the target value is τ.

$$L = c[\hat{y}(\underline{x},\underline{z}) - \tau]^2$$

where c is the cost coefficient. In the case of multiple response variables, that is, when there are several response variables, the loss function is as follows when the estimation equation for $\underline{y}(\underline{x}, \underline{z})$ is $\hat{\underline{y}}(\underline{x}, \underline{z})$ and the target value is $\underline{\tau} = (\tau_1, \tau_2, \ldots, \tau_m)$.

$$\underline{L} = [\hat{\underline{y}}(\underline{x},\underline{z}) - \underline{\tau}]'C[\hat{\underline{y}}(\underline{x},\underline{z}) - \tau] \tag{8}$$

where $\underline{L} = (L_1, L_2, \ldots, L_r)'$ is the vector of loss function, $\hat{\underline{y}}(\underline{x},\underline{z}) = (\hat{y}_1(\underline{x},\underline{z}), \hat{y}_2(\underline{x},\underline{z}), \ldots, \hat{y}_r(\underline{x},\underline{z}))'$ is the vector of predicted responses at the point $(\underline{x}, \underline{z})$, and C is the cost matrix, which is positive definite. c_{ij} represents the economic loss cost caused by the target value and deviation corresponding to the estimated equations $\hat{y}_i$ and $\hat{y}_j$. The expected loss function of Eq. (6) is as follows.

$$E[\underline{L}] = [E[\hat{\underline{y}}(\underline{x},\underline{z})] - \underline{\tau}]'C[E[\hat{\underline{y}}(\underline{x},\underline{z})] - \tau] + tr[C\Sigma_{\hat{y}}(x,z)] \tag{9}$$

Assuming that $\hat{\underline{m}}(\underline{x}) = (\hat{m}_1(\underline{x}), \hat{m}_2(\underline{x}), \ldots, \hat{m}_r(\underline{x}))'$, from Eq. (5), Eq. (7) can be expressed as follows.

$$E[\underline{L}] = [\hat{\underline{m}}(\underline{x}) - \underline{\tau}]'C[\hat{\underline{m}}(\underline{x}) - \tau] + E[tr[C\Sigma_{\hat{y}}(x,z)]] \tag{10}$$

In order to obtain the optimal condition of the control factor that minimizes Eq. (10), the unbiased estimator of a is estimated using Eq. (6) as follows.

$$E[\underline{L}] = [\hat{\underline{m}}(\underline{x}) - \underline{\tau}]'C[\hat{\underline{m}}(\underline{x}) - \tau] + E[tr[C\Sigma(x,z)]] \tag{11}$$

Using the estimated expected loss function Eq. (11), we propose the optimal condition of the control factor that minimizes the expected loss function as a simultaneous optimization method for robust design when there are multiple response variables in the combined array. In other words, a multi-response simultaneous optimization method is proposed as follows.

$$M_L = \underset{\underline{x} \in R_x}{\text{Min}}\ \hat{E}[\underline{L}] \tag{12}$$

Here, R_x is the area of interest of the control factor $\underline{x}$. The multi-response optimization performance measure M_L proposed above can be optimized by using the "const" function in the toolbox of MATLAB 6.0 for Windows.

3. Numerical Example

In the combined array, the control factors x_1, x_2 and the noise factor z are experimentally arranged as shown in Table 1. Table 1 is the experimental arrangement according to the central composite design. y_1 is the nominal-is-best characteristics and y_2 is the larger-the-better characteristics.

Table 1. Experimental data by combined array

#	x_1	x_2	z	y_1	y_2
1	-1	-1	-1	80.6	81.4
2	-1	-1	1	74.9	95.9
3	-1	1	-1	83.1	105.0
4	-1	1	1	71.2	103.0
5	1	-1	-1	66.8	74.0
6	1	-1	1	74.2	76.8
7	1	1	-1	38.1	81.2
8	1	1	1	36.8	76.9
9	-1.41	0	0	80.9	100.0
10	-1.41	0	0	42.4	50.5
11	0	-1.41	0	73.4	71.2
12	0	-1.41	0	45.0	101.0
13	0	0	0	77.4	102.0
14	0	0	0	74.6	104.0

From Table 1, the estimated quadratic regression model considering the interaction between the control factor and the noise factor is as follows.

$$\hat{y}_1(x,z) = 76.00 - 12.37x_1 - 8.96x_2 - 7.22x_1^2 - 8.45x_2^2 - 8.11x_1x_2 + 5.38z^2 - 1.44z + 2.96x_1z - 1.86x_2z. \quad (13)$$

$$\hat{y}_2(x,z) = 103.00 - 12.21x_1 - 6.68x_2 - 13.96x_1^2 - 8.50x_2^2 - 2.93x_1x_2 + 6.23z^2 - 1.38z + 1.75x_1z - 2.95x_2z. \quad (14)$$

Using Eq. (7) from Eq. (13) and Eq. (14), the following estimated mean model is obtained.

$$\hat{m}_1(x) = 77.79 - 12.37x_1 - 8.96x_2 - 7.22x_1^2 - 8.45x_2^2 - 8.11x_1x_2.$$

$$\hat{m}_2(x) = 105.08 - 12.21x_1 - 6.68x_2 - 13.96x_1^2 - 8.50x_2^2 - 2.93x_1x_2.$$

Region of interest R_x is $-1 \leq x_1, x_2 \leq 1$, since y_1 is the nominal-is-best characteristics, the target value τ_1 of $\hat{m}_1(x)$ is set to a specific value of 75.00, and since y_2 is the larger-the-better characteristics, the maximum value of 109.65 in R_x phase is set as the target value τ_2. The variance-covariance matrix estimated according to Eq. (3) is as follows.

$$\hat{\Sigma} = \begin{bmatrix} 5.4367 & 2.3361 \\ 2.3361 & 68.2638 \end{bmatrix}$$

The cost matrix was determined as follows considering the loss cost of the two response variables.

$$C = \begin{bmatrix} 49 & 21 \\ 21 & 9 \end{bmatrix}$$

If only individual losses are taken into account when determining the cost factor, if the loss is A_i at the consumer tolerance point $\tau_i \pm \Delta_i$, it is as follows.

$$c_{ii} = A_i / \Delta_i^2$$

Cost matrix C is determined by considering not only the cost of loss when the response variable deviates from the target value, but also the difference in the size of the loss cost between the response variables and consumers, producers, and technical problems using prior information. It serves as a weight for the response variable.

When using the "const" function in the toolbox of MATLAB 6.0 to find the optimal condition, it may be a local minimum, so you need to compare the initial values with different initial values. When Eq. (12), which is a simultaneous optimization method, is applied, the optimum condition of the control factor is as follows.

x_1=0.03, x_2=0.29

In the optimal condition of the control factors, $\hat{m}_1(x)$ and $\hat{m}_2(x)$ are 74.92 and 106.68, respectively, and the expected loss cost is 100.44. The consumer loss is 89.68, and the error according to the estimation of the response function is 10.76, which is 12% of the consumer loss.

4. Conclusions

In this paper, we propose a multi-response simultaneous optimization method using a lost function when there are several response variables to be considered. Finding the optimal conditions for control factors in multiple characteristics is in most cases very complex depending on various cases, unlike the case of single quality characteristics. Therefore, in the case of multi-characteristics, the optimization performance measure for finding simultaneous optimal conditions should be simple and clear to be useful in practice. The experimental arrangement can significantly reduce the number of experiments compared to the product array proposed by Taguchi by using the combined array in the robust design. It could be an alternative solution. The simultaneous optimization method proposed in this paper considers variance-covariance between response variables and considers economic loss costs by using a cost matrix.

Acknowledgments

This study was supported(in part) by research funds from Chosun University, 2019.

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