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# A Classification of the Torsion-free Extensions

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## Abstract

The purpose of this paper is to classify the torsion-free extensions $1\to\mathbb{Z}^3\to\Pi\to\mathbb{Z}_\Phi\to 1$ with injective abstract kernel $\phi$: $\mathbb{Z}_\Phi\to \mathrm{Aut}(\mathbb{Z}^3)$. From this classification, we handle the sufficient conditions so as to classify the crystallographic groups of $\mathrm{Sol}^4_{m,n}$.

**Keywords:** Torsion-free extensions, Crystallographic group, Bieberbach group, $\mathrm{Sol}^4_{m,n}$, $A_{m,n}$

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## 1. Introduction

Let $X$ be a complete connected, simply connected Riemannian manifold, and let $G$ be a group of isometries of $X$. A pair $(X, G)$ is called a *geometry* in the sense of Thurston[1,2] if $G$ acts transitively on $X$ and $G$ contains a discrete subgroup $I$ with the coset space $\Gamma\backslash X$ of finite volume.

Let $G$ be a connected, simply connected solvable Lie group and let $C$ be any maximal compact subgroup of Aff($G$). A discrete cocompact subgroup $\Pi$ of $G\times G$ is called a *crystallographic group* of $G$. The coset space $\Pi\backslash G$ is an *infra-solvmanifold* of $G$, when $\Pi$ is a *Bieberbach group* (i.e., a torsion-free crystallographic group) of $G$. The maximal compact subgroup $C$ can be chosen so that $G\times C$ is equal to Isom($G$). Therefore, the Bieberbach groups of $G$ are exactly the fundamental groups of compact infra-solvmanifolds of $G$. Consequently, a closed manifold has a $(X, G)$-geometry if and only if it is an infra-solvmanifold of $G$. The crystallographic groups of $\mathrm{Sol}^3$ and $\mathrm{Sol}^4_1$ are classified in [3] and [4], respectively. All the closed four-manifolds with $\mathrm{Sol}^4_1$-geometry were studied in [5].

There are infinite but countable number of the Lie groups $\mathrm{Sol}^4_\lambda$ that admit a lattice. Such Lie groups are denoted by $\mathrm{Sol}^4_{m,n}$. In [6], we showed that $\mathrm{Sol}^4_{m,n}$ has a unique lattice up to isomorphism and studied the necessary conditions for the crystallographic groups of $\mathrm{Sol}^4_{m,n}$.

In this paper, we classify the torsion-free extensions $1\to\mathbb{Z}^3\to\Pi\to\mathbb{Z}_\Phi\to 1$ with injective abstract kernel $\phi$: $\mathbb{Z}_\Phi\to \mathrm{Aut}(\mathbb{Z}^3)$, and handle the classification problem to classify the crystallographic groups of $\mathrm{Sol}^4_{m,n}$.

## 2. Extensions $1\to\mathbb{Z}^3\to\Pi\to\mathbb{Z}_\Phi\to 1$

In this section, we achieve our classification problem by classifying the torsion-free extensions $1\to\mathbb{Z}^3\to\Pi\to\mathbb{Z}_\Phi\to 1$ with *injective* abstract kernel $\phi$: $\mathbb{Z}_\Phi\to \mathrm{Aut}(\mathbb{Z}^3)$, see for example [5, Lemma 1.2].

**2.1. Case $\Phi$={1}**. If $\Phi$={1} then $\Pi=\Gamma$ is a lattice of $\mathrm{Sol}^4_{m,n}$.

**2.2. Case $\Phi=\mathbb{Z}_2$** From [6], we may assume that

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$$\Gamma=\langle(x_1,0),(x_2,0),(x_3,0),(0,s)\rangle=\mathbb{Z}^3\times_{A_{m,n}}\mathbb{Z}$$

where $s=\ln\alpha_2$ and the vectors $x_i$ are given in (3-5) of [6], i.e.,

$$x_1=\begin{bmatrix}\alpha_2\alpha_3\\-(\alpha_2+\alpha_3)\\1\end{bmatrix},\ x_2=\begin{bmatrix}\alpha_3\alpha_1\\-(\alpha_3+\alpha_1)\\1\end{bmatrix},\ x_3=\begin{bmatrix}\alpha_1\alpha_2\\-(\alpha_1+\alpha_2)\\1\end{bmatrix}.$$

Consider $\Phi=\langle X\rangle$ Choose a lift $(r,t,X)\in\Pi$ of $X$ where $t=0$ or $\frac{s}{2}$, and $r\in\mathbb{R}^3$ Then $\Pi=\langle\Gamma,(r,t,X)\rangle$. Let $t=0$. This is exactly the case where $\mathbb{Z}_\Phi=\langle(s,I),(0,X)\rangle=\mathbb{Z}\times\mathbb{Z}_2$. Since $\mathbb{Z}^3$ must be a normal subgroup of $\Pi$, we have

$(r,0,X)(x_i,0,I)(r,0,X)^{-1}=(X(x_i),0,I)\in\mathbb{Z}^3$.

Hence,

$$X(x_i)=p_{1i}x_1+p_{2i}x_2+p_{3i}x_3\ \ (i=1,2,3) \qquad (2\text{-}1)$$

for some integers $p_{ij}$ This means that

$$[x_1\ x_2\ x_3]^{-1}X[x_1\ x_2\ x_3]=[p_{ij}]\in GL(3,\mathbb{Z}). \qquad (2\text{-}2)$$

A direct computation of (2-2) shows that

$$[p_{ij}]=\begin{bmatrix}\frac{-\alpha_1^2+n}{(\alpha_3-\alpha_1)(\alpha_1-\alpha_2)} & \frac{2\alpha_3\alpha_1}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)} & \frac{2\alpha_1\alpha_2}{(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)}\\ \frac{2\alpha_2\alpha_3}{(\alpha_3-\alpha_1)(\alpha_1-\alpha_2)} & \frac{-\alpha_2^2+n}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)} & \frac{2\alpha_1\alpha_2}{(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)}\\ \frac{2\alpha_2\alpha_3}{(\alpha_3-\alpha_1)(\alpha_1-\alpha_2)} & \frac{2\alpha_3\alpha_1}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)} & \frac{-\alpha_3^2+n}{(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)}\end{bmatrix}$$

By (2-1), we have $X(x_1)=p_{11}x_1+p_{21}x_2+p_{31}x_3$ or

$$\begin{bmatrix}-\alpha_2\alpha_3\\-(\alpha_2+\alpha_3)\\1\end{bmatrix}=p_{11}\begin{bmatrix}\alpha_2\alpha_3\\-(\alpha_2+\alpha_3)\\1\end{bmatrix}+p_{21}\begin{bmatrix}\alpha_3\alpha_1\\-(\alpha_3+\alpha_1)\\1\end{bmatrix}+p_{31}\begin{bmatrix}\alpha_1\alpha_2\\-(\alpha_1+\alpha_2)\\1\end{bmatrix}$$

From the middle entries above, we have

$$0=(p_{11}-1)(\alpha_2+\alpha_3)+p_{21}(\alpha_3+\alpha_1)+p_{31}(\alpha_1+\alpha_2)$$
$$=\frac{4n\alpha_2\alpha_3}{(\alpha_3-\alpha_1)(\alpha_1-\alpha_2)}\neq 0$$

where the second identity is obtained by a direct computation. In conclusion, when $\Phi=\langle X\rangle$, $t$ cannot be 0. By the similar argument as above, we can show that unless $\Phi=\langle -I\rangle$, $t$ cannot be 0.

Now consider $\Phi=\langle -I\rangle$ with $t=0$ In this case

$(r,0,-I)(x_i,0,I)(r,0,-I)^{-1}$
$=(-I(x_i),0,I)=(-x_i,0,I)\in\mathbb{Z}^3$.

Moreover, $(r,0,-I)^2=(r-I(r),0,I)=(0,0,I)$.

Consequently,

$$\Pi=\langle\Gamma,(r,0,-I)\rangle=\Gamma\times\mathbb{Z}_2.$$

It is clear that the groups $\langle\Gamma,(r,0,-I)\rangle$ and $\langle\Gamma,(r',0,-I)\rangle$ with any $r,r'\in\mathbb{R}^3$ are isomorphic to each other.

Let $t=\frac{s}{2}$ Then $\mathbb{Z}_\Phi=\langle(\frac{s}{2},X)\rangle\cong\mathbb{Z}$ and so $\Pi$ is necessarily torsion-free. The abstract kernel $\phi:\mathbb{Z}_\Phi\to Aut(\mathbb{Z}^3)=GL(3,\mathbb{Z})$ with respect to the ordered generators $\{x_1,x_2,x_3\}$ of $\mathbb{Z}^3$ is determined by the image $\phi\left(\frac{s}{2},X\right)$ Note that $\left(\frac{s}{2},X\right)^2=(s,I)$, $\phi(s)=A_{m,n}$ and

$$\varphi'\left(\frac{s}{2},X\right)=\varphi\left(\frac{s}{2}\right)X=\begin{bmatrix}-\sqrt{\alpha_1}&0&0\\0&\sqrt{\alpha_2}&0\\0&0&\sqrt{\alpha_3}\end{bmatrix}.$$

Thus, we must have that $\phi\left(\frac{s}{2},X\right)\in GL(3,\mathbb{Z})$ is a square root of $A_{m,n}$ (a matrix whose square is $A_{m,n}$) with eigenvalues $-\sqrt{\alpha_1}$, $\sqrt{\alpha_2}$, and $\sqrt{\alpha_3}$. We denote such an integer matrix by $\sqrt[X]{A}$. Let

$$x^3-kx^2+lx+1=0$$

be the characteristic equation of $\sqrt[X]{A}$ Then we have

$$m=k^2-2l,\quad n=l^2+2k \qquad (2\text{-}3)$$

With $(k,l)$ satisfying (2-3), we form the group

$$\Pi=\mathbb{Z}^3\times_{\sqrt[X]{A}}\left(\frac{1}{2}\mathbb{Z}\right).$$

This group fits the following diagram (2-4). (see (4-1) in [6]).

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^3 & \xrightarrow{=} & \mathbb{Z}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\qquad (2\text{-}4)
$$

In the above argument, if $X$ is replaced with $Y$, $Z$, or $-I$, then

- $\sqrt[X]{A}$ should be replaced with $\sqrt[Y]{A}$, $\sqrt[Z]{A}$ or $\sqrt[-I]{A}$ and
- the eigenvalues $-\sqrt{\alpha_1}$, $\sqrt{\alpha_2}$, $\sqrt{\alpha_3}$ of $\sqrt[X]{A}$ should be replaced with the eigenvalues $\sqrt{\alpha_1}$, $-\sqrt{\alpha_2}$, $\sqrt{\alpha_3}$ of $\sqrt[Y]{A}$, $\sqrt{\alpha_1}$, $\sqrt{\alpha_2}$, $-\sqrt{\alpha_3}$ of $\sqrt[Z]{A}$ or $-\sqrt{\alpha_1}$, $-\sqrt{\alpha_2}$, $-\sqrt{\alpha_3}$ of $\sqrt[-I]{A}$.

Let $x^3-kx^2+lx+1=0$ be the characteristic equation of $\sqrt[X]{A}$, $\sqrt[Y]{A}$, $\sqrt[Z]{A}$ or $\sqrt[-I]{A}$ Then the pair $(k, l)$ satisfies (2-3). With $(k, l)$ satisfying (2-3), the group

$$\Pi=\mathbb{Z}^3\times_{\sqrt[Y]{A}}\left(\frac{1}{2}\mathbb{Z}\right)$$

$$\Pi=\mathbb{Z}^3\times_{\sqrt[Z]{A}}\left(\frac{1}{2}\mathbb{Z}\right)$$

or

$$\Pi=\mathbb{Z}^3\times_{\sqrt[-I]{A}}\left(\frac{1}{2}\mathbb{Z}\right)$$

fits the diagram (2-4).

In a similar way, we can replace $X$ a generator of $\Phi$ with another generator $-X$, $-Y$, or $-Z$ In this case, the characteristic equation of the corresponding square matrix $\sqrt[-X]{A}$, $\sqrt[-Y]{A}$ or $\sqrt[-Z]{A}$ is

$$x^3-kx^2+lx-1=0$$

where the pair $(k, l)$ satisfies

$$\begin{aligned} m &= k^2+2l \\ n &= l^2-2k \end{aligned} \qquad (2\text{-}5)$$

For pairs $(k, l)$ satisfying (2-5), we can form the group

$$\Pi=\mathbb{Z}^3\times_{\sqrt[-X]{A}}\left(\frac{1}{2}\mathbb{Z}\right)$$

$$\Pi=\mathbb{Z}^3\times_{\sqrt[-Y]{A}}\left(\frac{1}{2}\mathbb{Z}\right)$$

or

$$\Pi=\mathbb{Z}^3\times_{\sqrt[-Z]{A}}\left(\frac{1}{2}\mathbb{Z}\right)$$

fitting the diagram (2-4).

Now we will show that every $A_{m,n}$ cannot admit a square root. First, we remark by (2-3) and (2-5) that both $m$ and $k$ and $n$ and $l$ share the even-odd parity.

**Theorem 2.1.** *The integer matrix $A_{m,n}$ cannot admit a square root if one of the following holds:*

(1) *$m$ is of the form $4M$ and $n$ is odd or of the form $4N+2$*

(2) *$m$ is of the form $4M+1$ and $n$ is odd or of the form $4N$*

(3) *$m$ is of the form $4M+2$ and $n$ is even or of the form $4N+3$*

(4) *$m$ is of the form $4M+3$ and $n$ is even or of the form $4N+1$*

If $A_{m,n}$ does not have a square root, then $\mathrm{Sol}^4_{m,n}$ has no Bieberbach group with holonomy group $\mathbb{Z}_2$.

*Proof.* First, let $m=4M$ and $n$ is odd. Then $k=2K$ and $l=2L+1$ for some integers $R$ and $L$. By substitution to (2-3), we have

$$4M=(2K)^2-2(2L+1)=4(K^2-L-1)+2,$$

which is impossible.

Let $m=4M$ and $n=4N+2$ Then $k=2K$ and $l=2L$ for some integers $R$ and $L$ By substitution to (2-3) and (2-5), we have

$$4N+2=(2L)^2\pm 2(2K)=4(L^2\pm K),$$

which is impossible.

The remaining cases can be considered in the same way. We can easily show that (2-3) and (2-5) are not satisfied in every remaining case. Hence $A_{m,n}$ cannot admit a square root. □

The following example shows that the complementary case of Theorem 2.1 does not guarantee the existence of a square root.

**Example 2.2.** Consider $(m, n) = (12, 8)$ This integer pair satisfies (3-4) in [6]. Assume that there exists an integer pair $(k, l)$ satisfying the equations (2-3). Then $k = 2K$ and $l = 2L$ for some integers $R$ and $L$ and by substitution we get

$$3 = K^2 - L,\ 2 = L^2 + K$$

which yield $K^4 - 6K^2 + K + 7 = 0$. However, by a simple inspection we can see that this equation has no integer root. Hence $A_{12,8}$ does not admit a square root, and $\mathrm{Sol}^4_{12,8}$ has no Bieberbach group with holonomy group $\mathbb{Z}_2$.

Now, we will examine the complementary case of Theorem 2.1 in detail, and obtain the "existence" conditions of a square root of $A_{m,n}$ We can immediately see that each case produces an hyperbola equation.

2.2.1. Case 1: $m = 4M$, $n = 4N$. With $k = 2K$ and $l = 2L$ we have $4M = (2K)^2 - 2(2L) = 4(K^2 - L)$ and $4N = (2L)^2 + 2(2K) = 4(L^2 + K)$ and hence (2-3) is equivalent to

$$M = K^2 - L,$$
$$N = L^2 - K.$$

Thus we have

$$(K-1)K - L(L+1) = M - N,$$

so $M - N = 2a$ is an even integer, and hence we have

$$(K+L)(K-L-1) = 2a$$

If $K+L = p$, $K-L-1 = q$ with $2a = pq$ (a multiple of two integers), then $2K-1 = p+q$ and $2L+1 = p-q$ so one of $p$ and $q$ is odd and the other is even. In this case, $K = \frac{p+q+1}{2}$, $L = \frac{p-q-1}{2}$. Hence

$$M = \frac{(p+q+1)(p+q+3)}{4} - p,$$

$$N = \frac{(p-q-1)(p-q-3)}{4} + q.$$

In conclusion, we have:

**Theorem 2.3.** *Let $m = 4M$ and $n = 4N$ be given integers.*

(1) *If $M - N$ is odd or is a product of even integers, then the equations (2-3) have no integer solution* $(k, l)$

(2) *If $M - N = pq$ is a product of an even integer and an odd integer such that*

$$M = \frac{(p+q+1)(p+q+3)}{4} - p,$$

$$N = \frac{(p-q-1)(p-q-3)}{4} + q,$$

*then $(k, l) = (p + q + 1, p - q - 1)$ is an integer solution of the equations (2-3).*

**Example 2.4.**

(1) Given $(m, n) = (12, 8)$ as in Example 2.2, $M - N = 3-2 = 1$ is odd. By Theorem 2.3, the equations (2-3) with $(m, n) = (12, 8)$ have no integer solution $(k, l)$

(2) Consider $(m, n) = (20, 12)$ Then

$$M - N = 5-3 = 2 = pq.$$

Taking $p = 1$ and $p = 2$ we can see that

$$M = \frac{(p+q+1)(p+q+3)}{4} - p$$

and

$$N = \frac{(p-q-1)(p-q-3)}{4} + q$$

hold. Notice that the other choices for $(p, q)$ do not satisfy the above identities. Hence by Theorem 2.3, $(k, l) = (4, -2)$ is an integer solution of the equations (2-3).

(3) Consider $(m, n) = (52, 20)$ Then $M = 13$, $N = 5$ and $M - N = 8$ Taking $p = -1$ and $q = -8$ we can see that

$$M = \frac{(p+q+1)(p+q+3)}{4} - p$$

and

$$N = \frac{(p-q-1)(p-q-3)}{4} + q$$

hold. By Theorem 2.3, $(k, l) = (-8, 6)$ is an integer solution of the equations (2-3).

2.2.2. Case 2: $m = 4M + 1$, $n = 4N + 2$. With $k = 2K + 1$ and $l = 2L$ in (2-3), we have

$$M = K^2 + K - L,\ N = L^2 - K$$

and so an hyperbola equation

$$K^2 = \left(L+\frac{1}{2}\right)^2 = M - N - \frac{1}{4}.$$

Hence

$$(2K)^2 - (2L+1)^2 = 4(M-N) - 1.$$

If $2K+2L+1 = p$, $2K-2L-1 = q$ with $4(M-N)-1 = pq$ then $4K = p+q$ and $4L = p-q-2$ Remark that $(p,q) \equiv (1,3)$ or (3, 1) mod 4. Hence

$$M = \left(\frac{p+q}{4}\right)^2 + \frac{q+1}{2},\ N = \left(\frac{p-q}{4}\right)^2 + \frac{2q+1}{4}.$$

In conclusion, we have:

**Theorem 2.5.** *Let $m = 4M + 1$ and $n = 4N + 2$ be given integers. Let $4(M-N)-1 = pq$ be a product of two odd integers such that*

$$M = \left(\frac{p+q}{4}\right)^2 + \frac{q+1}{2},\ N = \left(\frac{p-q}{4}\right)^2 + \frac{2q+1}{4}.$$

*Then $(k, l) = \left(\frac{p+q}{2}+1, \frac{p-q}{2}-1\right)$ is an integer solution of the equations* (2-3).

**Example 2.6.** Consider $(m, n) = (9, 6)$ Then $M = 2$, $N = 1$ and $M - N = 1$ and $4(M-N)-1 = 3$ With $p = 3$ and $q = 1$ we can see that

$$M = \left(\frac{p+q}{4}\right)^2 + \frac{q+1}{2} \text{ and } N = \left(\frac{p-q}{4}\right)^2 + \frac{2q+1}{4} \text{ hold.}$$

By Theorem 2.5, $(k, l) = (3, 0)$ is an integer solution of the equations (2-3).

2.2.3.Case 3: $m = 4M + 2$, $n = 4N + 1$ With $k = 2K$ and $l = 2L + 1$ in (2-3), we have

$$M = K^2 - L - 1,\ N = L^2 + L + R$$

and so an hyperbola equation

$$\left(K - \frac{1}{2}\right)^2 - (L+1)^2 = M - N + \frac{1}{4}.$$

Hence

$$(2K-1)^2 - (2(L+1))^2 = 4(M-N) + 1.$$

If $2K+2L+1 = p$, $2K-2L-3 = q$ with $4(M-N)+1 = pq$ then $4K = p+q+2$ and $4L = p-q-4$ Remark that $(p,q) \equiv (1,1)$ or (3, 3) mod 4. Hence

$$M = \left(\frac{p+q}{4}\right)^2 + \frac{2q+1}{4},\ N = \left(\frac{p-q}{4}\right)^2 + \frac{q+1}{2}.$$

In conclusion, we have:

**Theorem 2.7.** *Let $m = 4M + 2$ and $m = 4M + 1$ be given integers. Let $4(M-N)+1 = pq$ be a product of two odd integers such that*

$$M = \left(\frac{p+q}{4}\right)^2 + \frac{2q+1}{4},\ N = \left(\frac{p-q}{4}\right)^2 + \frac{q+1}{2}.$$

*Then $(k, l) = \left(\frac{p+q}{2}+1, \frac{p-q}{2}-1\right)$ is an integer solution of the equations* (2-3).

**Example 2.8.** Consider $(m, n) = (6, 5)$ Then $M = 1$, $N = 1$ and $4(M-N)+1 = 1$ Taking $p = 1$ and $q = 1$ we can see that

$$M = \left(\frac{p+q}{4}\right)^2 + \frac{2q+1}{4} \text{ and } N = \left(\frac{p-q}{4}\right)^2 + \frac{q+1}{2} \text{ hold.}$$

By Theorem 2.5, $(k, l) = (2, -1)$ is an integer solution of the equations (2-3).

2.2.4. Case 4: $m = 4M + 3$, $n = 4N + 3$. With $k = 2K + 1$ and $l = 2L - 1$ in (2-3), we have

$$M = K^2 + K - L,\ N = L^2 - L + K$$

and so an hyperbola equation

$$K^2 - L^2 = M - N.$$

If $K+L = p$, $K-L = q$ with $M-N = pq$ (a multiple of two integers), then $2K = p+q$ and $2L = p-q$ so $p$ and $q$ share the even-odd parity. In this case,

$$K = \frac{p+q}{2},\ L = \frac{p-q}{2}. \text{ Hence}$$

$$M = \left(\frac{p+q}{2}\right)^2 + q,\ N = \left(\frac{p-q}{2}\right)^2 + q.$$

In conclusion, we have:

**Theorem 2.9.** *Let* $m = 4M + 3$ *and* $n = 4N + 3$ *be given integers.*

(1) *If* $M-N = pq$ *is a product of two even integers or a product of two odd integers such that*

$$M = \left(\frac{p+q}{2}\right)^2 + q\ ,\ N = \left(\frac{p-q}{4}\right)^2 + q$$

*then* $(k, l) = (p+q+1, p-q-1)$ *is an integer solution of the equations* (2-3).

(2) *If* $M - N$ *cannot be expressed as a product of two even integers or as a product of two odd integers, then the equations* (2-3) *have no integer solution* $(k, l)$.

**Example 2.10.**

(1) Consider $(m, n) = (11, 7)$ Then $M = 2$ and $N = 1$, hence $M - N = 1 = pq$, $p = q = \pm 1$. With $p = 1$ and $q = 1$ we can see that

$$M = \left(\frac{p+q}{2}\right)^2 + q \ \text{ and } \ N = \left(\frac{p-q}{4}\right)^2 + q \ \text{ hold.}$$

By Theorem 2.9, $(k, l) = (3, -1)$ is an integer solution of the equations (2-3).

(2) Consider $(m, n) = (19, 11)$ Then $M = 4$ and $N = 2$ hence $M - N = 2$ By Theorem 2.9, the equations (2-3) have no integer solution $(k, l)$.

The following theorem states about the "uniqueness" of square root of $A_{m,n}$ whenever it exists.

**Theorem 2.11.**

(1) *If the pair* $(k, l)$ *satisfies the equations* (2-3), *then* $(-k, l)$ *satisfies the equations* (2-5), *and the vice versa.*

(2) *If the pair* $(k, l)$ *satisfies the equations* (2-3), *then* $(k, l)$ *is unique.*

(3) *If the pair* $(k, l)$ *satisfies the equations* (2-3), *then the associated equation* $x^3 - kx^2 + lx + 1 = 0$ *has 3 distinct roots.*

(4) *If* $\beta$ *is a root of* $x^3 - kx^2 + lx + 1 = 0$ *then* $-\beta$ *is a root of* $x^3 + kx^2 + lx - 1 = 0$

*Proof.* The proofs of (1) and (4) are trivial. □

For (2), assume that

$$k^2 - 2l = m = p^2 - 2q\ ,\ l^2 + 2k = n = q^2 + 2p\ .$$

Then

$$(k+p)(k-p) = 2(l-q)\ ,\ (l+q)(l-q) = -2(k-p)\ , \quad (2\text{-}6)$$

hence $k = p \Leftrightarrow l = q$ Assume further that $k \neq p$ and $l \neq q$ Thus we obtain

$$(k+p)(l+q) = -4\ .$$

If $k + p = \pm 1$ and $l + q = \mp 4$ then by (2-6), $\pm 2k - 1 = 4(l \pm 2)$ a contradiction. If $k + p = 2$ and $l + q = -2$ then by (2-6), $k = l + 2$ hence $m = k^2 - 2l = l^2 + 2l + 4$ and $n = l^2 + 2k = l^2 + 2l + 4$ a contradiction as $m > n$ If $k + p = -2$ and $l + q = 2$ then by (2-6), $k = -1$ hence $\mathrm{m} = k^2 - 2l = l^2 - 2l = l^2 + 2k = n$ a contradiction. Hence (2) is proved.

Let $B$ be the companion matrix of the equation $x^3 - kx^2 + lx + 1 = 0$. By (2-3), $B^2$ has the characteristic equation $x^3 - mx^2 + nx - 1 = 0$. Thus $B^2$ has 3 distinct positive real eigenvalues. This implies that $B$ has 3 distinct eigenvalues, which are roots of the equation $x^3 - kx^2 + lx + 1 = 0$. This proves (3).

**Remark 2.12.** Suppose that $(p, q)$ is a pair of integers satisfying a condition in Theorem 2.3, 2.5, 2.7 or 2.9. Then this pair must be unique, depending only on $(m, n)$ because of Theorem 2.11. (2).

Theorem 2.11 above says that for the given Lie group $\mathrm{Sol}^4_{m,n}$ if there exists $(k, l)$ satisfying (2-3) and hence (2-5) then there is exactly one pair $(\sqrt[U]{A}, -\sqrt[U]{A})$ of two square roots of $A_{m,n}$ which is one of the following:

$$(\sqrt[U]{A}, -\sqrt[U]{A}) = (\sqrt[X]{A}, -\sqrt[X]{A})\ ,\ (\sqrt[Y]{A}, -\sqrt[Y]{A})$$
$$(\sqrt[Z]{A}, -\sqrt[Z]{A})\ ,\ (-\sqrt[I]{A}, \sqrt[I]{A})\ .$$

Each pair except $(-\sqrt[I]{A}, \sqrt[I]{A})$ will give rise to one Bieberbach group of $\mathrm{Sol}^4_{m,n}$ with trivial holonomy group (i.e., a lattice of $\mathrm{Sol}^4_{m,n}$), and two Bieberbach groups of $\mathrm{Sol}^4_{m,n}$ with holonomy group $\mathbb{Z}_2$ For example, if $(\sqrt[U]{A}, -\sqrt[U]{A}) = (\sqrt[X]{A}, -\sqrt[X]{A})$ then

$$\Gamma = \mathbb{Z}^3 \times_{A_{m,n}} \mathbb{Z} \cong \mathbb{Z}^3 \times_A \mathbb{Z}\ ,$$

$$\Pi_X \cong \mathbb{Z}^3 \times_{\sqrt[X]{A}} \left(\frac{1}{2}\mathbb{Z}\right),\ \Pi_{-X} \cong \mathbb{Z}^3 \times_{-\sqrt[X]{A}} \left(\frac{1}{2}\mathbb{Z}\right)$$

with holonomy groups $\{1\}$, $<X>$, and $<-X>$, respectively. When $A_{m,n}$ has a pair of square roots $(-{}^{I}\!\sqrt{A}, {}^{I}\!\sqrt{A})$ $\mathrm{Sol}^4_{m,n}$ has always a lattice, and has one Bieberbach group

$$\Pi_{-I} \cong \mathbb{Z}^3 \times_{-{}^{I}\!\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)$$

with holonomy group $<-I> = \{\pm I\}$. Since ${}^{I}\!\sqrt{A}$ has 3 distinct real eigenvalues, the associated group

$$\Pi_{I} \cong \mathbb{Z}^3 \times_{{}^{I}\!\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)$$

is a lattice of $\mathrm{Sol}^4_{k,l}$ This is not a Bieberbach group of $\mathrm{Sol}^4_{m,n}$ (with holonomy group $\mathbb{Z}_2$).

**Example 2.13.** If $(m, n) = (6, 5)$ then it satisfies the conditions (3-4) in [6] and $(k, l) = (2, -1)$ satisfies the equations (2-3), see Example 2.8. Take

$$\sqrt{A} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad A = (\sqrt{A})^2 = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & -1 \\ 1 & -2 & 5 \end{bmatrix}$$

Note that $A$ is conjugate to $A_{m,n} = A_{6.5}$. By a simple observation, we can see that the characteristic equation $x^3 - 2x^2 - x + 1 = 0$ of $\sqrt{A}$ has eigenvalues $\sqrt{\alpha_1}$, $-\sqrt{\alpha_2}$, and $\sqrt{\alpha_3}$. This shows that $\sqrt{A} = {}^{Y}\!\sqrt{A}$ and $\mathrm{Sol}^4_{9,6}$ has Bieberbach group $\Pi_Y$ with holonomy group $\mathbb{Z}_2$ By (1) in Theorem 2.11, $(k, l) = (-2, -1)$ satisfies the equations (2-5). By (4) in Theorem 2.11, the associated equation $x^3 + 2x^2 - x - 1 = 0$ has three distinct roots $-\sqrt{\alpha_1}$, $\sqrt{\alpha_2}$, and $-\sqrt{\alpha_3}$ This implies $\mathrm{Sol}^4_{6.5}$ has Bieberbach group $\Pi_{-Y}$ with holonomy group $\mathbb{Z}_2$ Finally, by (2) in Theorem 2.11, the Bieberbach groups $\Pi_Y$ and $\Pi_{-Y}$ of $\mathrm{Sol}^4_{9,6}$ are the only Bieberbach groups with holonomy group $\mathbb{Z}_2$.

**Example 2.14.** For other examples, if $(m, n) = (9, 6)$ then it satisfies the conditions (3-4) in [6] and $(k, l) = (3, 0)$ satisfies the equations (2-3), see Example 2.6. Hence we can see that $\mathrm{Sol}^4_{9,6}$ two Bieberbach groups $\Pi_Z$ and $\Pi_{-Z}$ with holonomy group $\mathbb{Z}_2$.

If $(m, n) = (37, 26)$ then it satisfies the conditions (3-4) in [6] and $(k, l) = (-5, -6)$ satisfies the equations (2-3). Hence we can see that $\mathrm{Sol}^4_{37,26}$ two Bieberbach groups $\Pi_X$ and $\Pi_{-X}$ with holonomy group $\mathbb{Z}_2$.

**Example 2.15.** Consider $(m, n) = (52, 20)$ This pair satisfies the conditions (3-4) in [6] and $(k, l) = (-8, 6)$ satisfies the equations (2-3), see Example 2.4. The associated equation $x^3 + 8x^2 + 6x + 1 = 0$ has 3 *negative* roots, hence the equation $x^3 - 8x^2 + 6x - 1 = 0$ associated with $(k, l) = (8, 6)$ has 3 *positive* roots. Consequently, $\mathrm{Sol}^4_{52,20}$ has only one Bieberbach group

$\Pi_{-I} \cong \mathbb{Z}^3 \times_{-{}^{I}\!\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)$ with holonomy group $\{\pm I\}$.

The group $\Pi_I \cong \mathbb{Z}^3 \times_{{}^{I}\!\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)$ is a lattice of $\mathrm{Sol}^4_{8,6}$

**Theorem 2.16.** *The Lie group* $\mathrm{Sol}^4_{m,n}$ *has a unique, up to isomorphism, crystallographic group with torsion element and with holonomy group* $\mathbb{Z}_2$.

$<\Gamma, (0, 0, -I)> = \Gamma \times \mathbb{Z}_2$.

*The Lie group* $\mathrm{Sol}^4_{m,n}$ *has a Bieberbach group with holonomy group* $\mathbb{Z}_2$ *if and only if the simultaneous equations* (2-3) *have a pair* $(k, l)$ *of integer solution. If this is the case, then* $\mathrm{Sol}^4_{m,n}$ *has two Bieberbach groups, up to isomorphism,*

$$\mathbb{Z}^3 \times_{{}^{U}\!\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right), \quad \mathbb{Z}^3 \times_{-{}^{U}\!\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)$$

where $U \in \{X, Y, Z\}$, or has one Bieberbach group, up to isomorphism,

$$\mathbb{Z}^3 \times_{-{}^{I}\!\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right).$$

**2.3. Case $\Phi = \mathbb{Z}_2^2$ or $\mathbb{Z}_2^3$** Let $\Pi$ be a crystallographic group of $\mathrm{Sol}^4_{m,n}$ fitting the diagram (2-4). Let $\Phi'$ be and index 2 subgroup of $\Phi'$ By pulling back the diagram (2-4) via the inclusion $i: \Phi' \to \Phi$ we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \uparrow{=} & & \uparrow & & \uparrow{i} & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi_{\Phi'} & \longrightarrow & \Phi' & \longrightarrow & 1 \end{array}$$

where $\Pi_\Phi$ is a crystallographic group of $\mathrm{Sol}^4_{m,n}$ with holonomy group $\Phi'$ Assume that $\Pi$ is a Bieberbach group. Then $\Pi_\Phi$ is also Bieberbach group for any subgroup $\Phi'$ of $\Phi$ Let $U$, $V$ be nontrivial generators of $\Phi$. With $\Phi' = <U>$ because $\Pi_\Phi$ is torsion-free, the discussion in Subsection 2.2 tells that $\left(\frac{s}{2}, U\right) \in$ $\mathbb{Z}_{\Phi'}$SUBSET$\mathbb{Z}_\Phi$. Similarly, we have $\left(\frac{s}{2}, V\right) \in \mathbb{Z}_\Phi$. Hence $\left(\frac{s}{2}, U\right)\left(\frac{s}{2}, V\right) = (s, UV) = (s, I)(0, UV)$. This forces $(0, UV) \in \mathbb{Z}_\Phi$ which implies from subsection 2.2 again that when $\Phi' = <UV>$, $\Pi_\Phi$ is not torsion-free. This is a contradiction. Consequently, we have

**Theorem 2.17.** *The Lie group* $\mathrm{Sol}^4_{m,n}$ *has no Bieberbach group with holonomy group* $\mathbb{Z}_2^2$ *or* $\mathbb{Z}_2^3$

Now, let $\Pi$ be a crystallographic group of $\mathrm{Sol}^4_{m,n}$ with holonomy group $\mathbb{Z}_2^2 = <U, V>$ so that for some proper $\Phi' \subset \Phi$, $\Pi_{\Phi'}$ is non-Bieberbach, crystallographic group. By Theorem 2.16, we must have $\Phi' = <-I>$, and $\Pi_{\Phi'} = \Gamma \times \mathbb{Z}_2$. We may assume that $V = -I$ Then $U$ must be one of $\pm X$, $\pm Y$, and $\pm Z$ By Theorem 2.16 again, when $\Phi' = <U>$ we must have $\Pi_{\Phi'} = \mathbb{Z}^3 \times_{U\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)$ Consequently, we have

**Theorem 2.18.**

(1) *If* $\mathrm{Sol}^4_{m,n}$ *admits a Bieberbach group* $\mathbb{Z}^3 \times_{U\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)$ *(and hence a Bieberbach group* $\mathbb{Z}^3 \times_{-U\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)$*) with holonomy group* $\mathbb{Z}_2 = <U>$ *(and* $\mathbb{Z}_2 = <-U>$ *respectively) where* $U \in \{\pm X, \pm Y, \pm Z\}$ *then* $\mathrm{Sol}^4_{m,n}$ *admits a unique, up to isomorphism, crystallographic group*

$$\mathbb{Z}^3 \times_{U\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right) \times \mathbb{Z}_2 \left( \cong \left(\mathbb{Z}^3 \times_{-U\sqrt{A}} \left(\frac{1}{2}\mathbb{Z}\right)\right) \times \mathbb{Z}_2 \right)$$

where the holonomy group is $\mathbb{Z}_2^2 = \{\pm I, \pm U\}$.

(2) *If* $\mathrm{Sol}^4_{m,n}$ *does not admit a Bieberbach group with holonomy group* $\mathbb{Z}_2$ *then* $\mathrm{Sol}^4_{m,n}$ *does not admit a crystallographic group with holonomy group* $\mathbb{Z}_2^2$.

**Theorem 2.19.** *The Lie group* $\mathrm{Sol}^4_{m,n}$ *does not admit a crystallographic group with holonomy group* $\mathbb{Z}_2^3$.

*Proof.* Assume that $\mathrm{Sol}^4_{m,n}$ admits a crystallographic group with holonomy group $\Phi = \{\pm I, \pm X, \pm Y, \pm Z\}$. By Theorem 2.16, every element $U$ of $\Phi - \{\pm I\}$ must be lifted as an element $\left(\frac{s}{2}, U\right)$ in $\mathbb{Z}_\Phi$ Since $\left(\frac{s}{2}, X\right)$, $\left(\frac{s}{2}, Y\right) \in \mathbb{Z}_\Phi$, we have $\left(\frac{s}{2}, X\right)\left(\frac{s}{2}, Y\right) = (s, -Z) = (s, I)(0, -Z)$. This implies that $-Z \in \Phi$ is lifted to $(0, -Z) \in \mathbb{Z}_\Phi$ a contradiction.

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## References

[1] P. Scott, "*The geometries of 3-manifolds*", Bull. London Math. Soc., vol. 15, p. 401-487, 1983.

[2] W. P. Thurston, *Three-dimensional Geometry and Topology*, vol. 1, Princeton University Press, Princeton, NJ, 1997.

[3] K. Y. Ha and J. B. Lee, "*Crystallographic groups of* Sol", Math. Nachr., vol. 286, p. 1614-1667, 2013.

[4] K. B. Lee and S. Thuong, "*Infra-solvmanifolds of* $\mathrm{Sol}^4_1$", J. Korean Math. Soc., vol. 52, p. 1209-1251, 2015.

[5] S. V. Thuong, "*Classification of closed manifolds with* $\mathrm{Sol}^4_1$*-geometry*", Geometriae Dedicata, vol. 199, p. 373-397, 2019.

[6] W. S. Yoo, "*On the crystallographic group of* $\mathrm{Sol}^4_{m,n}$", J. Chungcheong Math. Soc., vol. 34, No. 1, p. 17-26, 2021.

[7] J. A. Hillman, *Four-manifolds, Geometries and Knots,* Geometry & Topology Monographs, 5, Geometry & Topology Publications, Coventry, 2002.