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EXISTENCE AND GENERAL DECAY FOR A VISCOELASTIC EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. In the present work, we investigate a viscoelastic equation involving a logarithmic nonlinear source term. After proving the existence of solutions, we establish a general decay estimate of the solution using energy estimates and theory of convex functions. This result extends and complements some previous results of [9, 21].

1. Introduction

We consider the viscoelastic equation with logarithmic nonlinear source

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds = |u|^{\gamma-2}u\ln|u| \quad \text{in } \Omega\times(0,\infty), \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{2}$$

$$u(0) = u_0, \;\; u_t(0) = u_1 \quad \text{in } \Omega, \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain having smooth boundary $\partial\Omega$.

One of the main concerns in the study of viscoelastic problems is to establish more general and optimal decay rates of solutions under minimal conditions of h. And many stability results have been established ([5, 10, 18–21, 23]). For instance, Messaoudi [18] obtained generalized decay rates of solutions to problem (1)-(3) without the logarithmic nonlinear source term when h verifies

$$h'(t) \leq -\varrho(t)h(t), \tag{4}$$

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where ϱ is positive, differentiable, and non-increasing. The authors of [19] proved a decay rate of general type for a quasilinear equation under the condition

$$h'(t) \leq -\varrho(t)h^p(t), \tag{5}$$

here $1 \leq p < \frac{3}{2}$. And then, a natural question 'Can the range of parameter p be extended from $1 \leq p < \frac{3}{2}$ to $1 \leq p < 2$?' was raised. Mustafa [21] answered for the question inspired by the ideas of [12] and [13]. He established more generalized and explicit decay results for (1)-(3) without the logarithmic nonlinearity when h fulfills

$$h'(t) \leq -\varrho(t)H(h(t)), \tag{6}$$

where H is a convex function meeting some conditions. He claimed that (5) with $1 \leq p < 2$ is only a special case of (6). For the articles associated with (6), we refer [11, 16].

In this article, we are concerned with a general energy decay rate for problem (1)-(3). This type of equations with logarithmic nonlinearity has its physical applications in the fields of hydrodynamics, thermodynamics and filtration theory and so on. For more detail applications, we refer to [2, 8]. Several authors considered nonlinear models of parabolic or hyperbolic equations with such nonlinearity [3, 4, 7, 14, 17, 22]. The authors of [4, 22] studied the equation

$$u_t - div(|\nabla u|^{p-2}\nabla u) - \Delta u_t = |u|^{\gamma-2}u \ln |u|.$$

They proved the existence of solutions by using the potential well method and the logarithmic Sobolev inequality. Di et al. [7] discussed the strongly damped wave equation

$$u_{tt} - \Delta u - \Delta u_t = |u|^{\gamma-2}u \ln |u|.$$

They used the modified potential well method to overcome difficulty created by the presence of $-\Delta u$ instead of $-div(|\nabla u|^{p-2}\nabla u)$. Most recent, Ha and Park [9] proved the existence and uniqueness of local solutions for a strongly damped viscoelastic wave equation with logarithmic nonlinearity, and investigated a blow-up result. Motivated by these results, we are concerned with a new energy decay result for solutions to problem (1)-(3). As we know, stability for viscoelastic wave equations involving logarithmic nonlinear source term is seldom studied.

2. Preliminaries

Let $(\varphi, \phi) = \int_\Omega \varphi(x)\phi(x)dx$. $||\cdot||_q$ denotes the norm of the space $L^q(\Omega)$, $1 \leq q \leq \infty$. For simplicity, we denote $||\cdot||_2$ by $||\cdot||$. Let B_q be the optimal constant with $||v||_q \leq B_q||\nabla v||$ for $v \in H_0^1(\Omega)$, where

$$2 \leq q \leq \frac{2N}{N-2}, \text{ if } N \geq 3; \ 2 \leq q < \infty, \text{ if } N = 1, 2.$$

We endow assumptions on γ and h as below:
(A_1) The exponent γ verifies

$$2 < \gamma < \infty, \text{ if } N = 1, 2; \; 2 < \gamma < \frac{2(N-1)}{N-2}, \text{ if } N \geq 3. \tag{7}$$

(A_2) Let $h : [0, \infty) \to (0, \infty)$ is a differentiable and nonincreasing function with

$$1 - \int_0^\infty h(s)ds := l > 0. \tag{8}$$

(A_3) As in [21], h satisfies

$$h'(t) \leq -\varrho(t)H(h(t)) \quad \text{for all } t \geq 0, \tag{9}$$

where $H : (0, \infty) \to (0, \infty)$ is a C^1-function, which is either linear or strictly increasing and strictly convex C^2-function on $(0, r]$, $r \leq h(0)$, with $H(0) = H'(0) = 0$, and ϱ is positive, nonincreasing, and differentiable.

The conditions (A_2) and (A_3) imply the existence of $t^* > 0$ with

$$h(t^*) = r. \tag{10}$$

Some examples of the kernel function h satisfying (A_2) and (A_3) are provided by Mustafa [21].

Let

$$(h\Box z)(t) = \int_0^t h(t-s)||z(t) - z(s)||^2 ds,$$

$$k_\beta(t) = \beta h(t) - h'(t),$$

and

$$C_\beta = \int_0^\infty \frac{h^2(s)}{k_\beta(s)} ds.$$

By the argument of [21], we have:

Lemma 2.1. *For any $\beta > 0$ and $z \in L^2_{loc}(0, \infty; L^2(\Omega))$,*

$$|| \int_0^t h(t-s)(z(t) - z(s))ds||^2 \leq C_\beta (k_\beta \Box z)(t) \tag{11}$$

and

$$\begin{aligned} &|| \int_0^t h'(t-s)(z(t) - z(s))ds||^2 \\ &\leq 2\Big(\int_0^t k_\beta(s)ds\Big)(k_\beta \Box z)(t) + 2\beta^2 C_\beta (k_\beta \Box z)(t). \end{aligned} \tag{12}$$

To deal with logarithmic source term, we define

$$J(w) = \frac{l}{2}||\nabla w||^2 - \frac{1}{\gamma}\int_\Omega |w(x)|^\gamma \ln |w(x)| dx + \frac{1}{\gamma^2}||w||_\gamma^\gamma, \tag{13}$$

$$I(w) = l||\nabla w||^2 - \int_\Omega |w(x)|^\gamma \ln |w(x)| dx, \tag{14}$$

$$d = \inf_{w\in H_0^1(\Omega)\setminus\{0\}} \sup_{\xi>0} J(\xi w). \tag{15}$$

Then, we know ([6, 7, 25])

$$d = \inf_{w\in\mathcal{N}} J(w), \tag{16}$$

here $\mathcal{N} = \{w \in H_0^1(\Omega) \setminus \{0\} \mid I(w) = 0\}$.

In this article, we use the auxiliary lemma below several times.

Lemma 2.2 (Lemma 2.1 in [9]). *For each $\mu > 0$, there hold*

$$|\ln\tau| \le \frac{\tau^{-\mu}}{e\mu} \quad \text{for} \quad 0 < \tau < 1 \quad \text{and} \quad 0 \le \ln\tau \le \frac{\tau^{\mu}}{e\mu} \quad \text{for} \quad \tau \ge 1.$$

3. Existence of solutions

Definition. It is said that a function u is a weak solution of problem (1)-(3) if it fulfills

$$u \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap C^2([0,T]; H^{-1}(\Omega)),$$

$$\langle u_{tt}, w\rangle + (\nabla u, \nabla w) - \int_0^t h(t-s)(\nabla u(s), \nabla w)ds = (|u|^{\gamma-2}u\ln|u|, w)$$

for $w \in H_0^1(\Omega)$, where $\langle\cdot,\cdot\rangle$ is the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, and $u(0) = u_0$ in $H_0^1(\Omega)$, $u_t(0) = u_1$ in $L^2(\Omega)$.

Theorem 3.1. *Assume that (A_1) and (A_2) hold. Let $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$, $I(u_0) > 0$, and*

$$E(u_0, u_1) := \frac{1}{2}||u_1||^2 + \frac{1}{2}||\nabla u_0||^2 - \frac{1}{\gamma}\int_\Omega |u_0(x)|^\gamma \ln|u_0(x)|dx + \frac{1}{\gamma^2}||u_0||_\gamma^\gamma < d.$$

Then, problem (1)-(3) admits a unique weak solution.

Proof. Existence. Let $\{v_j\}_{j\in\mathbb{N}}$ be a basis of $H_0^1(\Omega)$, which is orthonormal in $L^2(\Omega)$. For a fixed $n \in \mathbb{N}$, we set $V_n = span\{v_1, v_2, \ldots, v_n\}$. Theory of ordinary differential equations provide a unique local solution $u^n(x,t) = \sum_{j=1}^n f_j^n(t)v_j(x)$ on a maximal interval $[0, t_n)$, $t_n \in (0, T]$, for the approximate problem

$$\begin{cases} (u_{tt}^n(t), v) + (\nabla u^n(t), \nabla v) - \int_0^t h(t-s)(\nabla u^n(s), \nabla v)ds \\ = (|u^n(t)|^{\gamma-2}u^n(t)\ln|u^n(t)|, v) \ \ v \in V_n, \\ u^n(0) = u_0^n = \sum_{j=1}^n (u_0, v_j)v_j \to u_0 \ \text{ in } H_0^1(\Omega), \\ u_t^n(0) = u_1^n = \sum_{j=1}^n (u_1, v_j)v_j \to u_1 \ \text{ in } L^2(\Omega). \end{cases} \tag{17}$$

Now, we show that $t_n = T$ and that the solution is bounded independent of n, t. Replacing v by $u_t^n(t)$ in the first equation of (17), one gets

$$\frac{d}{dt}E(u^n(t), u_t^n(t)) = \frac{1}{2}(h'\Box\nabla u^n)(t) - \frac{h(t)}{2}||\nabla u^n(t)|| \le 0, \tag{18}$$

where

$$E(u^n(t), u_t^n(t)) = \frac{1}{2}||u_t^n(t)||^2 + \frac{1}{2}\Big(1 - \int_0^t h(s)ds\Big)||\nabla u^n(t)||^2 + \frac{1}{2}(h\Box\nabla u^n)(t) - \frac{1}{\gamma}\int_\Omega |u^n(x,t)|^\gamma \ln|u^n(x,t)|dx + \frac{1}{\gamma^2}||u^n(t)||_\gamma^\gamma.$$

Thus, we have

$$E(u^n(t), u_t^n(t)) \le E(u^n(0), u_t^n(0)) = \frac{1}{2}||u_1^n||^2 + \frac{1}{2}||\nabla u_0^n||^2 - \frac{1}{\gamma}\int_\Omega |u_0^n(x)|^\gamma \ln|u_0^n(x)|dx + \frac{1}{\gamma^2}||u_0^n||_\gamma^\gamma. \tag{19}$$

Since the functions E and I are continuous,

$$E(u_0^n, u_1^n) < d \text{ and } I(u_0^n) > 0 \text{ for appropriately large } n. \tag{20}$$

This asserts

$$E(u^n(t), u_t^n(t)) < d \tag{21}$$

and

$$I(u^n(t)) > 0 \text{ for appropriately large } n \text{ and } t \ge 0. \tag{22}$$

Indeed, from (19) and (20), it is clear

$$E(u^n(t), u_t^n(t)) < d \text{ for appropriately large } n \text{ and } t \ge 0. \tag{23}$$

In order to show $I(u^n(t)) > 0$ for $t \ge 0$, let us suppose that there exists $t_0 > 0$ such that $I(u^n(t_0)) = 0$. Then,

$$E(u^n(t_0), u_t^n(t_0)) \ge J(u^n(t_0)) \ge \inf_{w\in\mathcal{N}} J(w) = d,$$

which contradicts (23). Thus, (22) is proved.

From (21), we obtain

$$0 < \frac{1}{2}||u_t^n(t)||^2 + \frac{l(\gamma-2)}{2\gamma}||\nabla u^n(t)||^2 + \frac{1}{\gamma}I(u^n(t)) < d. \tag{24}$$

Therefore, we can choose a subsequence of $\{u^n\}$, we still denote it by $\{u^n\}$, with

$$\begin{cases} u^n \to u \quad \text{weak}^* \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ u_t^n \to u_t \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{25}$$

Aubin-Lions compactness theorem ensures

$$u^n(x,t) \to u(x,t) \text{ a.e. } (x,t) \in \Omega \times (0,T) \tag{26}$$

and

$$|u^n(x,t)|^{\gamma-2}u^n(x,t)\ln|u^n(x,t)| \to |u(x,t)|^{\gamma-2}u(x,t)\ln|u(x,t)| \text{ a.e. } (x,t) \in \Omega\times(0,T). \tag{27}$$

Next, we show that

$$|u^n|^{\gamma-1}\ln|u^n| \text{ is bounded in } L^\infty(0,T;L^{\frac{\gamma}{\gamma-1}}(\Omega)). \tag{28}$$

For this, we let

$$\Omega_1 = \{x \in \Omega : |u^n(x,t)| < 1\} \text{ and } \Omega_2 = \{x \in \Omega : |u^n(x,t)| \geq 1\}.$$

Thanks to $2 < \gamma < \frac{2N}{N-2}$, we can take $\mu_1 > 0$ such that $2 < \gamma + \frac{\mu_1 \gamma}{\gamma - 1} < \frac{2N}{N-2}$. So, by the same argument of Eq. (3.7) of [9], we know

$$\int_\Omega \left(\left| |u^n(x,t)|^{\gamma-1} \ln |u^n(x,t)| \right| \right)^{\frac{\gamma}{\gamma-1}} dx$$

$$\leq \left(\frac{1}{e(\gamma-1)} \right)^{\frac{\gamma}{\gamma-1}} |\Omega_1| + B_{\frac{\gamma(\gamma-1+\mu_1)}{\gamma-1}}^{\frac{\gamma(\gamma-1+\mu_1)}{\gamma-1}} \left(\frac{1}{e\mu_1} \right)^{\frac{\gamma}{\gamma-1}} ||\nabla u^n(t)||^{\frac{\gamma(\gamma-1+\mu_1)}{\gamma-1}} \leq c \tag{29}$$

for some $c > 0$. By (27), (28) and Lemma 1.3 in [15], one gets

$$|u^n|^{\gamma-2} u^n \ln |u^n| \to |u|^{\gamma-2} u \ln |u| \quad \text{weak}^* \text{ in } L^\infty(0,T; L^{\frac{\gamma}{\gamma-1}}(\Omega)).$$

The rest of the proof can be done as the proofs of Lemma 3.1 and Theorem 3.1 of [9]. □

4. New general decay

The energy of problem (1)-(3) is defined by

$$E(t) =: E(u(t), u_t(t)) = \frac{1}{2}||u_t(t)||^2 + \frac{1}{2}\Big(1 - \int_0^t h(s)ds\Big)||\nabla u(t)||^2 + \frac{1}{2}(h\Box\nabla u)(t)$$

$$- \frac{1}{\gamma}\int_\Omega |u(x,t)|^\gamma \ln |u(x,t)| dx + \frac{1}{\gamma^2}||u(t)||_\gamma^\gamma. \tag{30}$$

Then, we find

$$E(t) \geq \frac{1}{2}||u_t(t)||^2 + \frac{1}{2}(h\Box\nabla u)(t) + J(u(t))$$

$$= \frac{1}{2}||u_t(t)||^2 + \frac{1}{2}(h\Box\nabla u)(t) + \frac{l(\gamma-2)}{2\gamma}||\nabla u(t)||^2 + \frac{1}{\gamma^2}||u(t)||_\gamma^\gamma + \frac{1}{\gamma}I(u(t)) \tag{31}$$

and

$$E'(t) = \frac{1}{2}(h'\Box\nabla u)(t) - \frac{h(t)}{2}||\nabla u(t)||^2 \leq 0. \tag{32}$$

By the same arguments of (21) and (22), if $I(u_0) > 0$ and $E(0) = E(u_0, u_1) < d$, then

$$I(u(t)) > 0 \text{ and } 0 < E(t) < d \text{ for } t \geq 0. \tag{33}$$

Following the ideas of [21], we define

$$L(t) = ME(t) + M_1\Phi(t) + M_2\Psi(t),$$

where $M, M_1, M_2 > 0$ and

$$\Phi(t) = (u_t(t), u(t)), \;\; \Psi(t) = -\int_0^t h(t-s)(u(t) - u(s), u_t(t))ds.$$

Lemma 4.1. *For every* $0 < \eta < 1$ *and* $\beta > 0$, *it holds*

$$\Phi'(t) \leq ||u_t(t)||^2 - \frac{l}{2}||\nabla u(t)||^2 + \int_\Omega |u(x,t)|^\gamma \ln |u(x,t)| dx + \frac{C_\beta}{2l}(k_\beta \Box \nabla u)(t) \tag{34}$$

and

$$\Psi'(t) \leq -\Big(\int_0^t h(s)ds - \eta\Big)||u_t(t)||^2 + \frac{c(C_\beta + 1)}{\eta}(k_\beta \Box \nabla u)(t) + \eta C_{E(0)}||\nabla u(t)||^2, \tag{35}$$

where

$$C_{E(0)} = 1 + \Big(\frac{B_{2(\gamma-1-\mu_3)}^{\gamma-1-\mu_3}}{e\mu_3}\Big)^2\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\gamma-2-\mu_3} + \Big(\frac{B_{2(\gamma-1+\mu_4)}^{\gamma-1+\mu_4}}{e\mu_4}\Big)^2\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\gamma-2+\mu_4}.$$

Proof. Using (1)-(2) and Young's inequality, one finds

$$\begin{aligned}\Phi'(t) &= ||u_t(t)||^2 - \Big(1 - \int_0^t h(s)ds\Big)||\nabla u(t)||^2 + \int_\Omega |u(x,t)|^\gamma \ln|u(x,t)|dx \\ &\quad + \int_0^t h(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t))ds \\ &\leq ||u_t(t)||^2 - \frac{l}{2}||\nabla u(t)||^2 + \int_\Omega |u(x,t)|^\gamma \ln|u(x,t)|dx \\ &\quad + \frac{1}{2l}||\int_0^t h(t-s)(\nabla u(s) - \nabla u(t))ds||^2. \end{aligned}\tag{36}$$

The formula (34) is observed applying (11) to the last term of (36). Similarly,

$$\begin{aligned}\Psi'(t) = &-\int_0^t h(s)ds||u_t(t)||^2 - \int_0^t h'(t-s)(u(t) - u(s), u_t(t))ds \\ &+ \Big(1 - \int_0^t h(s)ds\Big)\int_0^t h(t-s)(\nabla u(t) - \nabla u(s), \nabla u(t))ds \\ &+ ||\int_0^t h(t-s)(\nabla u(t) - \nabla u(s))ds||^2 \\ &- \int_0^t h(t-s)\int_\Omega (u(x,t) - u(x,s))|u(x,t)|^{\gamma-2}u(x,t)\ln|u(x,t)|dxds \\ := &-\int_0^t h(s)ds||u_t(t)||^2 + \sum_{i=3}^{6} K_i. \end{aligned}\tag{37}$$

By (11) and (12), it holds, for $0 < \eta < 1$ and $\beta > 0$,

$$|K_3| \leq \eta||u_t(t)||^2 + \frac{1}{2\eta}\Big(\int_0^t k_\beta(s)ds\Big)(k_\beta \Box u)(t) + \frac{1}{2\eta}\beta^2 C_\beta (k_\beta \Box u)(t)$$

$$\leq \eta||u_t(t)||^2 + \frac{(\beta(1-l)+h(0))B_2^2}{2\eta}(k_\beta\Box\nabla u)(t) + \frac{B_2^2\beta^2}{2\eta}C_\beta(k_\beta\Box\nabla u)(t),$$

$$|K_4| \leq \eta||\nabla u(t)||^2 + \frac{\left(1-\int_0^t h(s)ds\right)^2}{4\eta}C_\beta(k_\beta\Box\nabla u)(t),$$

$$|K_5| \leq C_\beta(k_\beta\Box\nabla u)(t) < \frac{C_\beta}{\eta}(k_\beta\Box\nabla u)(t),$$

and

$$|K_6| \leq \eta\int_\Omega \left(|u(x,t)|^{\gamma-1}\ln|u(x,t)|\right)^2 dx + \frac{B_2^2C_\beta}{4\eta}(k_\beta\Box\nabla u)(t).$$

Let

$$\Omega_3 = \{x\in\Omega : |u(x,t)|<1\} \text{ and } \Omega_4 = \{x\in\Omega : |u(x,t)|\geq 1\}.$$

Due to $2 < 2(\gamma-1) < \frac{2N}{N-2}$, there exist $\mu_3 > 0$ and $\mu_4 > 0$ such that $2 < 2(\gamma-1)-2\mu_3 < \frac{2N}{N-2}$ and $2 < 2(\gamma-1+\mu_4) < \frac{2N}{N-2}$, respectively. So, adapting Lemma 2.2, we get

$$\begin{aligned}
&\int_\Omega \left(|u(x,t)|^{\gamma-1}\ln|u(x,t|\right)^2 dx \\
&\leq \left(\frac{1}{e\mu_3}\right)^2\int_{\Omega_3}|u(x,t)|^{2(\gamma-1)-2\mu_3}dx + \left(\frac{1}{e\mu_4}\right)^2\int_{\Omega_4}|u(x,t)|^{2(\gamma-1+\mu_4)}dx \\
&\leq \left(\frac{B_{2(\gamma-1-\mu_3)}^{\gamma-1-\mu_3}}{e\mu_3}\right)^2\left(\frac{2\gamma E(0)}{l(\gamma-2)}\right)^{\gamma-2-\mu_3}||\nabla u(t)||^2 \\
&\quad + \left(\frac{B_{2(\gamma-1+\mu_4)}^{\gamma-1+\mu_4}}{e\mu_4}\right)^2\left(\frac{2\gamma E(0)}{l(\gamma-2)}\right)^{\gamma-2+\mu_4}||\nabla u(t)||^2.
\end{aligned}$$

Collecting these and (37), the inequality (35) is deduced. □

Lemma 4.2. *Let the conditions of Theorem 3.1 be fulfilled. Moreover, assume that*

$$E(0) < \min\left\{d, \frac{l(\gamma-2)}{2\gamma}\left(\frac{le\kappa}{2B_{\gamma+\kappa}^{\gamma+\kappa}}\right)^{\frac{2}{\gamma+\kappa-2}}\right\},$$

where $\kappa > 0$ with $2 < \gamma+\kappa < \frac{2N}{N-2}$. Then, there exist $\rho > 0$, $c_1 > 0$, and $c_2 > 0$ such that

$$L'(t) \leq -\rho E(t) + \frac{1}{2}(h\Box\nabla u)(t) - 3(1-l)||\nabla u(t)||^2 \quad \text{for } t\geq t^*. \tag{38}$$

Moreover, $E(t)$ is equivalent to $L(t)$.

Proof. Recalling $h' = \beta h - k_\beta$, (33), (1), and (A_2), we get

$$\begin{aligned} L'(t) \le &\frac{M\beta}{2}(h\Box\nabla u)(t) - \frac{M}{2}(k_\beta\Box\nabla u)(t) - \frac{M}{2}h(t)||\nabla u(t)||^2 \\ &- \Big\{M_2\Big(\int_0^t h(s)ds - \eta\Big) - M_1\Big\}||u_t(t)||^2 \\ &+ M_1\int_\Omega |u(x,t)|^\gamma \ln|u(x,t)|dx - \Big\{\frac{M_1 l}{2} - M_2\eta C_{E(0)}\Big\}||\nabla u(t)||^2 \\ &- \Big\{-\frac{M_1C_\beta}{2l} - \frac{M_2c(C_\beta+1)}{\eta}\Big\}(k_\beta\Box\nabla u)(t), \end{aligned}$$

and hence

$$\begin{aligned} L'(t) \le &-\rho E(t) + \Big\{\frac{M\beta}{2} + \frac{\rho}{2}\Big\}(h\Box\nabla u)(t) \\ &- \Big\{M_2\left(\int_0^t h(s)ds - \eta\right) - M_1 - \frac{\rho}{2}\Big\}||u_t(t)||^2 \\ &+ \Big(M_1 - \frac{\rho}{\gamma}\Big)\int_\Omega |u(x,t)|^\gamma \ln|u(x,t)|dx + \frac{\rho}{\gamma^2}||u(t)||_\gamma^\gamma \\ &- \Big\{\frac{M_1 l}{2} - M_2\eta C_{E(0)} - \frac{\rho}{2}\Big(1 - \int_0^t h(s)ds\Big)\Big\}||\nabla u(t)||^2 \\ &- \Big\{\frac{M}{2} - \frac{M_1C_\beta}{2l} - \frac{M_2c(C_\beta+1)}{\eta}\Big\}(k_\beta\Box\nabla u)(t) \quad \text{for any } \rho > 0. \end{aligned} \tag{39}$$

Owing to the assumption (A_1), we can pick $\kappa > 0$ with

$$2 < \gamma + \kappa < \infty, \text{ if } N = 1, 2; \; 2 < \gamma + \kappa < \frac{2N}{N-2}, \text{ if } N \ge 3.$$

From this, Lemma 2.2, (31), and (32), we find

$$\begin{aligned} \int_\Omega |u(x,t)|^\gamma \ln|u(x,t)|dx &\le \frac{1}{e\kappa}\int_{|u(x,t)|\ge 1} |u(x,t)|^{\gamma+\kappa}dx \\ &\le \frac{1}{e\kappa}||u(t)||_{\gamma+\kappa}^{\gamma+\kappa} \\ &\le \frac{B_{\gamma+\kappa}^{\gamma+\kappa}}{e\kappa}(||\nabla u(t)||^2)^{\frac{\gamma+\kappa-2}{2}}||\nabla u(t)||^2 \\ &\le \frac{B_{\gamma+\kappa}^{\gamma+\kappa}}{e\kappa}\Big(\frac{2\gamma E(t)}{l(\gamma-2)}\Big)^{\frac{\gamma+\kappa-2}{2}}||\nabla u(t)||^2 \\ &\le \frac{B_{\gamma+\kappa}^{\gamma+\kappa}}{e\kappa}\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\frac{\gamma+\kappa-2}{2}}||\nabla u(t)||^2 \end{aligned}$$

and

$$||u(t)||_\gamma^\gamma \le B_\gamma^\gamma||\nabla u(t)||^\gamma \le B_\gamma^\gamma\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\frac{\gamma-2}{2}}||\nabla u(t)||^2.$$

Substituting these into (39), letting $h^* = \int_0^{t^*} h(s)ds$, and selecting $M_1 > \frac{\rho}{\gamma}$, we observe

$$\begin{aligned} L'(t) \le & -\rho E(t) + \Big\{\frac{M\beta}{2} + \frac{\rho}{2}\Big\}(h\Box\nabla u)(t) - \Big\{M_2(h^* - \eta) - M_1 - \frac{\rho}{2}\Big\}||u_t(t)||^2 \\ & - \Big\{M_1\Big(\frac{l}{2} - \frac{B_{\gamma+\kappa}^{\gamma+\kappa}}{e\kappa}\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\frac{\gamma+\kappa-2}{2}}\Big) - M_2\eta C_{E(0)} \\ & - \frac{\rho}{2}\Big(1 - \int_0^t h(s)ds\Big) - \frac{\rho B_\gamma^\gamma}{\gamma^2}\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\frac{\gamma-2}{2}}\Big\}||\nabla u(t)||^2 \\ & - \Big\{\frac{M}{2} - \frac{M_1 C_\beta}{2l} - \frac{M_2 c(C_\beta + 1)}{\eta}\Big\}(k_\beta\Box\nabla u)(t) \quad \text{for } t \ge t^*. \end{aligned}$$

Taking $\eta = \frac{l}{4M_2}$, we have

$$\begin{aligned} L'(t) \le & -\rho E(t) + \Big\{\frac{M\beta}{2} + \frac{\rho}{2}\Big\}(h\Box\nabla u)(t) - \Big\{M_2 h^* - \frac{l}{4} - M_1 - \frac{\rho}{2}\Big\}||u_t(t)||^2 \\ & - \Big\{M_1\Big(\frac{l}{2} - \frac{B_{\gamma+\kappa}^{\gamma+\kappa}}{e\kappa}\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\frac{\gamma+\kappa-2}{2}}\Big) - \frac{lC_{E(0)}}{4} \\ & - \frac{\rho}{2}\Big(1 - \int_0^t h(s)ds\Big) - \frac{\rho B_\gamma^\gamma}{\gamma^2}\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\frac{\gamma-2}{2}}\Big\}||\nabla u(t)||^2 \\ & - \Big\{\frac{M}{4} - \frac{4cM_2^2}{l} + \frac{M}{4} - C_\beta\Big(\frac{M_1}{2l} + \frac{4cM_2^2}{l}\Big)\Big\}(k_\beta\Box\nabla u)(t) \quad \text{for } t \ge t^*. \end{aligned}$$

We fix $M_1 > \frac{\rho}{\gamma}$ suitably large again so that

$$M_1\Big(\frac{l}{2} - \frac{B_{\gamma+\kappa}^{\gamma+\kappa}}{e\kappa}\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\frac{\gamma+\kappa-2}{2}}\Big) - \frac{lC_{E(0)}}{4} > 4(1-l) \tag{40}$$

and pick $M_2 > \frac{l}{4}$ appropriately large such that

$$M_2 h^* - \frac{l}{4} - M_1 > 1. \tag{41}$$

Since $\frac{\beta h^2(s)}{k_\beta(s)} < h(s)$, $\lim_{\beta\to 0^+} \beta C_\beta = \lim_{\beta\to 0^+} \int_0^\infty \frac{\beta h^2(s)}{k_\beta(s)} ds = 0$. Thus, there exists $0 < \beta_0 < 1$ such that

$$\beta C_\beta < \frac{1}{8(\frac{M_1}{2l} + \frac{4cM_2^2}{l})} \quad \text{for } \beta < \beta_0. \tag{42}$$

Now, we take $\beta = \frac{1}{2M}$ and $M > 0$ appropriately so that

$$\beta = \frac{1}{2M} < \beta_0 \quad \text{and} \quad \frac{M}{4} - \frac{4cM_2^2}{l} > 0. \tag{43}$$

From (42) and (43), we obtain

$$\frac{M}{4} - C_\beta\Big(\frac{M_1}{2l} + \frac{4cM_2^2}{l}\Big) > 0. \tag{44}$$

Considering (40), (41), (43), (44), we arrive at

$$L'(t) \leq -\rho E(t) + \Big\{\frac{1}{4} + \frac{\rho}{2}\Big\}(h\Box\nabla u)(t) - \Big\{1 - \frac{\rho}{2}\Big\}||u_t(t)||^2$$
$$- \Big\{4(1-l) - \frac{\rho}{2}\Big(1 - \int_0^t h(s)ds\Big) - \frac{\rho B_\gamma^\gamma}{\gamma^2}\Big(\frac{2\gamma E(0)}{l(\gamma-2)}\Big)^{\frac{\gamma-2}{2}}\Big\}||\nabla u(t)||^2.$$

(38) can be obtained selecting $\rho > 0$ appropriately small. Furthermore, Young inequality, (33), and (31) provide

$$\begin{aligned}
&|L(t) - ME(t)| \\
\leq\ & \frac{M_1+M_2}{2}||u_t(t)||^2 + \frac{M_2B_2^2}{2}\Big(\int_0^t h(s)ds\Big)(h\Box\nabla u)(t) + \frac{M_1B_2^2}{2}||\nabla u(t)||^2 \\
=\ & \frac{M_1+M_2}{2}||u_t(t)||^2 + \frac{M_2B_2^2}{2}\Big(\int_0^t h(s)ds\Big)(h\Box\nabla u)(t) \\
&+ \frac{M_1B_2^2\gamma}{l(\gamma-2)}\Big(J(u(t)) - \frac{1}{\gamma}I(u(t)) - \frac{1}{\gamma^2}||u(t)||_\gamma^\gamma\Big) \\
\leq\ & \max\Big\{\frac{M_1+M_2}{4}, \frac{M_2B_2^2(1-l)}{4}, \frac{M_1B_2^2\gamma}{l(\gamma-2)}\Big\}E(t).
\end{aligned}$$

Again, taking $M > \max\Big\{\frac{M_1+M_2}{4}, \frac{M_2B_2^2(1-l)}{4}, \frac{M_1B_2^2\gamma}{l(\gamma-2)}\Big\}$, we complete the proof. □

Theorem 4.3. *Let the conditions of Lemma 4.5 and* (A_3) *be satisfied. Then, the energy of problem* (1)-(3) *verifies*

$$E(t) \leq C_0\tilde{H}^{-1}\Big(\omega\int_{h^{-1}(r)}^t \varrho(s)ds\Big) \quad \text{for } t \geq t^*,$$

where $\omega > 0$, $C_0 > 0$, *and*

$$\tilde{H}(s) = \int_s^r \frac{1}{\tau H'(\tau)}d\tau. \tag{45}$$

Proof. The proof is similar to those of [21,24]. But we state the proof here for completeness. Since h and ϱ are nonincreasing and continuous on $[0,\infty)$, they are bounded on $[0,t^*]$. Thus, for some $c_3 > 0$ and $c_4 > 0$, it holds

$$c_3 \leq \varrho(t)H(h(t)) \leq c_4 \quad \text{for } t \in [0,t^*].$$

Moreover, we observe that

$$h'(t) \leq -\varrho(t)H(h(t)) \leq -c_3 \leq -\frac{c_3}{h(0)}h(t) \quad \text{for } t \in [0,t^*].$$

From this, (38), and (32), we get

$$L'(t) \leq -\rho E(t) - \frac{h(0)}{2c_3}(h'\Box\nabla u)(t) + \frac{1}{2}\int_{t^*}^t h(s)||\nabla u(t) - \nabla u(t-s)||^2ds$$

$$\le -\rho E(t) - \frac{h(0)}{c_3}E'(t) + \frac{1}{2}\int_{t^*}^{t} h(s)||\nabla u(t) - \nabla u(t-s)||^2 ds, \ t \ge t^*.$$

Letting $F(t) = L(t) + \frac{h(0)}{c_3}E(t)$, we have

$$F'(t) \le -\rho E(t) + \frac{1}{2}\int_{t^*}^{t} h(s)||\nabla u(t) - \nabla u(t-s)||^2 ds \ \text{ for } t \ge t^*. \tag{46}$$

Case 1: *H is linear, that is, $H(s) = as$ for some $a > 0$.*
Using $\varrho'(t) \le 0$, (46), (9), and (32), we get

$$\begin{aligned}
&(\varrho(t)F(t) + \frac{1}{a}E(t))' \\
\le\ & -\rho\varrho(t)E(t) + \frac{1}{2}\int_{t^*}^{t} \varrho(s)h(s)||\nabla u(t) - \nabla u(t-s)||^2 ds + \frac{1}{a}E'(t) \\
\le\ & -\rho\varrho(t)E(t) - \frac{1}{2a}\int_{t^*}^{t} h'(s)||\nabla u(t) - \nabla u(t-s)||^2 ds + \frac{1}{a}E'(t) \\
\le\ & -\rho\varrho(t)E(t) \ \text{ for } t \ge t^*. \qquad (47)
\end{aligned}$$

Case 2: *H is nonlinear.*
Let $\Xi(t) = \int_0^t \left(\int_{t-s}^{\infty} h(\tau)d\tau\right)||\nabla u(s)||^2 ds$. Then it meets (see Lemma 3.4 in [21])

$$\Xi'(t) \le -\frac{1}{2}(h\Box\nabla u)(t) + 3(1-l)||\nabla u(t)||^2.$$

From Lemma 4.2 and this, we get

$$(L(t) + \Xi(t))' \le -\rho E(t). \tag{48}$$

This and (33) ensure

$$\begin{aligned}
0 < \int_0^{\infty} E(s)ds &\le \int_{t^*}^{t} E(s)ds \\
&\le -\frac{1}{\rho}\int_{t^*}^{t}(L'(s) + \Xi'(s))ds \le \frac{L(t^*) + \Xi(t^*)}{\rho} < \infty. \qquad (49)
\end{aligned}$$

Now, we define

$$\Gamma(t) := m\int_{t^*}^{t} ||\nabla u(t) - \nabla u(t-s)||^2 ds$$

and

$$\chi(t) := -\int_{t^*}^{t} h'(s)||\nabla u(t) - \nabla u(t-s)||^2 ds.$$

From (49), we can select $0 < m < 1$ such that

$$\Gamma(t) < 1 \ \text{ for } t \ge t^*. \tag{50}$$

It is also observed that

$$\chi(t) \le -(h'\Box\nabla u)(t) \le -2E'(t). \tag{51}$$

Making use of (A_3), the fact $H(\lambda y) \leq \lambda H(y)$ for $0 \leq \lambda \leq 1$ and $y \in (0, r]$, and Jensen's inequality, we infer

$$
\begin{aligned}
\chi(t) &= -\frac{1}{m\Gamma(t)} \int_{t^*}^{t} \Gamma(t) h'(s) m ||\nabla u(t) - \nabla u(t-s)||^2 ds \\
&\geq \frac{1}{m\Gamma(t)} \int_{t^*}^{t} \Gamma(t) \varrho(s) H(h(s)) m ||\nabla u(t) - \nabla u(t-s)||^2 ds \\
&\geq \frac{\varrho(t)}{m\Gamma(t)} \int_{t^*}^{t} H\Big(\Gamma(t) h(s)\Big) m ||\nabla u(t) - \nabla u(t-s)||^2 ds \\
&\geq \frac{\varrho(t)}{m} H\Big(m \int_{t^*}^{t} h(s) ||\nabla u(t) - \nabla u(t-s)||^2 ds\Big) \\
&= \frac{\varrho(t)}{m} \overline{H}\Big(m \int_{t^*}^{t} h(s) ||\nabla u(t) - \nabla u(t-s)||^2 ds\Big), \qquad (52)
\end{aligned}
$$

where $\overline{H}$ is an extension of H as $\overline{H}$ is a strictly increasing and strictly convex C^2-function on $(0, \infty)$ and the fact $m \int_{t^*}^{t} h(s) ||\nabla u(t) - \nabla u(t-s)||^2 ds < r$ is used in the last equality. Thus, we see from (52) that

$$
\int_{t^*}^{t} h(s) ||\nabla u(t) - \nabla u(t-s)||^2 ds \leq \frac{1}{m} \overline{H}^{-1}\Big(\frac{m\chi(t)}{\varrho(t)}\Big).
$$

Adapting this to (46), we find

$$
F'(t) \leq -\rho E(t) + \frac{1}{2m} \overline{H}^{-1}\Big(\frac{m\chi(t)}{\varrho(t)}\Big) \quad \text{for } t \geq t^*. \qquad (53)
$$

On the other hand, for the convex function $\overline{H}$, it is known that

$$
yz \leq \overline{H}^*(y) + \overline{H}(z) \quad \text{for } y, z \geq 0 \qquad (54)
$$

and

$$
\overline{H}^*(y) = y(\overline{H}')^{-1}(y) - \overline{H}((\overline{H}')^{-1}(y)) \quad \text{for } y \geq 0, \qquad (55)
$$

where $\overline{H}^*$ is the conjugate function of the convex function $\overline{H}$ (see [1]).

Next, let $0 < \theta < r$, $\mathcal{E}(t) = \frac{E(t)}{E(0)}$, and $c_5 > 0$. Since $\overline{H}'(s) > 0$, $\overline{H}''(s) > 0$, and $\overline{H}(0) = \overline{H}'(0) = 0$, we observe from (53), (54), and (55) that

$$
\begin{aligned}
&\Big[\overline{H}'(\theta\mathcal{E}(t)) F(t) + c_5 E(t)\Big]' \\
\leq& -\rho \overline{H}'(\theta\mathcal{E}(t)) E(t) + \frac{1}{2m} \overline{H}'(\theta\mathcal{E}(t)) \overline{H}^{-1}\Big(\frac{m\chi(t)}{\varrho(t)}\Big) + c_5 E'(t) \\
\leq& -\rho \overline{H}'(\theta\mathcal{E}(t)) E(t) + \frac{1}{2m} \overline{H}^*\Big(\overline{H}'(\theta\mathcal{E}(t))\Big) + \frac{\chi(t)}{2\varrho(t)} + c_5 E'(t) \\
\leq& -\rho E(0) H'(\theta\mathcal{E}(t)) \mathcal{E}(t) + \frac{\theta}{2m} \mathcal{E}(t) H'(\theta\mathcal{E}(t)) + \frac{\chi(t)}{2\varrho(t)} + c_5 E'(t), \qquad (56)
\end{aligned}
$$

we used $\theta\mathcal{E}(t) < r$ in the last equality. Using $\varrho'(t) \le 0$ and (51), we have

$$\begin{aligned} &\Big[\varrho(t)\Big\{\overline{H}'(\theta\mathcal{E}(t))F(t) + c_5E(t)\Big\} + E(t)\Big]' \\ (57) \qquad &\le -\varrho(t)\Big(\rho E(0) - \frac{\theta}{2m}\Big)H'(\theta\mathcal{E}(t))\mathcal{E}(t). \end{aligned}$$

Choosing $\theta > 0$ sufficiently small such that $c_6 := \rho E(0) - \frac{\theta}{2m} > 0$, we arrive at

$$\begin{aligned} &\Big[\varrho(t)\Big\{\overline{H}'(\theta\mathcal{E}(t))F(t) + c_5E(t)\Big\} + E(t)\Big]' \\ (58) \qquad &\le -c_6\varrho(t)H'(\theta\mathcal{E}(t))\mathcal{E}(t). \end{aligned}$$

Letting

$$(59) \qquad \mathcal{F}(t) = \begin{cases} \varrho(t)F(t) + \frac{1}{a}E(t) & \text{if } H \text{ is linear;} \\ \varrho(t)\Big\{\overline{H}'(\theta\mathcal{E}(t))F(t) + c_5E(t)\Big\} + E(t) & \text{if } H \text{ is nonlinear,} \end{cases}$$

we obtain from (47) and (58) that

$$(60) \qquad \mathcal{F}'(t) \le -c_7\varrho(t)H_0(\mathcal{E}(t)) \quad \text{for } t \ge t^*,$$

where $c_7 = \min\{\frac{\rho E(0)}{a}, c_6\}$ and

$$(61) \qquad H_0(s) = \begin{cases} s & \text{if } H \text{ is linear;} \\ sH'(\theta s) & \text{if } H \text{ is nonlinear.} \end{cases}$$

Due to $\mathcal{F}(t) \sim E(t)$, there exist $c_8, c_9 > 0$ such that

$$(62) \qquad c_8\mathcal{F}(t) \le E(t) \le c_9\mathcal{F}(t).$$

Finally, putting

$$(63) \qquad \mathcal{L}(t) = \frac{c_8\mathcal{F}(t)}{E(0)},$$

we see that

$$(64) \qquad \mathcal{L}(t) \le \mathcal{E}(t) \le 1.$$

Due to the fact H_0 is increasing on $(0,1]$, (63), (60) and (64), we deduce

$$(65) \qquad \mathcal{L}'(t) \le -c_{10}\varrho(t)H_0(\mathcal{L}(t)) \quad \text{for } t \ge t^*,$$

where $c_{10} = \frac{c_7c_8}{E(0)}$. Thus, we get

$$\begin{aligned} \int_{t^*}^t c_{10}\varrho(s)ds &\le -\int_{t^*}^t \frac{\mathcal{L}'(s)}{H_0(\mathcal{L}(s))}ds = -\int_{t^*}^t \frac{\mathcal{L}'(s)}{\mathcal{L}(s)H'(\theta\mathcal{L}(s))}ds \\ &= \int_{\theta\mathcal{L}(t)}^{\theta\mathcal{L}(t^*)} \frac{1}{sH'(s)}ds \\ (66) \qquad &\le \int_{\theta\mathcal{L}(t)}^{r} \frac{1}{sH'(s)}ds = \tilde{H}(\theta\mathcal{L}(t)), \end{aligned}$$

here $\tilde{H}$ is the function defined in (45). Since $\tilde{H}$ is strictly decreasing on $(0, r]$, we conclude, for some $\omega > 0$,

$$\mathcal{L}(t) \leq \frac{1}{\theta}\tilde{H}^{-1}\Big(\omega \int_{t^*}^{t} \varrho(s)ds\Big) \quad \text{for } t \geq t^*.$$

This completes the proof. □

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