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BERGMAN TYPE OPERATORS ON SOME GENERALIZED CARTAN-HARTOGS DOMAINS

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Abstract. For $\mu = (\mu_1, \dots, \mu_t)$ $(\mu_j > 0)$, $\xi = (z_1, \dots, z_t, w) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_t} \times \mathbb{C}^m$, define

$$\Omega(\mu, t) = \{\xi \in \mathbb{B}_{n_1} \times \cdots \times \mathbb{B}_{n_t} \times \mathbb{C}^m : \|w\|^2 < C(\chi, \mu) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{\mu_j}\},$$

where $\mathbb{B}_{n_j}$ is the unit ball in $\mathbb{C}^{n_j}$ $(1 \le j \le t)$, $C(\chi, \mu)$ is a constant only depending on $\chi = (n_1, \dots, n_t)$ and $\mu = (\mu_1, \dots, \mu_t)$, which is a special type of generalized Cartan-Hartogs domain. We will give some sufficient and necessary conditions for the boundedness of some type of operators on $L^p(\Omega(\mu, t), \omega)$ (the weighted L^p space of $\Omega(\mu, t)$ with weight ω, $1 < p < \infty$). This result generalizes the works from certain classes of generalized complex ellipsoids to the generalized Cartan-Hartogs domain $\Omega(\mu, t)$.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and $\omega(z)$ be a non-negative measurable function on Ω. For $p \in [1, \infty)$, let $L^p(\Omega, \omega)$ denote the set of all complex measurable functions f for which

$$\left(\int_\Omega |f(z)|^p \omega(z) dV(z) \right)^{\frac{1}{p}} < \infty,$$

where $dV(z)$ is the ordinary Lebesgue measure on Ω. We call $\omega(z)$ a weight on Ω and $L^p(\Omega, \omega)$ the weighted L^p space of Ω. The norm of $L^p(\Omega, \omega)$ is defined as follows:

$$\|f\|_{p,\omega}^p = \int_\Omega |f(z)|^p \omega(z) dV(z).$$

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If $p = 2$, $L^2(\Omega, \omega)$ is a Hilbert space with the inner product:

$$\langle f, g\rangle_\omega = \int_\Omega f(z)\overline{g(z)}\omega(z)dV(z).$$

The weighted Bergman space of Ω with weight ω is defined by

$$A^p(\Omega, \omega) := \mathcal{O}(\Omega) \cap L^p(\Omega, \omega),$$

where $\mathcal{O}(\Omega)$ is the space of all holomorphic functions on Ω. $A^2(\Omega, \omega)$ is a subspace of holomorphic functions in $L^2(\Omega, \omega)$. By the result in [16] we know that if ω is continuous and never vanishes inside Ω, then $A^2(\Omega, \omega)$ is a closed subspace of $L^2(\Omega, \omega)$ and there exists the orthogonal projection, called the weighted Bergman projection:

$$P_\omega : L^2(\Omega, \omega) \to A^2(\Omega, \omega).$$

This projection is an integral operator with the weighted Bergman kernel, denoted by $K_\omega(z, w)$:

$$P_\omega f(z) := \int_\Omega K_\omega(z, w)f(w)\omega(w)dV(w).$$

When $\omega(z) \equiv 1$, the weighted Bergman kernel K_ω and the weighted Bergman projection P_ω will degenerate to the ordinary Bergman kernel K and the ordinary Bergman projection P, respectively. If $\{\phi_j : j = 0, 1, 2, \dots\}$ is an orthonormal basis of $A^2(\Omega, \omega)$, then the weighted kernel function satisfies

$$K_\omega(z, w) = \sum_{j=0}^{\infty} \phi_j(z)\overline{\phi_j(w)}.$$

It is of great interest to find an explicit form of the Bergman kernel for a domain Ω. Many results have been obtained when Ω has nice properties. Gindikin [7] obtained explicit forms of the Bergman kernels for homogeneous domains. For non-homogeneous domains, there are many results on the explicit forms of the Bergman kernels obtained in [1, 3, 8, 12–15, 18–22].

Let Ω be a bounded domain in $\mathbb{C}^n$ and $p : \Omega \to (0, \infty)$ a continuous function. Define a Hartogs type domain $\widehat{\Omega}$ by

$$\widehat{\Omega} := \{(z, w) \in \Omega \times \mathbb{C}^m : \|w\|^2 < p(z)\},$$

where $\|\cdot\|$ is the standard Hermitian norm on $\mathbb{C}^m$. By the Forelli-Rudin construction, Ligocka [10] studied the properties of holomorphic functions on Hartogs type domains and obtained the regularity of weighted Bergman projections on weakly regular domains. Using similar method, Charpentier-Dupain-Mounkaila [4] investigated the regularity properties of weighted Bergman projections for smoothly bounded pseudoconvex domains of finite type in $\mathbb{C}^n$.

Let Ω be an irreducible bounded symmetric domain in $\mathbb{C}^n$ of genus γ, the generic norm of Ω is defined by

$$N(z, w) := (V(\Omega)K(z, w))^{-\frac{1}{\gamma}},$$

where $V(\Omega)$ is the volume of Ω with respect to the Euclidean measure of $\mathbb{C}^n$ and $K(z,w)$ is the Bergman kernel. Ahn-Park [1] introduced the generalized Cartan-Hartogs domain as follows:

$$\widehat{\Omega}_m := \left\{(z,w) \in \Omega_1 \times \cdots \times \Omega_t \times \mathbb{C}^m : \|w\|^2 < \prod_{j=1}^{t} N_{\Omega_j}(z_j,z_j)^{\mu_j}, \mu_j > 0\right\},$$

where $z_j \in \Omega_j$, Ω_j is one of bounded symmetric domains of six types, and $N_{\Omega_j}(z_j,z_j)$ is the corresponding generic norm of Ω_j $(1 \le j \le t)$. This type of domains generalises the Cartan-Hartogs domain introduced by Yin and Roos [19–21]. Using the virtual Bergman kernel, Ahn-Park [1] obtained the Bergman kernel for the generalized Cartan-Hartogs domain in explicit form.

For $t \in \mathbb{N}_+$, let $\mu = (\mu_1, \ldots, \mu_t)$, $\xi = (z_1, \ldots, z_t, w)$, where $\mu_j > 0$, $w \in \mathbb{C}^m$, $z_j \in \mathbb{C}^{n_j}$, $n_j \in \mathbb{N}_+$, $1 \le j \le t$. Denote

$$\Omega(\mu,t) = \left\{\xi \in \mathbb{B}_{n_1} \times \cdots \times \mathbb{B}_{n_t} \times \mathbb{C}^m : \|w\|^2 < C(\chi,\mu) \prod_{j=1}^{t} (1-\|z_j\|^2)^{\mu_j}\right\},$$

where $\mathbb{B}_{n_j}$ is the unit ball in $\mathbb{C}^{n_j}$, $C(\chi,\mu) = \prod_{j=1}^{t} V(\mathbb{B}_{n_j})^{\frac{\mu_j}{n_j+1}}$ is a constant only depending on $\chi = (n_1, \ldots, n_t)$ and $\mu = (\mu_1, \ldots, \mu_t)$. The domain $\Omega(\mu,t)$ is a special generalized Cartan-Hartogs domain $\widehat{\Omega}_m$ with $\Omega_j = \mathbb{B}_{n_j}$ for $1 \le j \le t$. If we take $t = 1, \mu_1 = 1$, then

$$\Omega(1,1) = \left\{\xi \in \mathbb{B}_{n_1} \times \mathbb{C}^m : \|w\|^2 < \left(\frac{\pi^{n_1}}{n_1!}\right)^{\frac{1}{n_1+1}} (1-\|z_1\|^2)\right\}.$$

We can see that $\Omega(1,1)$ is biholomorphic to the unit ball in $\mathbb{C}^{n_1+m}$ with a constant Jacobian determinant. Therefore, the Bergman projection is bounded on the $L^p(\Omega(1,1))$ if and only if the Bergman projection is bounded on the $L^p(\mathbb{B}_{n_1+m})$.

For the following type of operators:

$$T_{a,b}f(z) = (1-\|z\|^2)^a \int_{\mathbb{B}_n} \frac{(1-\|z'\|^2)^b}{(1-\langle z,z'\rangle)^{a+b+n+1}} f(z')dV(z'),$$

Forelli-Rudin [6] and Zhu [24] studied the boundedness of this type of operators on the weighted L^p space of the unit ball $\mathbb{B}_n$ with $a = 0$, $b \in \mathbb{C}$, and Zhu [25] studied the boundedness of this type of operators on the weighted L^p space of the unit disc $\mathbb{B}$ with $n = 1$, $a, b \in \mathbb{R}$. Kaptanoğlu [9] studied the boundedness of this type of operators from L^p spaces to the Besov spaces on the unit ball. Zhao [23] studied the boundedness of this type of operators from L^p spaces to L^q spaces on the unit ball with possibly different measures, for $1 \le p \le q < \infty$. Liu-Stoll [11] and Shi [17] generalized the results to the generalized complex ellipsoids as follows:

$$\Omega_a := \{\xi = (z,w) \in \mathbb{C}^{n+m} : z \in \mathbb{C}^n, w \in \mathbb{C}^m, \|z\|^2 + \|w\|^{\frac{2}{a}} < 1\},$$

where $a > 0$. Motivated by their works, in this paper we study the mapping properties for the Bergman type operators on the generalized Cartan-Hartogs domain $\Omega(\mu, t)$, which can naturally be viewed as complements to Liu-Stoll [11] and Shi [17].

Define

$$\rho(\xi) = 1 - \frac{\|w\|^2}{C(\chi,\mu)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j}}.$$

For $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_t)$, $\sigma_0 > -1$, $\sigma_j > -\mu_j m - 1$, $1 \le j \le t$, define

$$\omega_\sigma(\xi) = \rho^{\sigma_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\sigma_j}.$$

We will first give a formula for the weighted Bergman kernel of $A^2(\Omega(\mu,t),\omega_\sigma)$ as follows.

Theorem 1.1. *Let $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_t)$, $\sigma_0 > -1$, $\sigma_j > -\mu_j m - 1$, $1 \le j \le t$. Then the weighted Bergman kernel of the Hilbert space $A^2(\Omega(\mu,t),\omega_\sigma)$ is given by*

$$\begin{aligned} &K_{\omega_\sigma}(\xi,\xi') \\ &= \sum_{k=0}^{n_1+\cdots+n_t} C_k \frac{\prod_{j=1}^{t}\left(1-\langle z_j, z_j'\rangle\right)^{\mu_j(k+\sigma_0+1)-\sigma_j-n_j-1}}{\left[C(\chi,\mu)\prod_{j=1}^{t}\left(1-\langle z_j, z_j'\rangle\right)^{\mu_j} - \langle w, w'\rangle\right]^{k+m+\sigma_0+1}}, \end{aligned} \tag{1}$$

where $\xi = (z_1, \ldots, z_t, w), \xi' = (z_1', \ldots, z_t', w') \in \Omega(\mu,t)$ and C_k are constants depending on m, n_j, σ_0, σ_j, $\mu_j (1 \le j \le t)$.

Let $s = (s_0, s_1, \ldots, s_t)$, $s_0 = \sigma_0 + ia_0$, $\sigma_0 > -1$, $a_0 \in \mathbb{R}$ and $s_j = \sigma_j + ia_j$, $\sigma_j > -\mu_j m - 1$, $a_j \in \mathbb{R}$, $1 \le j \le t$. Let K_{ω_s} be the kernel obtained from K_{ω_σ} by replacing σ with s in (1).

Let $v = (v_1, \ldots, v_t)$. For $u, v_1, \ldots, v_t \ge 0$, we define

$$\begin{aligned} &K_{s,u,v}(\xi,\xi') \\ &= \sum_{k=0}^{n_1+\cdots+n_t} C_k \frac{\prod_{j=1}^{t}\left(1-\langle z_j, z_j'\rangle\right)^{\mu_j(k+s_0+1)-s_j-n_j-1-v_j}}{\left[C(\chi,\mu)\prod_{j=1}^{t}\left(1-\langle z_j, z_j'\rangle\right)^{\mu_j} - \langle w, w'\rangle\right]^{k+m+s_0+1+u}}. \end{aligned} \tag{2}$$

We can see that when $u = v_1 = \cdots = v_t = 0$, $K_{s,u,v}(\xi,\xi') = K_{\omega_s}(\xi,\xi')$.

Using this kernel, we will study the action of the following operator $T_{s,u,v,\epsilon}$ on the space $L^p(\Omega(\mu,t),\omega_\lambda)$:

$$(T_{s,u,v,\epsilon}f)(\xi) = \rho^{u+\epsilon}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j u+v_j}\int_{\Omega(\mu,t)}\omega_s(\xi')K_{s,u,v}(\xi,\xi')f(\xi')dV(\xi'),$$

where $\epsilon \geq 0$ and we will obtain the following results.

Theorem 1.2. *Let $s = (s_0, s_1, \ldots, s_t)$, $s_0 = \sigma_0 + ia_0$, $\sigma_0 > -1$, $a_0 \in \mathbb{R}$ and $s_j = \sigma_j + ia_j$, $\sigma_j > -\mu_j m - 1$, $a_j \in \mathbb{R}$, $1 \leq j \leq t$.*

(i) *For $1 < p < \infty$, $T_{s,u,v,\epsilon}$ is a bounded operator on $L^p(\Omega(\mu,t),\omega_\lambda)$ if and only if*

$$\begin{cases} -p(u+\epsilon) < \lambda_0 + 1 < p(\sigma_0 + 1), \\ -p(\mu_j u + v_j) < \lambda_j + \mu_j m + 1 < p(\sigma_j + \mu_j m + 1), \quad 1 \leq j \leq t. \end{cases} \tag{3}$$

(ii) *For $p = 1$, if $T_{s,u,v,\epsilon}$ is a bounded operator on $L^1(\Omega(\mu,t),\omega_\lambda)$, then*

$$\begin{cases} -u - \epsilon < \lambda_0 + 1 \leq \sigma_0 + 1, \\ -\mu_j u - v_j < \lambda_j + \mu_j m + 1 \leq \sigma_j + \mu_j m + 1, \quad 1 \leq j \leq t. \end{cases} \tag{4}$$

Conversely, if

$$\begin{cases} -u - \epsilon < \lambda_0 + 1 < \sigma_0 + 1, \\ -\mu_j u - v_j < \lambda_j + \mu_j m + 1 < \sigma_j + \mu_j m + 1, \quad 1 \leq j \leq t, \end{cases} \tag{5}$$

then $T_{s,u,v,\epsilon}$ is a bounded operator on $L^1(\Omega(\mu,t),\omega_\lambda)$.

Note that we do not make any assumptions on λ, so that $\omega_\lambda dV$ may not be a finite measure.

Remark 1.3. (i) Suppose $t = \mu_1 = 1$, $m = \epsilon = 0$. Then $C_k = 0$ for $k = 0, 1, \ldots, n_1 - 1$ and $C_{n_1} = \frac{\Gamma(n_1+1)}{\pi^{n_1}}$. Therefore,

$$\begin{aligned}(T_{s,u,v,\epsilon}f)(\xi) = {} & \frac{\Gamma(n_1+1)}{\pi^{n_1}} V(\mathbb{B}_{n_1})^{\frac{n_1+s_0+u+1}{2}} (1 - \|z_1\|^2)^{u+v_1} \\ & \times \int_{\mathbb{B}_{n_1}} \frac{(1 - \|z_1'\|^2)^{s_1} f(z_1')}{(1 - \langle z_1, z_1' \rangle)^{s_1+n_1+1+u+v_1}} dV(z_1').\end{aligned}$$

If $u = v_1 = 0$, then $T_{s,u,v,\epsilon}$ is the operator studied by Forelli-Rudin [6] and Zhu [24]. If $n_1 = 1$, $s_1 = \sigma_1$, then $T_{s,u,v,\epsilon}$ is the operator studied by Zhu [25].

(ii) Suppose $t = 1$, $\epsilon = \frac{v_1}{\mu_1}$, $s_1 = \mu_1 s_0$. Then $T_{s,u,v,\epsilon}$ is the operator studied by Shi [17].

If A and B are functions of several variables, we use $A \lesssim B$ to denote that $A \leq K \cdot B$ for a positive constant K. Also we use $A \approx B$ to denote that $A \lesssim B \lesssim A$.

2. Preliminaries

Lemma 2.1 (See [2]). *For $\alpha > -1$, the following multiple integration exists and*

$$\int_0^1 dx_m \cdots \int_0^{1-\sum_{i=2}^m x_i} \left(1 - \sum_{i=1}^m x_i\right)^\alpha \prod_{i=1}^m x_i^{q_i} dx_1 = \frac{\prod_{i=1}^m \Gamma(q_i+1)\Gamma(\alpha+1)}{\Gamma(\alpha + \sum_{i=1}^m q_i + m + 1)},$$

where $q = (q_1, \dots, q_m) \in (\mathbb{R}_+)^m$, here $\mathbb{R}_+$ denotes the set of positive real numbers.

Lemma 2.2 (See [5]). *For $\|x\| < 1$ and $s > 0$, we have*

$$\sum_{\alpha \in \mathbb{R}_+^m} \frac{\Gamma(|\alpha| + s)}{\Gamma(s) \prod_{i=1}^m \Gamma(\alpha_i + 1)} x^{2\alpha} = \frac{1}{(1 - \|x\|^2)^s}.$$

Lemma 2.3 (See [5]). *Suppose that q_d is a polynomial of degree d in one variable. Then there are constants c_k such that*

$$\sum_{\gamma \in \mathbb{R}_+^m} \frac{\Gamma(|\gamma| + s)}{\prod_{i=1}^m \Gamma(\gamma_i + 1)} q_d(|\gamma|) x^{2\gamma} = \sum_{k=0}^d c_k \left(\frac{1}{1 - \|x\|^2} \right)^{s+k}.$$

Remark 2.4. Although the parameter α in Lemma 2.1 is a real number bigger than -1, we can see that Lemma 2.1 is also valid for $\alpha \in \mathbb{C}$ with $\Re\alpha > -1$. Similarly, Lemma 2.2 and Lemma 2.3 are also valid for $s \in \mathbb{C}, \Re s > 0$.

Lemma 2.5. *Let $s = (s_0, s_1, \dots, s_t)$, $s_0 = \sigma_0 + ia_0$, $\sigma_0 > -1$, $a_0 \in \mathbb{R}$ and $s_j = \sigma_j + ia_j$, $\sigma_j > -\mu_j m - 1$, $a_j \in \mathbb{R}$, $1 \le j \le t$. $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn_j}) \in \mathbb{N}^{n_j}$, $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$. Then*

$$\begin{aligned} &\int_{\Omega(\mu,t)} \rho^{s_0}(\xi) \prod_{j=1}^t (1 - \|z_j\|^2)^{s_j} |z_1^{\alpha_1} \cdots z_t^{\alpha_t} w^\beta|^2 dV(\xi) \\ &= \frac{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^{m + |\beta|} \beta! \Gamma(s_0 + 1)}{\Gamma(m + s_0 + |\beta| + 1)} \\ &\quad \times \prod_{j=1}^t \frac{(\alpha_j)! \Gamma(\mu_j(m + |\beta|) + s_j + 1)}{\Gamma(\mu_j(m + |\beta|) + s_j + |\alpha_j| + n_j + 1)}, \end{aligned}$$

where $(\alpha_j)! = \alpha_{j1}! \cdots \alpha_{jn_j}!$, $1 \le j \le t$.

Proof. By using the polar coordinates $z_{jl} = r_{jl} e^{i\theta_{jl}}$ $(1 \le j \le t, 1 \le l \le n_j)$, we have

$$\begin{aligned} &\int_{\Omega(\mu,t)} \rho^{s_0}(\xi) \prod_{j=1}^t (1 - \|z_j\|^2)^{s_j} |z_1^{\alpha_1} \cdots z_t^{\alpha_t} w^\beta|^2 dV(\xi) \\ &= (2\pi)^{(n_1 + \cdots + n_t)} \int_{\substack{\|r_1\| < 1 \\ r_{1l} \ge 0}} r_1^{2\alpha_1 + 1} \cdots \int_{\substack{\|r_t\| < 1 \\ r_{tl} \ge 0}} r_t^{2\alpha_t + 1} \prod_{j=1}^t (1 - \|r_j\|^2)^{s_j} \\ &\quad \times \int_W \left(1 - \frac{\|w\|^2}{C(\chi, \mu) \prod_{j=1}^t (1 - \|r_j\|^2)^{\mu_j}} \right)^{s_0} |w^\beta|^2 dV(w) dV(r_1, \dots, r_t), \end{aligned} \tag{6}$$

where $W = \{w \in \mathbb{C}^m : \|w\|^2 < C(\chi, \mu) \prod_{j=1}^t (1 - \|r_j\|^2)^{\mu_j}\}$. Denote the integral in the last line of the above equation by I. Let $\mathbb{S}_m$ denote the surface of the

unit ball in $\mathbb{C}^m$ and $d\sigma$ denote the normalized measure of $\mathbb{S}_m$. Using the polar coordinates we have

$$\begin{aligned} I &= \frac{2\pi^m}{\Gamma(m)} \int_0^{[C(\chi,\mu)\prod_{j=1}^t (1-\|r_j\|^2)^{\mu_j}]^{\frac{1}{2}}} r^{2m-1} dr \\ &\quad \times \int_{\mathbb{S}_m} \left(1 - \frac{r^2}{C(\chi,\mu)\prod_{j=1}^t (1-\|r_j\|^2)^{\mu_j}}\right)^{s_0} r^{2|\beta|} |\zeta^\beta|^2 d\sigma(\zeta) \\ &= \frac{2\pi^m}{\Gamma(m)} \frac{\Gamma(m)\beta!}{\Gamma(m+|\beta|)} \int_0^{[C(\chi,\mu)\prod_{j=1}^t (1-\|r_j\|^2)^{\mu_j}]^{\frac{1}{2}}} r^{2m+2|\beta|-1} \\ &\quad \times \left(1 - \frac{r^2}{C(\chi,\mu)\prod_{j=1}^t (1-\|r_j\|^2)^{\mu_j}}\right)^{s_0} dr. \end{aligned}$$

By setting $\tilde{r} = C(\chi,\mu)^{-1}\prod_{j=1}^t (1-\|r_j\|^2)^{-\mu_j} r^2$, we have

$$\begin{aligned} I &= \frac{\pi^m \beta!}{\Gamma(m+|\beta|)} \left(C(\chi,\mu)\prod_{j=1}^t (1-\|r_j\|^2)^{\mu_j}\right)^{m+|\beta|} \int_0^1 \tilde{r}^{m+|\beta|-1}(1-\tilde{r})^{s_0} d\tilde{r} \\ &= \frac{\pi^m \beta! \Gamma(s_0+1)}{\Gamma(m+s_0+|\beta|+1)} C(\chi,\mu)^{m+|\beta|} \prod_{j=1}^t (1-\|r_j\|^2)^{\mu_j(m+|\beta|)}. \end{aligned} \tag{7}$$

Putting (7) into (6), and setting $\tilde{r}_j = (\tilde{r}_{j1},\ldots,\tilde{r}_{jn_j}) = (r_{j1}^2,\ldots,r_{jn_j}^2)$, $1 \le j \le t$, we obtain

$$\begin{aligned} &\int_{\Omega(\mu,t)} \rho^{s_0}(\xi) \prod_{j=1}^t (1-\|z_j\|^2)^{s_j} |z_1^{\alpha_1}\cdots z_t^{\alpha_t} w^\beta|^2 dV(\xi) \\ &= \frac{\pi^{n_1+\cdots+n_t+m} C(\chi,\mu)^{m+|\beta|}\beta!\Gamma(s_0+1)}{\Gamma(m+s_0+|\beta|+1)} \\ &\quad \times \prod_{j=1}^t \int_{\sum_{\tilde{r}_{jl}<1}, \tilde{r}_{jl}\ge 0} \left(1 - \sum_{l=1}^{n_1} \tilde{r}_{jl}\right)^{\mu_j(m+|\beta|)+s_j} \tilde{r}_j^{\alpha_j} d\tilde{r}_j \\ &= \frac{\pi^{n_1+\cdots+n_t+m} C(\chi,\mu)^{m+|\beta|}\beta!\Gamma(s_0+1)}{\Gamma(m+s_0+|\beta|+1)} \\ &\quad \times \prod_{j=1}^t \frac{(\alpha_j)!\Gamma(\mu_j(m+|\beta|)+s_j+1)}{\Gamma(\mu_j(m+|\beta|)+s_j+|\alpha_j|+n_j+1)}, \end{aligned}$$

where the last equality holds by Lemma 2.1. □

Let $\xi = (z_1,\ldots,z_t,w)$, $\xi' = (z_1',\ldots,z_t',w') \in \Omega(\mu,t)$, and denote

$$\psi_{l,c}(\xi,\xi') = \frac{\langle w,w'\rangle^l}{\prod_{j=1}^t \left(1-\langle z_j,z_j'\rangle\right)^{c_j}},$$

where l is a non-negative integer and $c = (c_1, \dots, c_t) \in \mathbb{C}^t$ with $\Re c_j > 0$ $(1 \le j \le t)$ and the complex power is taken to be the principal branch.

Lemma 2.6. *Let $b, c \in \mathbb{C}^t$, $s = (s_0, s_1, \dots, s_t)$, $s_0 = \sigma_0 + ia_0$, $\sigma_0 > -1$, $a_0 \in \mathbb{R}$ and $s_j = \sigma_j + ia_j$, $\sigma_j > -\mu_j m - 1$, $a_j \in \mathbb{R}$, $\Re b_j > 0$, $\Re c_j > 0$, $1 \le j \le t$.*

(i) *If $l \neq k$, then*

$$\int_{\Omega(\mu,t)} \rho^{s_0}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} \psi_{l,c}(\xi, \xi') \overline{\psi_{k,b}(\xi, \xi')} dV(\xi') = 0. \tag{8}$$

(ii)

$$\begin{aligned} &\int_{\Omega(\mu,t)} \rho^{s_0}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} |\psi_{l,c}(\xi, \xi')|^2 dV(\xi') \\ &= B(s_0, l, c) \|w\|^{2l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \frac{|\Gamma(l_j + c_j)|^2 \|z_j\|^{2l_j}}{\Gamma(l_j + 1)\Gamma(\mu_j(m + l) + s_j + l_j + n_j + 1)}, \end{aligned} \tag{9}$$

where

$$B(s_0, l, c) = \frac{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^{m+l} l! \Gamma(s_0 + 1)}{\Gamma(m + s_0 + l + 1)} \prod_{j=1}^{t} \frac{\Gamma(\mu_j(m + l) + s_j + 1)}{|\Gamma(c_j)|^2}.$$

Proof. Using polar coordinates, it is easy to obtain (8). Now we prove (9).

$$\begin{aligned} &\int_{\Omega(\mu,t)} \rho^{s_0}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} |\psi_{l,c}(\xi, \xi')|^2 dV(\xi') \\ &= \int_{\Omega(\mu,t)} \rho^{s_0}(\xi) \sum_{|\beta|=l} \frac{\Gamma^2(l+1)}{\prod_{i=1}^{m} \Gamma^2(\beta_i + 1)} |w^\beta \overline{w'}^\beta|^2 \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} \\ &\quad \times \sum_{l_j=0}^{\infty} \sum_{|\alpha_j|=l_j} \left| \frac{\Gamma(l_j + c_j)}{\prod_{i=1}^{n_j} \Gamma(\alpha_{ji} + 1)\Gamma(c_j)} \right|^2 |z_j^{\alpha_j} \overline{z_j'}^{\alpha_j}|^2 dV(\xi'). \end{aligned}$$

By Lemma 2.2 and Lemma 2.5, we have

$$\begin{aligned} &\int_{\Omega(\mu,t)} \rho^{s_0}(\xi) \prod_{j=1}^{t} (1 - \|z_j\|^2)^{s_j} |\psi_{l,c}(\xi, \xi')|^2 dV(\xi') \\ &= \sum_{|\beta|=l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \sum_{|\alpha_j|=l_j} \frac{\Gamma^2(l+1)}{\prod_{i=1}^{m} \Gamma^2(\beta_i + 1)} |w^\beta|^2 |z_j^{\alpha_j}|^2 \left| \frac{\Gamma(l_j + c_j)}{\prod_{i=1}^{n_j} \Gamma(\alpha_{ji} + 1)\Gamma(c_j)} \right|^2 \\ &\quad \times \frac{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^{m + |\beta|} \beta! \Gamma(s_0 + 1)}{\Gamma(m + s_0 + |\beta| + 1)} \frac{(\alpha)! \Gamma(\mu_j(m + |\beta|) + s_j + 1)}{\Gamma(\mu_j(m + |\beta|) + s_j + |\alpha_j| + n_j + 1)} \\ &= \frac{\pi^{n_1 + \cdots + n_t + m} C(\chi, \mu)^{m+l} l! \Gamma(s_0 + 1)}{\Gamma(m + s_0 + l + 1)} \|w\|^{2l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \frac{\Gamma(\mu_j(m + l) + s_j + 1)}{|\Gamma(c_j)|^2} \end{aligned}$$

$$\times \frac{|\Gamma(l_j+c_j)|^2\|z_j\|^{2l_j}}{\Gamma(l_j+1)\Gamma(\mu_j(m+l)+s_j+l_j+n_j+1)}. \qquad \square$$

Let

$$\begin{aligned} &G_{k,s,u,v}(\xi,\xi') \\ &= C(\chi,\mu)^{\frac{d}{2}} \prod_{j=1}^{t} \left(1-\langle z_j, z_j^{'}\rangle\right)^{-\frac{b_j}{2}} \left(1-\frac{\langle w, w^{'}\rangle}{C(\chi,\mu)\prod_{j=1}^{t}(1-\langle z_j, z_j^{'}\rangle)^{\mu_j}}\right)^{-\frac{d}{2}}, \end{aligned}$$

where $b_j = \mu_j(m+u)+s_j+n_j+v_j+1$, $1 \le j \le t$, $d = s_0+m+k+u+1$. Then

$$K_{s,u,v}(\xi,\xi') = \sum_{k=0}^{n_1+\cdots+n_t} C_k G^2_{k,s,u,v}(\xi,\xi').$$

Lemma 2.7. *If* $-1 < x_0 < u+\sigma_0$, $-\mu_j m-1 < x_j < \mu_j u+\sigma_j+v_j$, $1 \le j \le t$, *then*

$$\begin{aligned} &\int_{\Omega(\mu,t)} \rho^{x_0}(\xi') \prod_{j=1}^{t} (1-\|z_j^{'}\|^2)^{x_j} |G_{k,s,u,v}(\xi,\xi')|^2 dV(\xi') \\ &\lesssim \rho^{x_0-\sigma_0-u}(\xi) \prod_{j=1}^{t} (1-\|z_j\|^2)^{x_j-\mu_j u-\sigma_j-v_j}. \end{aligned}$$

Proof. Since $\xi = (z,w), \xi' = (z',w') \in \Omega_\mu$, we can see that

$$\left|\langle w, w^{'}\rangle\right| < \left|C(\chi,\mu)\prod_{j=1}^{t}\left(1-\langle z_j, z_j^{'}\rangle\right)^{\mu_j}\right|.$$

Therefore $G_{k,s,u,v}$ can be written as

$$G_{k,s,u,v}(\xi,\xi') = \sum_{l=0}^{\infty} \frac{\Gamma(\frac{d}{2}+l)}{\Gamma(\frac{d}{2})l!} C(\chi,\mu)^{\frac{d}{2}-l} \psi_{l,c}(\xi,\xi'),$$

where $c_j = \mu_j l + \frac{b_j}{2}, 1 \le j \le t$. By Lemma 2.6, we have

$$\begin{aligned} &\int_{\Omega(\mu,t)} \rho^{x_0}(\xi') \prod_{j=1}^{t} (1-\|z_j^{'}\|^2)^{x_j} |G_{k,s,u,v}(\xi,\xi')|^2 dV(\xi') \\ &= \sum_{l=0}^{\infty} \frac{|\Gamma(\frac{d}{2}+l)|^2}{|\Gamma(\frac{d}{2})|^2(l!)^2} |C(\chi,\mu)^{d-2l}| \int_{\Omega(\mu,t)} \rho^{x_0}(\xi') \prod_{j=1}^{t} (1-\|z_j^{'}\|^2)^{x_j} \\ &\quad \times |\psi_{l,c}(\xi,\xi')|^2 dV(\xi'). \end{aligned} \tag{10}$$

Let $J_l(\xi)$ denote the integral on the right hand side of the above equation. By Lemma 2.6, we have

$$J_l(\xi) = B(x_0,l,c)\|w\|^{2l} \prod_{j=1}^{t} \sum_{l_j=0}^{\infty} \frac{|\Gamma(l_j+c_j)|^2\|z_j\|^{2l_j}}{\Gamma(l_j+1)\Gamma(\mu_j(m+l)+x_j+l_j+n_j+1)}. \tag{11}$$

Using Stirling's formula $\Gamma(z) = \sqrt{2\pi}e^{-z}z^{z-\frac{1}{2}}(1+O(\frac{1}{|z|}))$ $(|arg(z)| < \pi, |z| \to \infty)$ in (11), we have

$$J_l(\xi) \approx B(x_0,l,c)\|w\|^{2l}\prod_{j=1}^{t}\sum_{l_j=0}^{\infty}\|z_j\|^{2l_j}\frac{\Gamma(l_j+\mu_j(l+u)+\sigma_j+v_j-x_j)}{l_j!}$$

$$\approx B(x_0,l,c)\|w\|^{2l}\prod_{j=1}^{t}\sum_{l_j=0}^{\infty}\Gamma(\eta_j)(1-\|z_j\|^2)^{-\eta_j}, \tag{12}$$

where $\eta_j = \mu_j(l+u)+\sigma_j+v_j-x_j$, $1 \le j \le t$. Putting (12) into (10), we have

$$\begin{aligned}
&\int_{\Omega(\mu,t)}\rho^{x_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{x_j}|G_{k,s,u,v}(\xi,\xi')|^2dV(\xi')\\
\approx\ &\sum_{l=0}^{\infty}\frac{|\Gamma(\frac{d}{2}+l)|^2}{|\Gamma(\frac{d}{2})|^2(l!)^2}B(x_0,l,c)C(\chi,\mu)^{-l}\|w\|^{2l}\prod_{j=1}^{t}\sum_{l_j=0}^{\infty}\Gamma(\eta_j)(1-\|z_j\|^2)^{-\eta_j}\\
\approx\ &\prod_{j=1}^{t}(1-\|z_j\|^2)^{x_j-\mu_j u-\sigma_j-v_j}\sum_{l=0}^{\infty}\frac{|\Gamma(\frac{d}{2}+l)|^2}{\Gamma(m+x_0+l+1)l!}\\
&\times\frac{\Gamma(\mu_j(m+l)+x_j+1)\Gamma(\eta_j)}{|\Gamma(\mu_j l+\frac{b_j}{2})|^2}\left(\frac{\|w\|^2}{C(\chi,\mu)\prod_{i=1}^{t}(1-\|z_i\|^2)^{\mu_i}}\right)^l\\
\approx\ &\prod_{j=1}^{t}(1-\|z_j\|^2)^{x_j-\mu_j u-\sigma_j-v_j}\\
&\times\sum_{l=0}^{\infty}\frac{\Gamma(l+\sigma_0+u-x_0)}{l!}\frac{1}{l^{n_1+\cdots+n_t-k}}\left(\frac{\|w\|^2}{C(\chi,\mu)\prod_{i=1}^{t}(1-\|z_i\|^2)^{\mu_i}}\right)^l.
\end{aligned} \tag{13}$$

Note that $k \le n_1+\cdots+n_t$, by (13) we obtain

$$\begin{aligned}
&\int_{\Omega(\mu,t)}\rho^{x_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{x_j}|G_{k,s,u,v}(\xi,\xi')|^2dV(\xi')\\
\lesssim\ &\prod_{j=1}^{t}(1-\|z_j\|^2)^{x_j-\mu_j u-\sigma_j-v_j}\left(1-\frac{\|w\|^2}{C(\chi,\mu)\prod_{i=1}^{t}(1-\|z_i\|^2)^{\mu_i}}\right)^{x_0-\sigma_0-u}.
\end{aligned}$$

□

Using Lemma 2.7, we have following results, which are needed in the proof of Theorem 1.2.

Lemma 2.8. *If* $-1 < x_0 < u+\sigma_0$, $-\mu_j m-1 < x_j < \mu_j u+\sigma_j+v_j$, $1 \le j \le t$, *then*

$$\int_{\Omega(\mu,t)}\rho^{x_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{x_j}|K_{s,u,v}(\xi,\xi')|dV(\xi')$$

$$\lesssim \rho^{x_0-\sigma_0-u}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{x_j-\mu_j u-\sigma_j-v_j}.$$

Proof. Since $K_{s,u,v}(\xi,\xi')=\sum_{k=0}^{n_1+\cdots+n_t}C_kG^2_{k,s,u,v}(\xi,\xi')$, by Lemma 2.7 we have

$$\begin{aligned}&\int_{\Omega(\mu,t)}\rho^{x_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{x_j}|K_{s,u,v}(\xi,\xi')|dV(\xi')\\ \leq&\sum_{k=0}^{n_1+\cdots+n_t}|C_k|\int_{\Omega(\mu,t)}\rho^{x_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{x_j}|G_{k,s,u,v}(\xi,\xi')|^2dV(\xi')\\ \lesssim&\ \rho^{x_0-\sigma_0-u}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{x_j-\mu_j u-\sigma_j-v_j}.\end{aligned}$$

□

Lemma 2.9. *If $-1<x_0<u+\sigma_0$, $-\mu_j m-1<x_j<\mu_j u+\sigma_j+v_j$, $1\leq j\leq t$, then*

$$\begin{aligned}&\int_{\Omega_\mu}\rho^{x_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{x_j}|K_{s,u,v}(\xi,\xi')|dV(\xi)\\ \lesssim&\ \rho^{x_0-\sigma_0-u}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{x_j-\mu_j u-\sigma_j-v_j}.\end{aligned}$$

Proof. Noting that $|K_{s,u,v}(\xi,\xi')|=|\overline{K_{\overline{s},u,v}(\xi',\xi)}|=|K_{s,u,v}(\xi',\xi)|$, by Lemma 2.8 with ξ and ξ' exchanged we obtain the desired inequality. □

Lemma 2.10. *If $s=(s_0,s_1,\ldots,s_t)$, $s_0=\sigma_0+ia_0$, $\sigma_0>-1$, $a_0\in\mathbb{R}$ and $s_j=\sigma_j+ia_j$, $\sigma_j>-\mu_j m-1$, $a_j\in\mathbb{R}$, $1\leq j\leq t$, then there exists a constant $\mathcal{K}$ independent of ξ such that*

$$\int_{\Omega(\mu,t)}\rho^{s_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{s_j}K_{s,u,v}(\xi,\xi')dV(\xi')=\mathcal{K}.$$

Proof. Since $K_{s,u,v}(\xi,\xi')=\sum_{k=0}^{n_1+\cdots+n_t}C_k\sum_{l=0}^{\infty}\frac{\Gamma(d+l)}{\Gamma(d)l!}\psi_{l,c}(\xi,\xi')$, where $c_j=\mu_j l+b_j$, $1\leq j\leq t$, by Lemma 2.6 we have

$$\begin{aligned}&\int_{\Omega(\mu,t)}\rho^{s_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{s_j}K_{s,u,v}(\xi,\xi')dV(\xi')\\ =&\int_{\Omega(\mu,t)}\rho^{s_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{s_j}\sum_{k=0}^{n_1+\cdots+n_t}C_k\sum_{l=0}^{\infty}\frac{\Gamma(d+l)}{\Gamma(d)l!}\psi_{0,c}(\xi,\xi')dV(\xi')\\ =&\sum_{k=0}^{n_1+\cdots+n_t}C_k\int_{\Omega_\mu}\rho^{s_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{s_j}\sum_{l_j=0}^{\infty}\sum_{|\alpha_j|=l_j}\frac{\Gamma(l_j+b_j)}{\prod_{i=1}^{n_j}\Gamma(\alpha_{ji}+1)\Gamma(b_j)}\end{aligned}$$

$$
\begin{aligned}
&\times z_j^{\alpha_j}\overline{z_j^{'}}^{\alpha_j} dV(\xi') \\
=& \sum_{k=0}^{n_1+\cdots+n_t} C_k \int_{\Omega(\mu,t)} \rho^{s_0}(\xi') \prod_{j=1}^{t} (1-\|z_j^{'}\|^2)^{s_j} dV(\xi') \\
=& \mathcal{K}.
\end{aligned}
$$

□

Finally, we will give the Schur's lemma, which is a common tool to prove boundedness of an integral operator.

Lemma 2.11 (See [26]). *Let (Ω,κ) be a measure space and K be a nonnegative measurable function on $\Omega\times\Omega$, T be the integral operator with kernel K, and $1<p<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. If there exist positive constants C_1 and C_2 and a positive measurable function h on Ω such that*

$$\int_\Omega K(x,y)h^q(y)d\kappa(y) \le C_1 h^q(x) \tag{14}$$

for almost every x in Ω and

$$\int_\Omega K(x,y)h^p(x)d\kappa(x) \le C_2 h^p(y) \tag{15}$$

for almost every y in Ω, then T is a bounded linear operator on $L^p(\Omega,\kappa)$.

3. Proof of the main result

3.1. Proof of Theorem 1.1

Since $\Omega(\mu,t)$ is a Reinhardt domain, $\left\{\frac{z_1^{\alpha_1}\cdots z_t^{\alpha_t}w^\beta}{\|z_1^{\alpha_1}\cdots z_t^{\alpha_t}w^\beta\|_{2,\omega_\sigma}}\right\}$ constitutes an orthonormal basis of $A^2(\Omega(\mu,t),\omega_\sigma)$. Then

$$K_{\omega_\sigma}(\xi,\xi) = \sum_{\substack{\alpha_j\in\mathbb{N}^{n_j},1\le j\le t\\ \beta\in\mathbb{N}^m}} \frac{|z_1^{\alpha_1}\cdots z_t^{\alpha_t}w^\beta|^2}{\|z_1^{\alpha_1}\cdots z_t^{\alpha_t}w^\beta\|_{2,\omega_\sigma}^2}.$$

By Lemma 2.2 and Lemma 2.5, we have

$$
\begin{aligned}
K_{\omega_\sigma}(\xi,\xi) =& \frac{1}{\pi^{n_1+\cdots+n_t+m}\Gamma(\sigma_0+1)} \prod_{j=1}^{t} \sum_{\alpha_j\in\mathbb{N}^{n_j},\beta\in\mathbb{N}^m} \frac{\Gamma(m+\sigma_0+|\beta|+1)}{C(\chi,\mu)^{m+|\beta|}\beta!} \\
&\times \frac{\Gamma(\mu_j(m+|\beta|)+\sigma_j+|\alpha_j|+n_j+1)}{\Gamma(\mu_j(m+|\beta|)+\sigma_j+1)\alpha_j!}|z_j^{\alpha_j}||w^\beta|^2 \\
=& \frac{1}{\pi^{n_1+\cdots+n_t+m}\Gamma(\sigma_0+1)} \prod_{j=1}^{t} \sum_{\beta\in\mathbb{N}^m} \frac{\Gamma(m+\sigma_0+|\beta|+1)}{\Gamma(\mu_j(m+|\beta|)+\sigma_j+1)\beta!}|w^\beta|^2 \\
&\times C(\chi,\mu)^{-m-|\beta|} \sum_{\alpha_j\in\mathbb{N}^{n_j}} \frac{\Gamma(\mu_j(m+|\beta|)+\sigma_j+|\alpha_j|+n_j+1)}{\alpha_j!}|z_j^{\alpha_j}|^2
\end{aligned}
$$

$$= \frac{1}{\pi^{n_1+\cdots+n_t+m}C(\chi,\mu)^m}\prod_{j=1}^{t}\sum_{\beta\in\mathbb{N}^m}\frac{\Gamma(m+\sigma_0+|\beta|+1)\Gamma(\zeta_j)}{\Gamma(\mu_j(m+|\beta|)+\sigma_j+1)\beta!}|w^\beta|^2 \times (1-\|z_j\|^2)^{-\zeta_j}C(\chi,\mu)^{-|\beta|}, \tag{16}$$

where $\zeta_j = \mu_j(m+|\beta|)+\sigma_j+n_j+1$, $j=1,\dots,t$.

We note that $\frac{1}{\pi^{n_1+\cdots+n_t+m}C(\chi,\mu)^m}\prod_{j=1}^{t}\prod_{l_j=1}^{n_j}(\mu_j(m+|\beta|)+\sigma_j+l_j)$ is a polynomial of degree $\sum_{j=1}^{t}n_j$ in $|\beta|$, and we denote it by $P_{n_1+\cdots+n_t}(|\beta|)$. By (16) and Lemma 2.3, we have

$$\begin{aligned}
K_{\omega_\sigma}(\xi,\xi) &= \sum_{\beta\in\mathbb{N}^m}\frac{\Gamma(m+\sigma_0+|\beta|+1)}{\beta!C(\chi,\mu)^{|\beta|}}P_{n_1+\cdots+n_t}(|\beta|)|w^\beta|^2\prod_{j=1}^{t}(1-\|z_j\|^2)^{-\zeta_j}\\
&= \prod_{j=1}^{t}(1-\|z_j\|^2)^{-\mu_j m-\sigma_j-n_j-1}\sum_{\beta\in\mathbb{N}^m}\frac{\Gamma(m+\sigma_0+|\beta|+1)}{\beta!}\\
&\quad\times P_{n_1+\cdots+n_t}(|\beta|)|w^\beta|^2\left(C(\chi,\mu)\prod_{i=1}^{t}(1-\|z_i\|^2)^{\mu_i}\right)^{-|\beta|}\\
&= \prod_{j=1}^{t}(1-\|z_j\|^2)^{-\mu_j m-\sigma_j-n_j-1}\\
&\quad\times\sum_{k=0}^{n_1+\cdots+n_t}C_k\left(1-\frac{\|w\|^2}{C(\chi,\mu)\prod_{i=1}^{t}(1-\|z_i\|^2)^{\mu_i}}\right)^{-k-m-\sigma_0-1}\\
&= \sum_{k=0}^{n_1+\cdots+n_t}C_k\frac{\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j(k+\sigma_0+1)-\sigma_j-n_j-1}}{[C(\chi,\mu)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j}-\|w\|^2]^{k+m+\sigma_0+1}}.
\end{aligned}$$

By exchanging the second variable with ξ' in $K_{\omega_\sigma}(\xi,\xi)$ in the above calculations, we obtain (1).

3.2. Proof of Theorem 1.2

If $1<p<\infty$, we first prove the necessity of condition (3) for the boundedness of $T_{s,u,v,\epsilon}$ on $L^p(\Omega(\mu,t),\omega_\lambda)$.

Choose $\delta_0 > \max\left(-1-\sigma_0, \frac{-1-\lambda_0}{p}\right)$, $\delta_j > \max\left(-1-\sigma_j-\mu_j m, \frac{-1-\lambda_j-\mu_j m}{p}\right)$, $1\le j\le t$. Let $g(\xi)=\rho^{\delta_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\delta_j}$. Then $g\in L^p(\Omega(\mu,t),\omega_\lambda)$. By the definition of $T_{s,u,v,\epsilon}$ we have

$$(T_{s,u,v,\epsilon}g)(\xi) = \rho^{u+\epsilon}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j u+v_j}$$

$$\times \int_{\Omega(\mu,t)} \rho^{s_0+\delta_0}(\xi') \prod_{i=1}^{t} (1-\|z_i'\|^2)^{s_i+\delta_i} K_{s,u,v}(\xi,\xi') dV(\xi').$$

By Lemma 2.10, there exists a constant $\mathcal{K}$ independent of ξ such that

$$(T_{s,u,v,\epsilon}g)(\xi) = \mathcal{K}\rho^{u+\epsilon}(\xi) \prod_{j=1}^{t} (1-\|z_j\|^2)^{\mu_j u+v_j}.$$

Since $T_{s,u,v,\epsilon}$ is bounded on $L^p(\Omega(\mu,t),\omega_\lambda)$, we have $p(u+\epsilon)+\lambda_0 > -1$, $p(\mu_j u+v_j)+\lambda_j > -\mu_j m-1$, $1 \le j \le t$. To prove the right side of (3) we will look at $T^*_{s,u,v,\epsilon}$, the adjoint of $T_{s,u,v,\epsilon}$. $T^*_{s,u,v,\epsilon}$ is given by

$$(T^*_{s,u,v,\epsilon}g)(\xi) = \rho^{\overline{s_0}-\lambda_0}(\xi) \prod_{j=1}^{t} (1-\|z_j\|^2)^{\overline{s_j}-\lambda_j} \int_{\Omega(\mu,t)} \rho^{u+\epsilon+\lambda_0}(\xi') \times \prod_{i=1}^{t} (1-\|z_i'\|^2)^{\mu_i u+v_i+\lambda_i} K_{\overline{s},u,v}(\xi,\xi') g(\xi') dV(\xi'). \tag{17}$$

We know that if $T_{s,u,v,\epsilon}$ is bounded on $L^p(\Omega(\mu,t),\omega_\lambda)$, then $T^*_{s,u,v,\epsilon}$ is bounded on $L^q(\Omega(\mu,t),\omega_\lambda)$, where $\frac{1}{p}+\frac{1}{q}=1$ (See [26]). Choose $\tilde{\delta}_0 > \max\big(-1-u-\lambda_0-\epsilon, \frac{-1-\lambda_0}{q}\big)$, $\tilde{\delta}_j > \max\big(-1-\lambda_j-v_j-\mu_j(m+u), \frac{-1-\lambda_j-\mu_j m}{q}\big)$, $1 \le j \le t$. Let $g(\xi) = \rho^{\tilde{\delta}_0}(\xi) \prod_{j=1}^{t} (1-\|z_j\|^2)^{\tilde{\delta}_j}$. Then $g \in L^q(\Omega(\mu,t),\omega_\lambda)$. By definition of $T^*_{s,u,v,\epsilon}$ and Lemma 2.10, there exists a constant $\mathcal{K}$ independent of ξ such that

$$(T^*_{s,u,v,\epsilon}g)(\xi) = \mathcal{K}\rho^{\overline{s_0}-\lambda_0}(\xi) \prod_{j=1}^{t} (1-\|z_j\|^2)^{\overline{s_j}-\lambda_j}.$$

Since $T^*_{s,u,v,\epsilon}$ is bounded on $L^q(\Omega(\mu,t),\omega_\lambda)$, we have $q(\sigma_0-\lambda_0)+\lambda_0 > -1$, $q(\sigma_j-\lambda_j)+\lambda_j > -\mu_j m-1$, $1 \le j \le t$.

Next we prove the sufficiency of condition (3) for the boundedness of $T_{s,u,v,\epsilon}$ on $L^p(\Omega(\mu,t),\omega_\lambda)$. We can see that conditions

$$\lambda_0+1 < p(\sigma_0+1),\ \lambda_j+\mu_j+1 < p(\sigma_j+\mu_j m+1),\ 1 \le j \le t$$

and $\frac{1}{p}+\frac{1}{q}=1$ imply that

$$\frac{-1-\sigma_0}{q} < \frac{\sigma_0-\lambda_0}{p},\ \frac{-1-\sigma_j-\mu_j m}{q} < \frac{\sigma_j-\lambda_j}{p},\ 1 \le j \le t.$$

Similarly, conditions

$$-p(u+\epsilon) < \lambda_0+1,\ -p(\mu_j u+v_j) < \lambda_j+\mu_j m+1,\ 1 \le j \le t$$

and $\frac{1}{p}+\frac{1}{q}=1$ imply that

$$\frac{-1-u-\lambda_0-\epsilon}{p} < \frac{u+\epsilon}{q},\ \frac{-1-v_j-\lambda_j-\mu_j(m+u)}{p} < \frac{\mu_j u+v_j}{q},\ 1 \le j \le t.$$

Therefore,

$$A_0 := \left(\frac{-1-\sigma_0}{q}, \frac{u+\epsilon}{q}\right) \cap \left(\frac{-1-u-\lambda_0-\epsilon}{p}, \frac{\sigma_0-\lambda_0}{p}\right) \neq \emptyset,$$

and for $j = 1, 2, \ldots, t$

$$A_j := \left(\frac{-1-\sigma_j-\mu_j m}{q}, \frac{\mu_j u+v_j}{q}\right) \cap \left(\frac{-1-v_j-\lambda_j-\mu_j(m+u)}{p}, \frac{\sigma_j-\lambda_j}{p}\right) \neq \emptyset.$$

Then we can take $\tau_0 \in A_0$, $\tau_j \in A_j$, $1 \leq j \leq t$. Let $g(\xi) = \rho^{\tau_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\tau_j}$ and write

$$(T_{s,u,v,\epsilon}f)(\xi) = \int_{\Omega(\mu,t)} \rho^{u+\epsilon}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j u+v_j}\rho^{s_0-\lambda_0}(\xi') \times \prod_{i=1}^{t}(1-\|z_i^{'}\|^2)^{s_i-\lambda_i}K_{s,u,v}(\xi,\xi')f(\xi')d\omega_\lambda(\xi').$$

Let T be the integral operator with kernel

$$K(\xi,\xi') = \rho^{u+\epsilon}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j u+v_j}\rho^{\sigma_0-\lambda_0}(\xi')\prod_{i=1}^{t}(1-\|z_i^{'}\|^2)^{\sigma_i-\lambda_i} \times |K_{s,u,v}(\xi,\xi')|.$$

It is easy to see that if T is bounded on $L^p(\Omega(\mu,t),\omega_\lambda)$, then $T_{s,u,v,\epsilon}$ is bounded on $L^p(\Omega(\mu,t),\omega_\lambda)$. We denote

$$A_{01}^{'} = \left(\frac{-1-\sigma_0}{q}, \frac{u}{q}\right) \cap \left(\frac{-1-u-\lambda_0-\epsilon}{p}, \frac{\sigma_0-\lambda_0-\epsilon}{p}\right),$$
$$A_{02}^{'} = \left(\frac{-1-\sigma_0}{q}, \frac{u}{q}\right) \cap \left(\frac{-1-u-\lambda_0}{p}, \frac{\sigma_0-\lambda_0}{p}\right),$$
$$A_{01}^{''} = \left(\frac{-1-\sigma_0+\epsilon}{q}, \frac{u+\epsilon}{q}\right) \cap \left(\frac{-1-u-\lambda_0-\epsilon}{p}, \frac{\sigma_0-\lambda_0-\epsilon}{p}\right),$$
$$A_{02}^{''} = \left(\frac{-1-\sigma_0+\epsilon}{q}, \frac{u+\epsilon}{q}\right) \cap \left(\frac{-1-u-\lambda_0}{p}, \frac{\sigma_0-\lambda_0}{p}\right).$$

We can see that $A_0 = A_{01}^{'} \cup A_{02}^{'} \cup A_{01}^{''} \cup A_{02}^{''}$.

If $\tau_0 \in A_{01}^{'}$, $\tau_j \in A_j$, $j = 1, 2, \ldots, t$, by Lemma 2.8, we obtain

$$\int_{\Omega(\mu,t)} K(\xi,\xi')g^q(\xi')d\omega_\lambda(\xi')$$
$$= \rho^{u+\epsilon}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j u+v_j}\int_{\Omega(\mu,t)}\rho^{\sigma_0+q\tau_0}(\xi')\prod_{i=1}^{t}(1-\|z_i^{'}\|^2)^{\sigma_i+q\tau_i} \times |K_{s,u,v}(\xi,\xi')|dV(\xi')$$

$$\lesssim \rho^{q\tau+\epsilon}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{q\tau_j}$$
$$\lesssim \rho^{q\tau}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{q\tau_j}$$
$$= g^q(\xi).$$

Similarly, by Lemma 2.9, we obtain

$$\int_{\Omega(\mu,t)} K(\xi,\xi')g^p(\xi)d\omega_\lambda(\xi) \lesssim g^p(\xi').$$

If $\tau_0 \in A_{02}^{'}$, $\tau_j \in A_j$, $j = 1, 2, \ldots, t$, for the same reason as the case $\tau_0 \in A_{01}^{'}$, $\tau_j \in A_j$, $j = 1, 2, \ldots, t$, we have

$$\int_{\Omega(\mu,t)} K(\xi,\xi')g^q(\xi')d\omega_\lambda(\xi') \lesssim g^q(\xi).$$

By Lemma 2.9, we obtain

$$\begin{aligned}
&\int_{\Omega(\mu,t)} K(\xi,\xi')g^p(\xi)d\omega_\lambda(\xi)\\
&\lesssim \rho^{\sigma_0-\lambda_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{\sigma_j-\lambda_j}\int_{\Omega(\mu,t)}\\
&\quad\times \rho^{u+\lambda_0+p\tau_0}(\xi)\prod_{i=1}^{t}(1-\|z_i\|^2)^{\mu_i u+v_i+\lambda_i+p\tau_i}|K_{s,u,v}(\xi,\xi')|dV(\xi)\\
&\lesssim \rho^{p\tau_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{p\tau_j}\\
&= g^p(\xi').
\end{aligned}$$

Similarly, we can check that (14) and (15) hold for $\tau_0 \in A_{01}^{''} \cup A_{02}^{''}$, $\tau_j \in A_j$, $j = 1, 2, \ldots, t$. Thus (14) and (15) hold for $\tau_0 \in A_0$, $\tau_j \in A_j$, $j = 1, 2, \ldots, t$. By Lemma 2.11, we know that T is bounded on $L^p(\Omega(\mu,t),\omega_\lambda)$. Therefore, if (3) holds, then $T_{s,u,v,\epsilon}$ is bounded on $L^p(\Omega(\mu,t),\omega_\lambda)$.

When $p = 1$, if $T_{s,u,v,\epsilon}$ is a bounded operator on $L^1(\Omega(\mu,t),\omega_\lambda)$, we will prove that (4) hold. By the same reason as in the case $1 < p < \infty$, we can prove the left side of (4). Since $T_{s,u,v,\epsilon}$ is bounded on $L^1(\Omega(\mu,t),\omega_\lambda)$, then $T^*_{s,u,v,\epsilon}$ is bounded on $L^\infty(\Omega(\mu,t))$. Choose $\delta_0 > \max(0, -1-u-\epsilon-\lambda_0)$, $\delta_j > \max(0, -v_j-\lambda_j-\mu_j(m+u))$, $1 \le j \le t$. Let $g(\xi) = \rho^{\delta_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\delta_j}$. Then $g \in L^\infty(\Omega(\mu,t))$. By (17),

$$(T^*_{s,u,v,\epsilon}g)(\xi) = \rho^{\overline{s_0}-\lambda_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\overline{s_j}-\lambda_j}\int_{\Omega(\mu,t)}\rho^{u+\epsilon+\lambda_0+\delta_0}(\xi')$$

$$\times \prod_{i=1}^{t}(1-\|z_i^{'}\|^2)^{\mu_i u+v_i+\lambda_i+\delta_i} K_{\overline{s},u,v}(\xi,\xi')dV(\xi').$$

By Lemma 2.10, there exists a constant $\mathcal{K}$ independent of ξ such that

$$(T^*_{s,u,v,\epsilon}g)(\xi) = \mathcal{K}\rho^{\overline{s_0}-\lambda_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\overline{s_j}-\lambda_j}.$$

Since $T^*_{s,u,v,\epsilon}$ is bounded on $L^\infty(\Omega(\mu,t))$, we have $\sigma_0 \geq \lambda_0$, $\sigma_j \geq \lambda_j$, $1 \leq j \leq t$.

Next we will prove that if (5) holds, then $T_{s,u,v,\epsilon}$ is a bounded operator on $L^1(\Omega(\mu,t),\omega_\lambda)$. Given $f \in L^1(\Omega(\mu,t),\omega_\lambda)$, we have

$$\begin{aligned}
&\int_{\Omega(\mu,t)} |(T_{s,u,v,\epsilon}f)(\xi)|d\omega_\lambda(\xi)\\
\leq &\int_{\Omega(\mu,t)} \rho^{u+\epsilon+\lambda_0}(\xi)\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j u+v_j+\lambda_j}\int_{\Omega(\mu,t)}\rho^{\sigma_0}(\xi')\\
&\times\prod_{i=1}^{t}(1-\|z_i^{'}\|^2)^{\sigma_i}|K_{s,u,v}(\xi,\xi')||f(\xi')|dV(\xi')dV(\xi)\\
= &\int_{\Omega(\mu,t)} \rho^{\sigma_0}(\xi')\prod_{i=1}^{t}(1-\|z_i^{'}\|^2)^{\sigma_i}|f(\xi')|\int_{\Omega(\mu,t)}\rho^{u+\epsilon+\lambda_0}(\xi) \qquad (18)\\
&\times\prod_{j=1}^{t}(1-\|z_j\|^2)^{\mu_j u+v_j+\lambda_j}|K_{s,u,v}(\xi,\xi')|dV(\xi)dV(\xi').
\end{aligned}$$

Since (5) holds, by Lemma 2.9 and (18) we obtain

$$\begin{aligned}
&\int_{\Omega(\mu,t)} |(T_{s,u,v,\epsilon}f)(\xi)|d\omega_\lambda(\xi)\\
\lesssim &\int_{\Omega(\mu,t)} \rho^{\sigma_0}(\xi')\prod_{i=1}^{t}(1-\|z_i^{'}\|^2)^{\sigma_i}|f(\xi')|\rho^{\epsilon+\lambda_0-\sigma_0}(\xi')\prod_{j=1}^{t}(1-\|z_j^{'}\|^2)^{\lambda_j-\sigma_j}dV(\xi')\\
\lesssim &\int_{\Omega(\mu,t)} |f(\xi')|d\omega_\lambda(\xi').
\end{aligned}$$

Thus $T_{s,u,v,\epsilon}$ is a bounded operator on $L^1(\Omega(\mu,t),\omega_\lambda)$.

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