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MULTIPLICITY OF SOLUTIONS FOR QUASILINEAR SCHRÖDINGER TYPE EQUATIONS WITH THE CONCAVE-CONVEX NONLINEARITIES

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Abstract. We deal with the following elliptic equations:

$$\begin{cases} -\operatorname{div}(\varphi'(|\nabla z|^2)\nabla z) + V(x)|z|^{\alpha-2}z = \lambda\rho(x)|z|^{r-2}z + h(x,z), \\ z(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \text{in } \mathbb{R}^N,$$

where $N \geq 2$, $1 < p < q < N$, $1 < \alpha \leq p^*q'/p'$, $\alpha < q$, $1 < r < \min\{p, \alpha\}$, $\varphi(t)$ behaves like $t^{q/2}$ for small t and $t^{p/2}$ for large t, and p' and q' the conjugate exponents of p and q, respectively. Here, $V : \mathbb{R}^N \to (0,\infty)$ is a potential function and $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. The present paper is devoted to the existence of at least two distinct non-trivial solutions to quasilinear elliptic problems of Schrödinger type, which provides a concave–convex nature to the problem. The primary tools are the well-known mountain pass theorem and a variant of Ekeland's variational principle.

1. Introduction

In this paper, we are working with existence and multiplicity of solutions for the following quasilinear elliptic equations:

$$\text{(1.1)} \quad \begin{cases} -\operatorname{div}(\varphi'(|\nabla z|^2)\nabla z) + V(x)|z|^{\alpha-2}z = \lambda\rho(x)|z|^{r-2}z + h(x,z), \\ z(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \text{in } \mathbb{R}^N,$$

where $N \geq 2$, $1 < p < q < N$, $1 < \alpha \leq p^*q'/p'$, $\alpha < q$, $1 < r < \min\{p, \alpha\}$, $V : \mathbb{R}^N \to (0,\infty)$, $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, the function $\varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies the following properties: for $\alpha < q$,

(A1) $\varphi(0) = 0$;

(A2) There exist $\tilde{c} > 0$ and $\tilde{C} > 0$ such that

$$\begin{cases} \tilde{c}t^{\frac{p}{2}} \leq \varphi(t) \leq \tilde{C}t^{\frac{p}{2}}, & t \geq 1, \\ \tilde{c}t^{\frac{q}{2}} \leq \varphi(t) \leq \tilde{C}t^{\frac{q}{2}}, & 0 \leq t \leq 1; \end{cases}$$

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(A3) There exists $0 < \mu < \frac{\alpha}{\gamma}$ such that

$$\varphi'(t)t \leq \frac{\gamma\mu}{2}\varphi(t)$$

for all $t \geq 0$, and γ is given in (B1) below;

(A4) The map $t \to \varphi(t^2)$ is strictly convex.

Recently, the study of quasilinear elliptic equations such as the problem (1.1), including nonhomogeneous operators of the type

$$-\mathrm{div}(\varphi'(|\nabla z|^2)\nabla z), \quad \text{where} \quad \varphi \in C^1(\mathbb{R}^+, \mathbb{R}^+),$$

has extensively been considered basing on the pure or the applied mathematical theory to explain some particular cases arising from nonlinear elasticity, fluid mechanics, generalized Newtonian fluids, plasticity theory, biophysics problems and plasma physics. For deep information, we refer the reader to [22–25,28]. As we know, the study on problems with non-homogeneous differential operators is on account of the theory of the Orlicz-Sobolev space which is a generalization of the classical Sobolev space. In this regard, variational problems for elliptic equations of this type have been extensively investigated in recent years; for instance, see [1,2,9,14–16,19–21,26,27,37] and their references.

Recently, by use of variational method, the authors in [5,6] took into account the existence of nonnegative radially symmetric solutions for a nonlinear elliptic equation related to a new class of differential operators in an Orlicz-Sobolev space for the case of a different behavior of φ approaching zero and infinity; for instance,

$$\varphi(t) = \frac{2}{p}\left[\left(1 + t^{\frac{q}{2}}\right)^{\frac{p}{q}} - 1\right], \quad 1 < p < q, \quad t \in \mathbb{R}^+.$$

As mentioned in [6], we cannot make use of the theory of classical Sobolev spaces directly, because the principal part φ is provided by the different growth. Hence, in order to overcome this difficulty, they take an appropriate functional framework by the paper [7] into consideration, namely, the sum of Lebesgue spaces, which is fresh from the preceding related works [1,2,9,15,16,21,26,27, 37], even if the functional setting related with (1.1) in an Orlicz-Sobolev space is considered. Based on the results in [5,6], N. Chorfi and V. D. Rădulescu [13] established that quasilinear Schrödinger equations admit at least one nontrivial solution by using the mountain pass theorem which is initially provided by the paper [4]. Very recently, the authors in [32] investigated the existence of multiple large- or small- energy solutions to a Schrödinger-Kirchhoff type problem with the various conditions on the nonlinear term h, by using the fountain theorem and the dual fountain theorem in [42], respectively.

The main intention of the present paper is to guarantee the multiplicity of nontrivial solutions for Schrödinger type problems in case where the nonlinear term is concave-convex, by means of the mountain pass theorem (see [4]), a variant of Ekeland's variational principle (see [8]). From a pure mathematical perspective, people have been deeply studied about the concave-convex-type elliptic problems (see [10–12, 29, 41, 43, 44]) since the pioneer work of A. Ambrosetti, H. Brezis and G. Cerami [3] for the Laplacian problem:

$$\begin{cases} -\triangle z = \lambda |z|^{q-2}v + |z|^{h-2}z & \text{in} \quad \Omega, \\ z = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $1 < q < 2 < h < 2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N > 2, \\ +\infty & \text{if } N = 1, 2. \end{cases}$

In particular, the existence of multiple solutions to elliptic problems driven by a nonlocal integro-differential operator involving the concave-convex nonlinearities has been built in [12]; see also [10, 31, 44] and [29] for $p(x)$-Laplacian equations. For quasilinear elliptic equations involving nonhomogeneous operators which subject to Dirichlet boundary conditions, by applying the well-known Nehari manifold method, the authors in [11] established the existence and multiplicity of solutions to the problem below

$$\begin{cases} -\text{div}(\phi(|\nabla z|)\nabla z) = \lambda a(x)\,|z|^{r-1}\,z + b(x)\,|z|^{q-1}\,z, \quad z > 0 \quad \text{in } \Omega; \\ z = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $a, b \in L^\infty(\Omega)$ with $a^+, b^+ \not\equiv 0$, the exponents r and q satisfy $0 < r < \ell - 1 < q \le \ell^* - 1$, $1 < \ell < N$, $\ell^* = \ell N/(N-\ell)$, λ is a positive parameter and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is of C^2 class with suitable conditions. Also the existence, multiplicity and asymptotic behaviour of nonnegative solutions for critical quasilinear elliptic equations with concave-convex nonlinear terms were established in [17] by making use of the concentration-compactness principle to recover the compactness required in variational methods. Recently, Kim *et al.* [33] have studied that concave-convex problems involving fractional $p(\cdot)$-Laplacian operator have the existence of two distinct nontrivial solutions when the convex term satisfies the condition of Ambrosetti-Rabinowitz type in [4], that is, there exists a constant $\theta > 0$ such that $\theta > \sup_{x\in\mathbb{R}^N} p(x)$ and

$$0 < \theta H(x,t) \le h(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus \{0\} \text{ and } x \in \mathbb{R}^N, \tag{1.2}$$

where $H(x,t) = \int_0^t h(x,s)\,ds$. Also they obtained this result when (1.2) was superseded by the condition introduced by Jeanjean in [30], namely, there is a constant $\theta \ge 1$ such that

$$\theta\mathcal{H}(x,t) \ge \mathcal{H}(x,st) \tag{1.3}$$

for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where

$$\mathcal{H}(x,t) = h(x,t)t - (\sup_{x\in\mathbb{R}^N} p(x))H(x,t).$$

For deriving the multiplicity result, the authors used the mountain pass theorem in [4] and a variant of Ekeland's variational principle (see [8]) which are the main tools. In this way, the first purpose of this paper is to get the existence of two distinct nontrivial solutions for the problem (1.1) by the case of a combined effect of concave-convex nonlinearities, as long as the convex term h fulfils the condition of Ambrosetti-Rabinowitz type. The other one is to obtain the multiplicity result when the condition on h does not satisfy the Ambrosetti-Rabinowitz condition in general, and different from (1.3) which is provided in [38] although the given domain is bounded. To our knowledge, this paper is the first endeavor to develop the multiplicity of nontrivial solutions for quasilinear elliptic problems of Schrödinger type with the concave-convex nonlinearities in these situations.

This paper's outline is the following: we firstly look back the well-known facts for the sum of Lebesgue spaces and Orlicz-Sobolev spaces. Under some conditions on the convex term h, we carry out various existence results of two distinct nontrivial solutions to the problem (1.1) by utilizing as the major tools the variational principle.

2. Preliminaries and main results

In this section, we briefly list some definitions and essential properties of the sum of Lebesgue spaces and Orlicz-Sobolev space. For a deeper treatment of these spaces, we refer to [1, 40].

Definition 2.1. Let $1 < p < q < N$. We denote by $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ the completion of $C_c^\infty(\mathbb{R}^N, \mathbb{R})$ in the norm

$$\|z\|_{L^p(\mathbb{R}^N)+L^q(\mathbb{R}^N)} = \inf\{\|v\|_{L^p(\mathbb{R}^N)} + \|w\|_{L^q(\mathbb{R}^N)} \mid v \in L^p(\mathbb{R}^N), \\ w \in L^q(\mathbb{R}^N), z = v + w\}.$$

We set $\|z\|_{L^{p,q}(\mathbb{R}^N)} := \|z\|_{L^p(\mathbb{R}^N)+L^q(\mathbb{R}^N)}$.

Now, we define the Orlicz-Sobolev space for our analysis (see [6]).

Definition 2.2. For $\alpha > 1$, $\mathcal{W}$ is denoted by the completion of $C_c^\infty(\mathbb{R}^N, \mathbb{R})$ in the norm

$$\|z\|_{\mathcal{W}} = \|z\|_{L^\alpha(\mathbb{R}^N)} + \|\nabla z\|_{L^{p,q}(\mathbb{R}^N)}.$$

Lemma 2.3. *The space $\mathcal{W}$ is continuously embedded into $L^{p^*}(\mathbb{R}^N)$ for every $1 < \alpha \le p^* \frac{q'}{p'}$. In addition, $\mathcal{W}$ is continuously embedded into $L^\tau(\mathbb{R}^N)$ for any $\alpha \le \tau \le p^*$.*

We consider the case that the potential function V satisfies

(V) $V \in L^1_{loc}(\mathbb{R}^N)$, $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0$ and $\lim_{|x|\to\infty} V(x) = +\infty$.

Definition 2.4. For $\alpha > 1$, $\mathcal{W}_V$ is denoted by the completion of $C_c^\infty(\mathbb{R}^N, \mathbb{R})$ in the norm

$$\|z\|_{\mathcal{W}_V} = \|\nabla z\|_{L^{p,q}(\mathbb{R}^N)} + \|z\|_{L^\alpha(V,\mathbb{R}^N)},$$

where

$$\|z\|_{L^\alpha(V,\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} V(x)\, |z|^\alpha \, dx\right)^{\frac{1}{\alpha}}.$$

In view of the paper [6], we note that $(\mathcal{W}_V, \|z\|_{\mathcal{W}_V})$ is a reflexive Banach space and recollects the following embedding results.

Lemma 2.5 ([45])**.** *Let* (V) *hold. Then we have*

(i) *the space* $\mathcal{W}_V$ *is continuously embedded into* $L^{p^*}(\mathbb{R}^N)$ *for every* $1 < \alpha \leq p^*\frac{q'}{p'}$;

(ii) $\mathcal{W}_V$ *is continuously embedded into* $L^\tau(\mathbb{R}^N)$ *for any* $\alpha \leq \tau \leq p^*$;

(iii) *For every* $\alpha \in (1, p^*\frac{q'}{p'}]$, *the space* $\mathcal{W}_V$ *is compactly embedded into* $L^\kappa(\mathbb{R}^N)$ *with* $\alpha \leq \kappa < p^*$,

where $p^* = \frac{Np}{N-p}$.

In the next lemma we provide a list of properties that will be useful for the rest of the paper.

Lemma 2.6 ([6])**.** *Let* $\Omega \subset \mathbb{R}^N$, $z \in L^p(\Omega)+L^q(\Omega)$ *and* $\bigwedge_z = \{x \in \Omega : |z(x)| > 1\}$. *We have*

(i) $meas(\bigwedge_z) < \infty$;

(ii) $z \in L^p(\bigwedge_z) \cap L^q(\bigwedge_z^c)$;

(iii) $\|z\|_{L^{p,q}(\Omega)} \leq \max\{\|z\|_{L^p(\bigwedge_z)}, \|z\|_{L^q(\bigwedge_z^c)}\}$;

(iv) $\max\left\{\frac{\|z\|_{L^p(\bigwedge_z)}}{\max\{1, meas(\bigwedge_z)^{1/p-1/q}\}}, \|z\|_{L^q(\bigwedge_z^c)} - C(p,q)\right\} \leq \|z\|_{L^{p,q}(\Omega)}$ $\leq \|z\|_{L^p(\bigwedge_z)} + \|z\|_{L^q(\bigwedge_z^c)}$, *where* $C(p,q) = (q-p)(p/q)^{q/(q-p)}/p$.

Throughout this paper, we assume that the conditions (A1)–(A4) are fulfilled.

Definition 2.7. We say that $z \in \mathcal{W}_V$ is a weak solution of the problem (1.1) if

$$\int_{\mathbb{R}^N} \varphi'(|\nabla z|^2)\nabla z \cdot \nabla w \, dx + \int_{\mathbb{R}^N} V(x) \, |z|^{\alpha-2} \, zw \, dx$$
$$= \lambda \int_{\mathbb{R}^N} \rho(x)|z|^{r-2} zw \, dx + \int_{\mathbb{R}^N} h(x,z) w \, dx$$

for any $w \in \mathcal{W}_V$.

Assume that

(B1) $1 < r < p < q < \gamma < p^*$;

(B2) $\rho \in L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\text{meas}\{x \in \mathbb{R}^N : \rho(x) \neq 0\} > 0$ for any γ_0 with $\max\{p, \alpha\} < \gamma_0 < p^*$;

(H1) $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition;

(H2) There is a nonnegative function $\sigma \in L^\infty(\mathbb{R}^N)$ such that
$$|h(x,t)| \leq \sigma(x) \, |t|^{\gamma-1}$$
for all $t \in \mathbb{R}^N$, a.e. $x \in \mathbb{R}^N$;

(H3) There exists a positive constant θ such that $\theta > q$ and $0 < \theta H(x,t) \leq h(x,t)t$ for all $t > 0$, a.e. $x \in \mathbb{R}^N$, where $H(x,t) = \int_0^t h(x,s) \, ds$;

(H4) $H(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;
(H5) $\lim_{|t|\to\infty} \frac{H(x,t)}{|t|^q} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$;
(H6) $H(x,t) = o(|t|^\alpha)$, as $t \to 0$, uniformly for $x \in \mathbb{R}^N$.

Next, we define the functional $\mathcal{E}_\lambda : \mathcal{W}_V \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{E}_\lambda(z) = \frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z|^2)\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} V(x)|z|^\alpha\,dx \\ - \frac{\lambda}{r}\int_{\mathbb{R}^N} \rho(x)|z|^r\,dx - \int_{\mathbb{R}^N} H(x,z)\,dx.\end{aligned}$$

Let the functional $\Psi_\lambda : \mathcal{W}_V \to \mathbb{R}$ be defined by

$$\Psi_\lambda(z) = \frac{\lambda}{r}\int_{\mathbb{R}^N} \rho(x)\,|z|^r\,dx + \int_{\mathbb{R}^N} H(x,z)\,dx.$$

Then, it is easy to check that $\Psi_\lambda \in C^1(\mathcal{W}_V, \mathbb{R})$, and its Fréchet derivative is

$$\langle \Psi'_\lambda(z), w\rangle = \lambda \int_{\mathbb{R}^N} \rho(x)|z|^{r-2}zw\,dx + \int_{\mathbb{R}^N} h(x,z)w\,dx$$

for any $z, w \in \mathcal{W}_V$. Then it follows that the functional $\mathcal{E}_\lambda \in C^1(\mathcal{W}_V, \mathbb{R})$ and its Fréchet derivative is

$$\begin{aligned}\langle \mathcal{E}'_\lambda(z), w\rangle = \int_{\mathbb{R}^N} \varphi'(|\nabla z|^2)\nabla z \cdot \nabla w\,dx + \int_{\mathbb{R}^N} V(x)\,|z|^{\alpha-2}\,zw\,dx \\ - \lambda\int_{\mathbb{R}^N} \rho(x)|z|^{r-2}zw\,dx - \int_{\mathbb{R}^N} h(x,z)w\,dx\end{aligned}$$

for any $z, w \in \mathcal{W}_V$.

Lemma 2.8. *Assume that* (V), (B1)–(B2) *and* (H1)–(H2) *hold. Then* Ψ_λ *and* Ψ'_λ *are weakly strongly continuous on* $\mathcal{W}_V$ *for any* $\lambda > 0$.

Proof. Let $\{z_n\}$ be a sequence in $\mathcal{W}_V$ such that $z_n \rightharpoonup z$ in $\mathcal{W}_V$ as $n \to \infty$. From the boundedness of $\{z_n\}$ and Lemma 2.5, there is a subsequence such that

$$z_{n_k} \to z \quad \text{a.e. in } \mathbb{R}^N \quad \text{and} \quad z_{n_k} \to z \quad \text{in } L^{\gamma_0}(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N) \tag{2.1}$$

as $k \to \infty$.

First we show that Ψ_λ is weakly strongly continuous in $\mathcal{W}_V$. Using the convergence principle, we can choose a function $g \in L^{\gamma_0}(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)$ such that $|z_{n_k}| \leq g$ for all $k \in \mathbb{N}$. Hence, due to (H2) and the Young inequality, one has

$$\begin{aligned}&\frac{\lambda}{r}\int_{\mathbb{R}^N} \rho(x)\,|z_{n_k}|^r\,dx + \int_{\mathbb{R}^N} |H(x,z_{n_k})|\,dx \\ &\leq \frac{\lambda}{r}\int_{\mathbb{R}^N} |\rho(x)|\,|z_{n_k}|^r\,dx + \frac{1}{\gamma}\int_{\mathbb{R}^N} \sigma(x)|z_{n_k}|^\gamma\,dx \\ &\leq \frac{\lambda}{r}\int_{\mathbb{R}^N} \frac{2(\gamma_0 - r)}{\gamma_0}\,|\rho(x)|^{\frac{\gamma_0}{\gamma_0 - r}} + \frac{r}{\gamma_0}\,|z_{n_k}|^{\gamma_0}\,dx + \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{\gamma}\int_{\mathbb{R}^N} |z_{n_k}|^\gamma\,dx\end{aligned}$$

$$\leq C_0 \left[\int_{\mathbb{R}^N} |\rho(x)|^{\frac{\gamma_0}{\gamma_0 - r}} + |g|^{\gamma_0} \, dx + \int_{\mathbb{R}^N} |g|^{\gamma} \, dx \right]$$

for a positive constant C_0, and thus the integral at the left-hand side is dominated by an integrable function. By (H1) and (2.1), it follows that

$$\frac{\rho(x)}{r} |z_{n_k}(x)|^r \to \frac{\rho(x)}{r} |z(x)|^r \quad \text{and} \quad H(x, z_{n_k}) \to H(x, z) \qquad \text{as} \quad k \to \infty$$

for almost all $x \in \mathbb{R}^N$. Therefore, Lebesgue's dominated convergence theorem yields that

$$\frac{\lambda}{r} \int_{\mathbb{R}^N} \rho(x) |z_{n_k}|^r \, dx + \int_{\mathbb{R}^N} H(x, z_{n_k}) \, dx \to \lambda \int_{\mathbb{R}^N} \frac{\rho(x)}{r} |z|^r \, dx + \int_{\mathbb{R}^N} H(x, z) \, dx$$

as $k \to \infty$, that is, $\Psi_\lambda(z_{n_k}) \to \Psi_\lambda(z)$ as $k \to \infty$. Thus Ψ_λ is weakly strongly continuous in $\mathcal{W}_V$.

It remains to prove that Ψ'_λ is weakly strongly continuous in $\mathcal{W}_V^*$. By the assumption of ρ, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left| \rho(x) |z_{n_k}|^{r-2} z_{n_k} - \rho(x) |z|^{r-2} z \right|^{r'} \, dx \\
&\leq C_1 \int_{\mathbb{R}^N} |\rho(x)|^{\frac{1}{r-1}} |\rho(x)| \left(|z_{n_k}|^r + |z|^r \right) dx \\
&\leq C_2 \int_{\mathbb{R}^N} |\rho(x)| \left(|z_{n_k}|^r + |z|^r \right) dx \\
&\leq C_2 \int_{\mathbb{R}^N} \frac{2(\gamma_0 - r)}{\gamma_0} |\rho(x)|^{\frac{\gamma_0}{\gamma_0 - r}} + \frac{r}{\gamma_0} |z_{n_k}|^{\gamma_0} + \frac{r}{\gamma_0} |z|^{\gamma_0} \, dx \qquad (2.2)
\end{aligned}$$

for some positive constants C_1, C_2. Due to (H2), we know

$$\begin{aligned}
\int_{\mathbb{R}^N} |h(x, z_{n_k}) - h(x, z)|^{\gamma'} \, dx &\leq C_3 \int_{\mathbb{R}^N} |h(x, z_{n_k})|^{\gamma'} + |h(x, z)|^{\gamma'} \, dx \\
&\leq C_4 \int_{\mathbb{R}^N} |z_{n_k}|^{\gamma} + |z|^{\gamma} \, dx \qquad (2.3)
\end{aligned}$$

for some positive constants C_3, C_4. Invoking (2.1)–(2.3) and the convergence principle, we deduce

$$\left| \rho(x) \, |z_{n_k}|^{r-2} - \rho(x) \, |z|^{r-2} \right|^{r'} \leq f_1(x)$$

and

$$|h(x, z_{n_k}) - h(x, z)|^{\gamma'} \leq f_2(x)$$

for almost all $x \in \mathbb{R}^N$ and for some $f_1, f_2 \in L^1(\mathbb{R}^N)$, and also $\rho(x)|z_{n_k}|^{r-2} z_{n_k} \to \rho(x)|z|^{r-2} z$ and $h(x, z_{n_k}) \to h(x, z)$ as $k \to \infty$ for almost all $x \in \mathbb{R}^N$. By virtue of the Lebesgue convergence theorem, one has

$$\|\Psi'_\lambda(z_{n_k}) - \Psi'_\lambda(z)\|_{\mathcal{W}_V^*} = \sup_{\|w\|_{\mathcal{W}_V} \leq 1} |\langle \Psi'_\lambda(z_{n_k}) - \Psi'_\lambda(z), w \rangle|$$

$$= \sup_{\|w\|_{\mathcal{W}_V} \le 1} \left| \lambda \int_{\mathbb{R}^N} (\rho(x)|z_{n_k}|^{r-2} z_{n_k} - \rho(x)|z|^{r-2} z) w \, dx \right.$$
$$\left. + \int_{\mathbb{R}^N} (h(x, z_{n_k}) - h(x,z)) w \, dx \right|$$
$$\le 2 \Big(\lambda \|\rho(x)|z_{n_k}|^{r-2} z_{n_k} - \rho(x)|z|^{r-2} z\|_{L^{r'}(\mathbb{R}^N)}$$
$$+ \|h(x, z_{n_k}) - h(x,z)\|_{L^{\gamma'}(\mathbb{R}^N)} \Big) \to 0$$

as $k \to \infty$. Consequently, we derive that $\Psi'_\lambda(z_{n_k}) \to \Psi'_\lambda(z)$ in $\mathcal{W}_V^*$ as $k \to \infty$. This completes the proof. $\square$

Before going to the proofs of our main consequences, we present some useful preliminary assertions.

Lemma 2.9. *Let* (V), (B1)–(B2) *and* (H1)–(H3) *hold. Then the functional* $\mathcal{E}_\lambda$ *satisfies the followings*:

(1) *There is* $\lambda^* > 0$ *such that for any* $\lambda \in (0, \lambda^*)$ *we can choose positive constants* R *and* $0 < \delta < 1$ *such that* $\mathcal{E}_\lambda(z) \ge R > 0$ *for all* $z \in \mathcal{W}_V$ *with* $\|z\|_{\mathcal{W}_V} = \delta$;
(2) *There is* $\phi \in C_c^\infty(\mathbb{R}^N)$, $\phi > 0$ *such that* $\mathcal{E}_\lambda(t\phi) \to -\infty$ *as* $t \to +\infty$;
(3) *There is* $\psi \in C_c^\infty(\mathbb{R}^N)$, $\psi > 0$ *such that* $\mathcal{E}_\lambda(t\psi) < 0$ *as* $t \to 0^+$.

Proof. Let us show the condition (1). According to (B1) and Lemma 2.5, we know that $\|z\|_{L^\gamma(\mathbb{R}^N)} \le C_5 \|z\|_{\mathcal{W}_V}$ for $q < \gamma < p^*$ and for a positive constant C_5. Assume that $\|z\|_{\mathcal{W}_V} < 1$. Then it follows from (A2), (H2), (iii) of Lemmas 2.6 that

$$\mathcal{E}_\lambda(z) = \frac{1}{2} \int_{\mathbb{R}^N} \varphi(|\nabla z|^2) \, dx + \frac{1}{\alpha} \int_{\mathbb{R}^N} V(x)|z|^\alpha \, dx$$
$$- \frac{\lambda}{r} \int_{\mathbb{R}^N} \rho(x)|z|^r \, dx - \int_{\mathbb{R}^N} H(x,z) \, dx$$
$$\ge \frac{\tilde{c}}{2} \int_{\wedge_{\nabla z}^c} |\nabla z|^q \, dx + \frac{\tilde{c}}{2} \int_{\wedge_{\nabla z}} |\nabla z|^p \, dx + \frac{1}{\alpha} \int_{\mathbb{R}^N} V(x)|z|^\alpha \, dx$$
$$- 2 \frac{\lambda}{r} \|\rho\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} C_5 \|z\|_{\mathcal{W}_V}^r - \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{\gamma} \|z\|_{L^\gamma(\mathbb{R}^N)}^\gamma$$
$$\ge \frac{\tilde{c}}{2} \max \left\{ \int_{\wedge_{\nabla z}^c} |\nabla z|^q \, dx, \int_{\wedge_{\nabla z}} |\nabla z|^p \, dx \right\} + \frac{1}{\alpha} \|z\|_{L^\alpha(V, \mathbb{R}^N)}^\alpha$$
$$- 2 \frac{\lambda}{r} \|\rho\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} C_5 \|z\|_{\mathcal{W}_V}^r - \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{\gamma} \|z\|_{L^\gamma(\mathbb{R}^N)}^\gamma$$
$$\ge \frac{\tilde{c}}{2} \|\nabla z\|_{L^{p,q}(\mathbb{R}^N)}^q + \frac{1}{\alpha} \|z\|_{L^\alpha(V, \mathbb{R}^N)}^\alpha$$
$$- 2 \frac{\lambda}{r} \|\rho\|_{L^{\frac{\gamma_0}{\gamma_0 - r}}(\mathbb{R}^N)} C_5 \|z\|_{\mathcal{W}_V}^r - \frac{\|\sigma\|_{L^\infty(\mathbb{R}^N)}}{\gamma} \|z\|_{L^\gamma(\mathbb{R}^N)}^\gamma$$

$$\geq \left(\widetilde{C}_6 \min\left\{\frac{\tilde{c}}{2}, \frac{1}{\alpha}\right\} - 2\frac{\lambda}{r}C_6\|z\|_{\mathcal{W}_V}^{r-q} - \frac{1}{\gamma}C_7\|z\|_{\mathcal{W}_V}^{\gamma-q}\right)\|z\|_{\mathcal{W}_V}^{q} \tag{2.4}$$

for positive constants $\widetilde{C}_6, C_6, C_7$. Let us define the function $g_\lambda : (0,\infty) \to \mathbb{R}$ by

$$g_\lambda(t) = 2C_6\frac{\lambda}{r}t^{r-q} + C_7\frac{1}{\gamma}t^{\gamma-q}.$$

Then it is immediate that g_λ has a local minimum at the point

$$t_0 = \left(\frac{\lambda 2C_6(q-r)\gamma}{C_7(\gamma-q)r}\right)^{\frac{1}{\gamma-r}}$$

and so

$$\lim_{\lambda\to 0^+} g_\lambda(t_0) = 0.$$

Thus there is a positive constant λ^* such that for each $\lambda \in (0,\lambda^*)$, there are a positive real number R and $\delta > 0$ small enough such that $\mathcal{E}_\lambda(z) \geq R > 0$ for any $z \in \mathcal{W}_V$ with $\|z\|_{\mathcal{W}_V} = \delta$.

Next we show the condition (2). Note that assumption (H3) means

$$H(x, s\eta) \geq s^\theta H(x,\eta) \tag{2.5}$$

for all $\eta \in \mathbb{R}$, $x \in \mathbb{R}^N$, and $s \geq 1$.

Take $\phi \in C_c^\infty(\mathbb{R}^N)$ with $\phi > 0$. It follows from (A2) and (2.5) that

$$\begin{aligned}
\mathcal{E}_\lambda(t\phi) &= \frac{1}{2}\int_{\mathbb{R}^N}\varphi(|\nabla t\phi|^2)\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N}V(x)|t\phi|^\alpha\,dx \\
&\qquad - \frac{\lambda}{r}\int_{\mathbb{R}^N}\rho(x)|t\phi|^r\,dx - \int_{\mathbb{R}^N}H(x,t\phi)\,dx \\
&\leq \frac{\tilde{C}}{2}\int_{\wedge^c_{\nabla(t\phi)}}|\nabla t\phi|^q\,dx + \frac{\tilde{C}}{2}\int_{\wedge_{\nabla(t\phi)}}|\nabla t\phi|^p\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N}V(x)|t\phi|^\alpha\,dx \\
&\qquad - \frac{\lambda}{r}\int_{\mathbb{R}^N}\rho(x)|t\phi|^r\,dx - \int_{\mathbb{R}^N}H(x,t\phi)\,dx \\
&\leq \frac{\tilde{C}}{2}\left(t^q\int_{\mathbb{R}^N}|\nabla\phi|^q\,dx + t^p\int_{\mathbb{R}^N}|\nabla\phi|^p\,dx\right) + \frac{t^\alpha}{\alpha}\int_{\mathbb{R}^N}V(x)|\phi|^\alpha\,dx \\
&\qquad - \frac{\lambda t^r}{r}\int_{\mathbb{R}^N}\rho(x)|\phi|^r\,dx - t^\theta\int_{\mathbb{R}^N}H(x,\phi)\,dx
\end{aligned}$$

for sufficiently large $t \geq 1$. Since $\theta > q$, we see that $\mathcal{E}_\lambda(t\phi) \to -\infty$ as $t \to \infty$.

Finally we remain to prove the condition (3). Let us choose $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $\psi > 0$, and let λ be fixed. For $t > 0$ small enough, from (A2), (H3), we obtain

$$\begin{aligned}
\mathcal{E}_\lambda(t\psi) &= \frac{1}{2}\int_{\mathbb{R}^N}\varphi(|\nabla t\psi|^2)\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N}V(x)|t\psi|^\alpha\,dx \\
&\qquad - \frac{\lambda}{r}\int_{\mathbb{R}^N}\rho(x)|t\psi|^r\,dx - \int_{\mathbb{R}^N}H(x,t\psi)\,dx
\end{aligned}$$

$$\leq \frac{\tilde{C}}{2}\int_{\Lambda^c_{\nabla(t\psi)}}|\nabla t\psi|^q\,dx+\frac{\tilde{C}}{2}\int_{\Lambda_{\nabla(t\psi)}}|\nabla t\psi|^p\,dx+\frac{1}{\alpha}\int_{\mathbb{R}^N}V(x)|t\psi|^\alpha\,dx$$
$$-\frac{\lambda}{r}\int_{\mathbb{R}^N}\rho(x)|t\psi|^r\,dx-\int_{\mathbb{R}^N}H(x,t\psi)\,dx$$
$$\leq \frac{\tilde{C}}{2}\left(t^q\int_{\mathbb{R}^N}|\nabla\psi|^q\,dx+t^p\int_{\mathbb{R}^N}|\nabla\psi|^p\,dx\right)+\frac{t^\alpha}{\alpha}\int_{\mathbb{R}^N}V(x)|\psi|^\alpha\,dx$$
$$-\frac{\lambda t^r}{r}\int_{\mathbb{R}^N}\rho(x)|\psi|^r\,dx.$$

Since $r<\alpha$, we see that $\mathcal{E}_\lambda(t\psi)<0$ as $t\to 0^+$. Therefore $\mathcal{E}_\lambda$ satisfies the geometry of the mountain pass theorem. □

With the help of Lemma 2.8, we prove that the energy functional $\mathcal{E}_\lambda$ fulfils the Palais-Smale condition ((PS)-condition for short). This plays an elementary role in guaranteeing the existence of a nontrivial weak solution for the given problem. The basic idea of proof of this assertion is due to [39].

Definition 2.10. Let X be a real Banach space. We say that $\mathcal{E}_\lambda$ satisfies the (PS)-condition in X, if any (PS)-sequence $\{z_n\}_n\subset X$, i.e., $\{\mathcal{E}_\lambda(z_n)\}$ is bounded and $\mathcal{E}'_\lambda(z_n)\to 0$ as $n\to\infty$, admits a strongly convergent subsequence in X.

Lemma 2.11. *If* (V), (B1)–(B2) *and* (H1)–(H3) *hold, then the functional* $\mathcal{E}_\lambda$ *satisfies the* (PS)*-condition for any* $\lambda>0$.

Proof. Let $\{z_n\}$ be a (PS)-sequence in $\mathcal{W}_V$, *i.e.*, there is a positive constant K such that $|\langle\mathcal{E}'_\lambda(z_n),z_n\rangle|\leq K\|z_n\|_{\mathcal{W}_V}$ and $|\mathcal{E}_\lambda(z_n)|\leq K$ for any $n\in\mathbb{N}$. First we will verify that the sequence $\{z_n\}$ is bounded in $\mathcal{W}_V$. Suppose to the contrary that $\|z_n\|_{\mathcal{W}_V}>1$ and $\|z_n\|_{\mathcal{W}_V}\to\infty$, in the subsequence sense, as $n\to\infty$. By assumptions (A2) and (A3), we deduce that

$$K+K\|z_n\|_{\mathcal{W}_V}$$
$$\geq \mathcal{E}_\lambda(z_n)-\frac{1}{\theta}\langle\mathcal{E}'_\lambda(z_n),z_n\rangle$$
$$\geq \frac{1}{2}\int_{\mathbb{R}^N}\varphi(|\nabla z_n|^2)\,dx-\frac{1}{\theta}\int_{\mathbb{R}^N}\varphi'(|\nabla z_n|^2)|\nabla z_n|^2\,dx$$
$$+\left(\frac{1}{\alpha}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N}V(x)\,|z_n|^\alpha\;dx-\lambda\left(\frac{1}{r}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N}\rho(x)\,|z_n|^r\;dx$$
$$+\int_{\mathbb{R}^N}\left(\frac{1}{\theta}h(x,z_n)z_n-H(x,z_n)\right)dx$$
$$\geq \frac{1}{2}\min\Big\{\left(1-\frac{\gamma\mu}{\alpha}\right),\left(\frac{1}{\alpha}-\frac{1}{\theta}\right)\Big\}\left(\int_{\mathbb{R}^N}\varphi(|\nabla z_n|^2)\,dx+\int_{\mathbb{R}^N}V(x)\,|z_n|^\alpha\;dx\right)$$
$$-\lambda\left(\frac{1}{r}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N}\rho(x)\,|z_n|^r\;dx-\int_{\mathbb{R}^N}\left(H(x,z_n)-\frac{1}{\theta}h(x,z_n)z_n\right)dx$$

$$
\begin{aligned}
&\geq \frac{\tilde{c}}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\theta}\Big)\Big\}\left(\max\Big\{\int_{\Lambda^c_{\nabla z_n}}|\nabla z_n|^q\,dx,\int_{\Lambda_{\nabla z_n}}|\nabla z_n|^p\,dx\Big\}\right.\\
&\qquad\left.+\int_{\mathbb{R}^N}V(x)\left|z_n\right|^{\alpha}\,dx\right)-\lambda\left(\frac{1}{r}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N}\rho(x)\left|z_n\right|^{r}\,dx\\
&\qquad-\int_{\mathbb{R}^N}\left(H(x,z_n)-\frac{1}{\theta}h(x,z_n)z_n\right)\,dx\\
&\geq \frac{\tilde{c}}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\theta}\Big)\Big\}\left(\min\Big\{\|\nabla z_n\|^q_{L^{p,q}(\mathbb{R}^N)},\|\nabla z_n\|^p_{L^{p,q}(\mathbb{R}^N)}\Big\}\right.\\
&\qquad\left.+\|z_n\|^{\alpha}_{L^{\alpha}(V,\mathbb{R}^N)}\right)-\lambda\left(\frac{1}{r}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N}\rho(x)\left|z_n\right|^{r}\,dx\\
&\qquad-\int_{\mathbb{R}^N}\left(H(x,z_n)-\frac{1}{\theta}h(x,z_n)z_n\right)\,dx\\
&\geq \frac{\tilde{c}}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\theta}\Big)\Big\}\|z_n\|^{\min\{p,\alpha\}}_{\mathcal{W}_V}\\
&\quad-\lambda\left(\frac{1}{r}-\frac{1}{\theta}\right)\|\rho\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|z_n\|^r_{L^{\gamma_0}(\mathbb{R}^N)}-\int_{\mathbb{R}^N}\left(H(x,z_n)-\frac{1}{\theta}h(x,z_n)z_n\right)\,dx,
\end{aligned}
$$

where θ is a positive constant from (H3). By condition (H3), we have

$$
\begin{aligned}
&\frac{\tilde{c}}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\theta}\Big)\Big\}\|z_n\|^{\min\{p,\alpha\}}_{\mathcal{W}_V}\\
&\qquad\qquad-\lambda\left(\frac{1}{r}-\frac{1}{\theta}\right)\|\rho\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|z_n\|^r_{L^{\gamma_0}(\mathbb{R}^N)}\leq K+K\|z_n\|_{\mathcal{W}_V}.
\end{aligned}
$$

Since $\min\{p,\alpha\}>r>1$ this is a contradiction. Hence the sequence $\{z_n\}$ is bounded in $\mathcal{W}_V$ and thus $\{z_n\}$ has a weakly convergent subsequence in $\mathcal{W}_V$. Without loss of generality, we suppose that

$$z_n \rightharpoonup z_0 \text{ in } \mathcal{W}_V \text{ as } n\to\infty.$$

By Lemma 2.8, we infer that Ψ'_λ is a compact operator, and so $\Psi'_\lambda(z_n)\to\Psi'_\lambda(z_0)$ in $\mathcal{W}_V^*$ as $n\to\infty$. Let us define the functional $\Phi':\mathcal{W}_V\to\mathcal{W}_V^*$ by

$$\langle\Phi'(z),w\rangle=\int_{\mathbb{R}^N}\varphi'(|\nabla z|^2)\nabla z\cdot\nabla w\,dx+\int_{\mathbb{R}^N}V(x)\left|z\right|^{\alpha-2}zw\,dx$$

for any $z,w\in\mathcal{W}_V$. Since $\mathcal{E}'_\lambda(z_n)\to 0$ as $n\to\infty$, we know that

$$\langle\mathcal{E}'_\lambda(z_n),z_n-z_0\rangle\to 0 \text{ and } \langle\mathcal{E}'_\lambda(z_0),z_n-z_0\rangle\to 0,$$

and thus

$$\langle\mathcal{E}'_\lambda(z_n)-\mathcal{E}'_\lambda(z_0),z_n-z_0\rangle\to 0$$

as $n\to\infty$. From this, we have

$$
\begin{aligned}
&\langle\Phi'(z_n)-\Phi'(z_0),z_n-z_0\rangle\\
&=\lambda\langle\Psi'(z_n)-\Psi'(z_0),z_n-z_0\rangle+\langle\mathcal{E}'_\lambda(z_n)-\mathcal{E}'_\lambda(z_0),z_n-z_0\rangle\to 0,
\end{aligned}
$$

namely, $\langle \Phi'(z_n), z_n - z_0\rangle \to 0$ as $n \to \infty$. From modifications of the proof of Lemma 3.4 in [36] it is easy to show that Φ' is a mapping of type (S_+). Since $\mathcal{W}_V$ is reflexive, we infer that

$$z_n \to z_0 \text{ in } \mathcal{W}_V \text{ as } n \to \infty.$$

This completes the proof. □

The boundedness of solutions of the problem (1.1) follows upon the same arguments as in Proposition 1 in [34].

Proposition 2.12. *Let* (V), (B1)–(B2) *and* (H1)–(H2) *hold. If z is a weak solution of the problem* (1.1)*, then $z \in L^\infty(\mathbb{R}^N)$ and there exist positive constants C, η independent of z such that*

$$\|z\|_{L^\infty(\mathbb{R}^N)} \le C\|z\|^{\eta}_{L^\gamma(\mathbb{R}^N)}.$$

We are now in a position to state our first main result.

Theorem 2.13. *Let* (V), (B1)–(B2) *and* (H1)–(H3) *hold. Then there exists a positive constant λ^* such that for any $\lambda \in (0,\lambda^*)$, the problem* (1.1) *possesses at least two nontrivial different solutions in $\mathcal{W}_V$ which belong to L^∞-space.*

Proof. According to Lemmas 2.9 and 2.11, there is a positive real number λ^* such that for all $\lambda \in (0,\lambda^*)$, $\mathcal{E}_\lambda$ assures the (PS)-condition and the geometric conditions in mountain pass theorem. By applying mountain pass theorem, we derive that there is a critical point $z_0 \in \mathcal{W}_V$ of $\mathcal{E}_\lambda$ with $\mathcal{E}_\lambda(z_0) = \overline{c} > 0 = \mathcal{E}_\lambda(0)$. Hence there exists a non-trivial weak solution of the problem (1.1). In view of Lemma 2.9, for a fixed $\lambda \in (0,\lambda^*)$, we can choose positive constants R and $0 < \delta < 1$ such that $\mathcal{E}_\lambda(z) \ge R > 0$ for all $z \in \mathcal{W}_V$ with $\|z\|_{\mathcal{W}_V} = \delta$. Let us denote $c := \inf_{z\in \bar{B}_\delta} \mathcal{E}_\lambda(z)$ where $B_\delta := \{z \in \mathcal{W}_V : \|z\|_{\mathcal{W}_V} < \delta\}$ with a boundary ∂B_δ. Then taking (2.4) and Lemma 2.9(3) into account, we have $-\infty < c < 0$. Putting $0 < \epsilon < \inf_{z\in\partial B_\delta} \mathcal{E}_\lambda(z) - c$, by Theorem 1.1 in [18] (see also [29]), we can find $z_\epsilon \in \overline{B}_\delta$ such that there are the well-known useful inequalities

$$\begin{cases} \mathcal{E}_\lambda(z_\epsilon) \le c + \epsilon, \\ \mathcal{E}_\lambda(z_\epsilon) < \mathcal{E}_\lambda(z) + \epsilon\|z - z_\epsilon\|_{\mathcal{W}_V} \quad \text{for all} \quad z \in \overline{B}_\delta \quad z \ne z_\epsilon. \end{cases} \tag{2.6}$$

This says that $z_\epsilon \in B_\delta$ since $\mathcal{E}_\lambda(z_\epsilon) \le c + \epsilon < \inf_{z\in\partial B_\delta} \mathcal{E}_\lambda(z)$. From these facts we know that z_ϵ is a local minimum of $\tilde{\mathcal{E}_\lambda}(z) = \mathcal{E}_\lambda(z) + \epsilon\|z - z_\epsilon\|_{\mathcal{W}_V}$. Now by taking $z = z_\epsilon + tw$ for $w \in B_1$ and $t > 0$ small enough, from (2.6), we deduce

$$0 \le \frac{\tilde{\mathcal{E}_\lambda}(z_\epsilon + tw) - \tilde{\mathcal{E}_\lambda}(z_\epsilon)}{t} = \frac{\mathcal{E}_\lambda(z_\epsilon + tw) - \mathcal{E}_\lambda(z_\epsilon)}{t} + \epsilon\|w\|_{\mathcal{W}_V}.$$

Therefore, letting $t \to 0+$, we get

$$\langle \mathcal{E}'_\lambda(z_\epsilon), w\rangle + \epsilon\|w\|_{\mathcal{W}_V} \ge 0.$$

Replacing w by $-w$ in the argument above, we have

$$-\langle \mathcal{E}'_\lambda(z_\epsilon), w\rangle + \epsilon\|w\|_{\mathcal{W}_V} \ge 0.$$

Thus, one has

$$|\langle \mathcal{E}'_\lambda(z_\epsilon), w\rangle| \le \epsilon \|w\|_{\mathcal{W}_V}$$

for any $w \in \overline{B}_1$. Hence we know

$$\|\mathcal{E}'_\lambda(z_\epsilon)\|_{\mathcal{W}_V^*} \le \epsilon. \tag{2.7}$$

Using (2.6) and (2.7), we can choose a sequence $\{z_n\} \subset B_\delta$ such that

$$\begin{cases} \mathcal{E}_\lambda(z_n) \to c \quad \text{as} \quad n \to \infty, \\ \|\mathcal{E}'_\lambda(z_n)\|_{\mathcal{W}_V^*} \to 0 \quad \text{as} \quad n \to \infty. \end{cases} \tag{2.8}$$

Thus, $\{z_n\}$ is a bounded (PS)-sequence in the reflexive Banach space $\mathcal{W}_V$. Since $\mathcal{E}_\lambda$ satisfies the (PS)-condition, $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ such that $\{z_{n_k}\} \to z_1$ in $\mathcal{W}_V$ as $k \to \infty$. This together with (2.8) yields that $\mathcal{E}_\lambda(z_1) = c$ and $\mathcal{E}'_\lambda(z_1) = 0$. Hence z_1 is a nontrivial solution of the given problem with $\mathcal{E}_\lambda(z_1) < 0$ which differs from z_0. In view of Proposition 2.12, these solutions belong to L^∞-space. This completes the proof. □

From now on we will obtain the existence of multiple solutions for (1.1) when (H3) is replaced with the following condition:

(H7) $\alpha \le p$ and there exist $\nu > \alpha$, $\varrho \ge 0$ and $M > 0$ such that

$$h(x,t)t - \nu H(x,t) \ge -\varrho\, |t|^\alpha - \beta(x)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| \ge M$ and for some $\beta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\beta(x) \ge 0$.

This condition originally comes from the works of B. T. K. Oanh and D. N. Phuong [38].

Definition 2.14. Let X be a real Banach space. We say that $\mathcal{E}_\lambda$ satisfies the Cerami condition ((C)-condition for short) in X, if any (C)-sequence $\{z_n\}_n \subset X$, i.e., $\{\mathcal{E}_\lambda(z_n)\}$ is bounded and $\|\mathcal{E}'_\lambda(z_n)\|_{X^*}(1 + \|z_n\|_X) \to 0$ as $n \to \infty$, has a convergent subsequence in X.

Lemma 2.15. *Assume that* (V), (B1)–(B2), (H1)–(H2), (H5) *and* (H7) *hold. Then, the functional $\mathcal{E}_\lambda$ satisfies the (C)-condition for any $\lambda > 0$.*

Proof. Let $\{z_n\}$ be a (C)-sequence in $\mathcal{W}_V$, *i.e.*, $\sup_{n\in\mathbb{N}} |\mathcal{E}_\lambda(z_n)| \le \mathcal{K}_1$ and $\langle \mathcal{E}'_\lambda(z_n), z_n\rangle = o(1) \to 0$, as $n \to \infty$, and $\mathcal{K}_1$ is a positive constant. As in the proof of Theorem 2.11, it is sufficient to prove that $\{z_n\}$ is bounded in $\mathcal{W}_V$. To this end, arguing by contradiction, we assume that $\|z_n\|_{\mathcal{W}_V} > 1$ and $\|z_n\|_{\mathcal{W}_V} \to \infty$ as $n \to \infty$, and a sequence $\{y_n\}$ is defined by $y_n = z_n/\|z_n\|_{\mathcal{W}_V}$. Then, up to a subsequence, still denoted by $\{y_n\}$, we get $y_n \rightharpoonup y_0$ in $\mathcal{W}_V$ as $n \to \infty$, and due to Lemma 2.5,

$$y_n \to y_0 \text{ a.e. in } \mathbb{R}^N, \quad y_n \to y_0 \text{ in } L^s(\mathbb{R}^N) \tag{2.9}$$

as $n \to \infty$, for any s with $\alpha \le s < p^*$. Notice that $V(x) \to +\infty$ as $|x| \to \infty$, then

$$
\begin{aligned}
(2.10)\qquad &\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^\alpha\,dx - C_8\int_{|z_n|\le M}(|z_n|^\alpha+\sigma(x)\,|z_n|^\gamma)\,dx\\
&\ge \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^\alpha\,dx - \mathcal{K}_0,
\end{aligned}
$$

where $C_8, \mathcal{K}_0$ are some positive constants. Indeed, we know that

$$
\begin{aligned}
&\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^\alpha\,dx - C_8\int_{|z_n|\le M}(|z_n|^\alpha+\sigma(x)\,|z_n|^\gamma)\,dx\\
&\ge \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^\alpha\,dx + \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{|z_n|\le 1} V(x)\,|z_n|^\alpha\,dx\\
&\quad - C_8\int_{|z_n|\le 1}(|z_n|^\alpha+\sigma(x)\,|z_n|^\gamma)\,dx - C_8\int_{1<|z_n|\le M}(|z_n|^\alpha+\sigma(x)\,|z_n|^\gamma)\,dx\\
&\ge \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^\alpha\,dx + \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{|z_n|\le 1} V(x)\,|z_n|^\alpha\,dx\\
&\quad - C_8(1+\|\sigma\|_\infty)\int_{|z_n|\le 1}|z_n|^\alpha\,dx - \widetilde{C}_8.
\end{aligned}
$$

Since $\text{meas}\{x \in \mathbb{R}^N : |z_n| > 1\} < \infty$, we know $\{x \in \mathbb{R}^N : |z_n| > 1\} = A \cup N$ where A is a bounded set and N is of measure zero. Without loss of generality, suppose that there exists $B_r \subseteq \mathbb{R}^N$ such that $\{x \in \mathbb{R}^N : |z_n| > 1\} \subset B_r$. Since $V(x) \to +\infty$ as $|x| \to \infty$, there is $r_0 > 0$ such that $|x| \ge r_0 > r$ implies $V(x) \ge 2C_8(1+\|\sigma\|_\infty)\frac{\alpha\nu}{\nu-\alpha}$. Hence one has

$$
\begin{aligned}
&\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^\alpha\,dx - C_8\int_{|z_n|\le M}(|z_n|^\alpha+\sigma(x)\,|z_n|^\gamma)\,dx\\
&\ge \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^\alpha\,dx + \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\{|z_n|\le 1\}\cap B_{r_0}^c} V(x)\,|z_n|^\alpha\,dx\\
&\quad + \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\{|z_n|\le 1\}\cap B_{r_0}} V(x)\,|z_n|^\alpha\,dx\\
&\quad - C_8(1+\|\sigma\|_\infty)\int_{\{|z_n|\le 1\}\cap B_{r_0}^c}|z_n|^\alpha\,dx\\
&\quad - C_8(1+\|\sigma\|_\infty)\int_{\{|z_n|\le 1\}\cap B_{r_0}}|z_n|^\alpha\,dx - \widetilde{C}_8\\
&\ge \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^\alpha\,dx + \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\{|z_n|\le 1\}\cap B_{r_0}^c} V(x)\,|z_n|^\alpha\,dx\\
&\quad - C_8(1+\|\sigma\|_\infty)\int_{\{|z_n|\le 1\}\cap B_{r_0}^c}|z_n|^\alpha\,dx - \mathcal{K}_0
\end{aligned}
$$

$$\geq \frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^{\alpha}\;dx-\mathcal{K}_0,$$

as claimed. Combining (2.10) with (A2), (A3) and (H7), one has

$$\begin{aligned}
&\mathcal{K}_1+o(1)\\
\geq\ & \mathcal{E}_\lambda(z_n)-\frac{1}{\nu}\,\langle \mathcal{E}'_\lambda(z_n),z_n\rangle\\
\geq\ & \frac{1}{2}\int_{\mathbb{R}^N}\varphi(|\nabla z_n|^2)\,dx-\frac{1}{\nu}\int_{\mathbb{R}^N}\varphi'(|\nabla z_n|^2)|\nabla z_n|^2\,dx\\
&+\left(\frac{1}{\alpha}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} V(x)\,|z_n|^{\alpha}\;dx-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\rho(x)|z_n|^r\,dx\\
&+\int_{\mathbb{R}^N}\left(\frac{1}{\nu}h(x,z_n)z_n-H(x,z_n)\right)dx\\
\geq\ & \frac{1}{2}\int_{\mathbb{R}^N}\varphi(|\nabla z_n|^2)\,dx-\frac{1}{\nu}\int_{\mathbb{R}^N}\varphi'(|\nabla z_n|^2)|\nabla z_n|^2\,dx\\
&+\left(\frac{1}{\alpha}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N} V(x)\,|z_n|^{\alpha}\;dx-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\rho(x)|z_n|^r\,dx\\
&+\int_{|z_n|>M}\left(\frac{1}{\nu}h(x,z_n)z_n-H(x,z_n)\right)dx\\
&-C_8\int_{|z_n|\leq M}(|z_n|^{\alpha}+\sigma(x)\,|z_n|^{\gamma})\,dx\\
\geq\ & \frac{1}{2}\int_{\mathbb{R}^N}\varphi(|\nabla z_n|^2)\,dx-\frac{1}{\alpha}\int_{\mathbb{R}^N}\varphi'(|\nabla z_n|^2)|\nabla z_n|^2\,dx\\
&+\frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^{\alpha}\;dx-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\rho(x)|z_n|^r\,dx\\
&-\frac{1}{\nu}\int_{\mathbb{R}^N}(\varrho\,|z_n|^{\alpha}+\beta(x))\;dx-\mathcal{K}_0\\
\geq\ & \frac{1}{2}\left(1-\frac{\gamma\mu}{\alpha}\right)\int_{\mathbb{R}^N}\varphi(|\nabla z_n|^2)\,dx+\frac{1}{2}\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\int_{\mathbb{R}^N} V(x)\,|z_n|^{\alpha}\;dx\\
&-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\rho(x)|z_n|^r\,dx-\frac{1}{\nu}\int_{\mathbb{R}^N}(\varrho\,|z_n|^{\alpha}+\beta(x))\;dx-\mathcal{K}_0\\
\geq\ & \frac{1}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\Big\}\left(\int_{\mathbb{R}^N}\varphi(|\nabla z_n|^2)\,dx+\int_{\mathbb{R}^N} V(x)\,|z_n|^{\alpha}\;dx\right)\\
&-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\rho(x)|z_n|^r\,dx-\frac{1}{\nu}\int_{\mathbb{R}^N}(\varrho\,|z_n|^{\alpha}+\beta(x))\;dx-\mathcal{K}_0\\
\geq\ & \frac{\tilde{c}}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\Big\}\Big(\min\Big\{\|\nabla z_n\|^q_{L^{p,q}(\mathbb{R}^N)},\|\nabla z_n\|^p_{L^{p,q}(\mathbb{R}^N)}\Big\}\\
&+\int_{\mathbb{R}^N} V(x)\,|z_n|^{\alpha}\;dx\Big)-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\int_{\mathbb{R}^N}\rho(x)|z_n|^r\,dx
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\nu}\int_{\mathbb{R}^N}\left(\varrho\,|z_n|^\alpha+\beta(x)\right)\,dx-\mathcal{K}_0\\
&\geq \frac{C_9}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\Big\}\|z_n\|_{\mathcal{W}_V}^\alpha\\
&\quad-\lambda\left(\frac{1}{r}-\frac{1}{\nu}\right)\|\rho\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|z_n\|_{L^{\gamma_0}(\mathbb{R}^N)}^r-\frac{\varrho}{\nu}\|z_n\|_{L^\alpha(\mathbb{R}^N)}^\alpha-\frac{1}{\nu}\|\beta\|_{L^1(\mathbb{R}^N)}-\mathcal{K}_0.
\end{aligned}
$$

Hence we know that

$$
\begin{aligned}
&\mathcal{K}_1+o(1)+\lambda\left(\frac{1}{r}+\frac{1}{\nu}\right)\|\rho\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|z_n\|_{L^{\gamma_0}(\mathbb{R}^N)}^r\\
&\quad+\frac{\varrho}{\nu}\|z_n\|_{L^\alpha(\mathbb{R}^N)}^\alpha+\frac{1}{\nu}\|\beta\|_{L^1(\mathbb{R}^N)}+\mathcal{K}_0\\
&\geq \frac{C_9}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\Big\}\|z_n\|_{\mathcal{W}_V}^\alpha.
\end{aligned}
$$

Dividing this by $\frac{C_9}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\Big\}\|z_n\|_{\mathcal{W}_V}^\alpha$, and then taking the limit supremum of this inequality as $n\to\infty$, we have

$$
\begin{aligned}
1&\leq\frac{\varrho}{\nu\frac{C_9}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\Big\}}\limsup_{n\to\infty}\|y_n\|_{L^\alpha(\mathbb{R}^N)}^\alpha\\
&=\frac{\varrho}{\nu\frac{C_9}{2}\min\Big\{\Big(1-\frac{\gamma\mu}{\alpha}\Big),\Big(\frac{1}{\alpha}-\frac{1}{\nu}\Big)\Big\}}\|y_0\|_{L^\alpha(\mathbb{R}^N)}^\alpha. \qquad (2.11)
\end{aligned}
$$

Hence, it follows from (2.11) that $y_0\neq 0$.

By the assumption (A2) and Lemma 2.6, we have

$$
\begin{aligned}
\mathcal{E}_\lambda(z_n)&=\frac{1}{2}\int_{\mathbb{R}^N}\varphi(|\nabla z_n|^2)\,dx+\frac{1}{\alpha}\int_{\mathbb{R}^N}V(x)|z_n|^\alpha\,dx\\
&\quad-\frac{\lambda}{r}\int_{\mathbb{R}^N}\rho(x)|z_n|^r\,dx-\int_{\mathbb{R}^N}H(x,z_n)\,dx\\
&\geq\frac{1}{2}\Big(\tilde{c}\int_{\Lambda^c_{\nabla z_n}}|\nabla z_n|^q\,dx+\tilde{c}\int_{\Lambda_{\nabla z_n}}|\nabla z_n|^p\,dx\Big)+\frac{1}{\alpha}\|z_n\|_{L^\alpha(V,\mathbb{R}^N)}^\alpha\\
&\quad-\frac{\lambda}{r}\|\rho\|_{L^{\frac{\gamma_0}{\gamma_0-r}}(\mathbb{R}^N)}\|z_n\|_{L^{\gamma_0}(\mathbb{R}^N)}^r-\int_{\mathbb{R}^N}H(x,z_n)\,dx\\
&\geq\frac{\mathcal{C}}{2}\min\Big\{\|\nabla z_n\|_{L^{p,q}(\mathbb{R}^N)}^q,\|\nabla z_n\|_{L^{p,q}(\mathbb{R}^N)}^p\Big\}+\frac{1}{\alpha}\|z_n\|_{L^\alpha(V,\mathbb{R}^N)}^\alpha\\
&\quad-C_{10}\frac{\lambda}{r}\|z_n\|_{\mathcal{W}_V}^r-\int_{\mathbb{R}^N}H(x,z_n)\,dx\\
&\geq\min\left\{\frac{\mathcal{C}}{2},\frac{1}{\alpha}\right\}\|z_n\|_{\mathcal{W}_V}^\alpha-C_{10}\frac{\lambda}{r}\|z_n\|_{\mathcal{W}_V}^r-\int_{\mathbb{R}^N}H(x,z_n)\,dx. \qquad (2.12)
\end{aligned}
$$

Since $\mathcal{E}_\lambda(z_n)\leq\mathcal{K}_1$ for all $n\in\mathbb{N}$ and $\|z_n\|_{\mathcal{W}_V}\to\infty$ as $n\to\infty$, we assert that

$$
\int_{\mathbb{R}^N}H(x,z_n)\,dx\geq\min\left\{\frac{\mathcal{C}}{2},\frac{1}{\alpha}\right\}\|z_n\|_{\mathcal{W}_V}^\alpha-C_{10}\frac{\lambda}{r}\|z_n\|_{\mathcal{W}_V}^r-\mathcal{E}_\lambda(z_n)\to\infty \qquad (2.13)
$$

as $n \to \infty$, with $r < \alpha$. Observe that

$$\begin{aligned}\mathcal{E}_\lambda(z_n) &= \frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} V(x)|z_n|^\alpha\,dx \\ &\quad - \frac{\lambda}{r}\int_{\mathbb{R}^N} \rho(x)|z_n|^r\,dx - \int_{\mathbb{R}^N} H(x,z_n)\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} V(x)|z_n|^\alpha\,dx - \int_{\mathbb{R}^N} H(x,z_n)\,dx.\end{aligned}$$

And so,

$$\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} V(x)|z_n|^\alpha\,dx \ge \mathcal{E}_\lambda(z_n) + \int_{\mathbb{R}^N} H(x,z_n)\,dx. \tag{2.14}$$

Owing to assumption (H5), we can choose a positive constant $\delta > 1$ such that $H(x,t) > |t|^q$ for all $x \in \mathbb{R}^N$ and $|t| > \delta$. Taking (H1) and (H2) into account, we get $|H(x,t)| \le \hat{\mathcal{K}}$ for all $(x,t) \in \mathbb{R}^N \times [-t_0, t_0]$ for a constant $\hat{\mathcal{K}} > 0$. Therefore, there exists a real number $\mathcal{K}_1$ such that $H(x,t) \ge \mathcal{K}_1$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and thus

$$\frac{H(x,z_n) - \mathcal{K}_1}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}} \ge 0 \tag{2.15}$$

for all $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$. Set $\Omega_1 = \{x \in \mathbb{R}^N : y_0(x) \neq 0\}$. By convergence (2.9), we know that $|z_n(x)| = |y_n(x)|\,\|z_n\|_{\mathcal{W}_V} \to \infty$ as $n \to \infty$ for all $x \in \Omega_1$. Thus, it follows from assumption (H5) and Lemma 2.6(iv), for all $x \in \Omega_1$,

$$\begin{aligned}&\lim_{n\to\infty} \frac{H(x,z_n)}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}} \\ &\ge \lim_{n\to\infty} \frac{H(x,z_n)}{\frac{\tilde{C}}{2}\int_{\wedge^c_{\nabla(z_n)}} |\nabla z_n|^q\,dx + \frac{\tilde{C}}{2}\int_{\wedge_{\nabla(z_n)}} |\nabla z_n|^p\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}} \\ &\ge \lim_{n\to\infty} \frac{H(x,z_n)}{\tilde{C}(\|\nabla z_n\|^q_{L^q(\wedge^c_{\nabla(z_n)})} + \|\nabla z_n\|^p_{L^p(\wedge_{\nabla(z_n)})}) + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}} \\ &\ge \lim_{n\to\infty} \frac{H(x,z_n)}{\tilde{C}\big((\|\nabla z_n\|^q_{L^{p,q}(\mathbb{R}^N)} + C_{11}) + C_{12}\|\nabla z_n\|^p_{L^{p,q}(\mathbb{R}^N)}\big) + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}} \\ &\ge \lim_{n\to\infty} \frac{H(x,z_n)}{\tilde{C}C_{13}\big(\|\nabla z_n\|_{L^{p,q}(\mathbb{R}^N)} + 1\big)^q + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}} \\ &\ge \lim_{n\to\infty} \frac{H(x,z_n)}{\max\left\{\tilde{C}C_{14}, \frac{1}{\alpha}\right\}\|z_n\|^q_{\mathcal{W}_V}} \\ &= \lim_{n\to\infty} \frac{H(x,z_n)}{\max\left\{\tilde{C}C_{14}, \frac{1}{\alpha}\right\}|z_n(x)|^q}\,|y_n(x)|^q = \infty,\end{aligned} \tag{2.16}$$

where C_i is a positive constant for $i = 11, \dots, 14$. Hence, we obtain that $\text{meas}(\Omega_1) = 0$. Indeed, if $\text{meas}(\Omega_1) \neq 0$, using relations (2.13)–(2.16) and the Fatou lemma, we get that

$$\begin{aligned}
1 &= \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} H(x,z_n)\,dx}{\int_{\mathbb{R}^N} H(x,z_n)\,dx + \mathcal{E}_\lambda(z_n)} \\
&\geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{H(x,z_n)}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}}\,dx \\
&\geq \liminf_{n\to\infty} \int_{\Omega_1} \frac{H(x,z_n)}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}}\,dx \\
&\qquad - \limsup_{n\to\infty} \int_{\Omega_1} \frac{\mathcal{K}_1}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}}\,dx \\
&= \liminf_{n\to\infty} \int_{\Omega_1} \frac{H(x,z_n)-\mathcal{K}_1}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}}\,dx \\
&\geq \int_{\Omega_1} \liminf_{n\to\infty} \frac{H(x,z_n)-\mathcal{K}_1}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}}\,dx \\
&= \int_{\Omega_1} \liminf_{n\to\infty} \frac{H(x,z_n)}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}}\,dx \\
&\qquad - \int_{\Omega_1} \limsup_{n\to\infty} \frac{\mathcal{K}_1}{\frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla z_n|^2)\,dx + \frac{1}{\alpha}\|z_n\|^\alpha_{L^\alpha(V,\mathbb{R}^N)}}\,dx = \infty, \qquad (2.17)
\end{aligned}$$

which is impossible. Thus, $y_0(x) = 0$ for almost all $x \in \mathbb{R}^N$. Therefore, we conclude a contradiction. Thus, $\{z_n\}$ is bounded in $\mathcal{W}_V$. This completes the proof. □

The following consequence which is a variant of the Ekeland variational principle is essential in obtaining our second main result.

Corollary 2.16 ([8, 35])**.** *Let X be a Banach space and x_0 be a fixed point of X. Suppose that $h : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous function, not identically $+\infty$, bounded from below. Then, for every $\varepsilon > 0$ and $y \in X$ such that*

$$h(y) < \inf_X h + \varepsilon,$$

and every $\lambda > 0$, there exists some point $z \in X$ such that

$$h(z) \leq h(y), \quad \|z - x_0\|_X \leq (1 + \|y\|_X)(e^\lambda - 1),$$

and

$$h(x) \geq h(z) - \frac{\varepsilon}{\lambda(1 + \|z\|_X)}\|x - z\|_X \quad \text{for all} \quad x \in X.$$

We are now in a position to state our second main result.

Theorem 2.17. *Let* (V), (B1)–(B2), (H1)–(H2) *and* (H4)–(H7) *hold. Then there exists a positive constant* λ^* *such that for any* $\lambda \in (0, \lambda^*)$, *the problem* (1.1) *possesses at least two nontrivial different solutions in* $\mathcal{W}_V$ *which belong to* L^∞*-space.*

Proof. To apply Lemma 2.9, we must show the condition (2). By the assumptions (H2) and (H4)–(H6), for any $M_0 > 0$, there exist positive constants C_{15} and $C_{16}(M)$ such that

$$H(x,t) \geq M_0 |t|^q - C_{15}|t|^\alpha - C_{16}(M)\sigma(x) \tag{2.18}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ where σ comes from (H2). Take $w(>0) \in \mathcal{W}_V$. Then, for large enough $t > 0$, the relation (2.18) and assumption (A2) imply that

$$\begin{aligned}
\mathcal{E}_\lambda(tw) &= \frac{1}{2}\int_{\mathbb{R}^N} \varphi(|\nabla tw|^2)\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} V(x)|tw|^\alpha\,dx \\
&\quad - \frac{\lambda}{r}\int_{\mathbb{R}^N} \rho(x)|tw|^r\,dx - \int_{\mathbb{R}^N} H(x,tw)\,dx \\
&\leq \frac{\tilde{C}}{2}\int_{\Lambda^c_{\nabla(tw)}} |\nabla tw|^q\,dx + \frac{\tilde{C}}{2}\int_{\Lambda_{\nabla(tw)}} |\nabla tw|^p\,dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} V(x)|tw|^\alpha\,dx \\
&\quad - \frac{\lambda}{r}\int_{\mathbb{R}^N} \rho(x)|tw|^r\,dx - \int_{\mathbb{R}^N} H(x,tw)\,dx \\
&\leq \frac{\tilde{C}t^q}{2}\left(\int_{\mathbb{R}^N} |\nabla w|^q\,dx + \int_{\mathbb{R}^N} |\nabla w|^p\,dx\right) + \frac{t^\alpha}{\alpha}\int_{\mathbb{R}^N} V(x)|w|^\alpha\,dx \\
&\quad - \frac{\lambda t^r}{r}\int_{\mathbb{R}^N} \rho(x)|w|^r\,dx - M_0 t^q \int_{\mathbb{R}^N} |w|^q\,dx \\
&\quad + C_{15}t^\alpha \int_{\mathbb{R}^N} |w|^\alpha dx + C_{17} < 0
\end{aligned}$$

for sufficiently large M_0 and for positive constant C_{17}. We deduce that $\mathcal{E}_\lambda(tw) \to -\infty$ as $t \to \infty$. Hence the functional $\mathcal{E}_\lambda$ is unbounded from below.

Thanks to Lemmas 2.9 and 2.15, there is a positive real number λ^* such that for all $\lambda \in (0, \lambda^*)$, $\mathcal{E}_\lambda$ assures the (C)-condition and the geometric conditions in mountain pass theorem. By invoking mountain pass theorem, we derive that there is a critical point $z_0 \in \mathcal{W}_V$ of $\mathcal{E}_\lambda$ with $\mathcal{E}_\lambda(z_0) = \overline{c} > 0 = \mathcal{E}_\lambda(0)$. Hence there exists a non-trivial weak solution of the problem (1.1). According to Lemma 2.9, for a fixed $\lambda \in (0, \lambda^*)$, we can choose positive constants R and $0 < \delta < 1$ such that $\mathcal{E}_\lambda(z) \geq R > 0$ for all $z \in \mathcal{W}_V$ with $\|z\|_{\mathcal{W}_V} = \delta$. Let us denote $c := \inf_{z \in \bar{B}_\delta} \mathcal{E}_\lambda(z)$ where $B_\delta := \{z \in \mathcal{W}_V : \|z\|_{\mathcal{W}_V} < \delta\}$ with a boundary ∂B_δ. Then by (2.4) and Lemma 2.9(3), we have $-\infty < c < 0$. Putting $0 < \epsilon < \inf_{z \in \partial B_\delta} \mathcal{E}_\lambda(z) - c$, by Corollary 2.16, we can find $z_\epsilon \in \overline{B}_\delta$ such that

$$\begin{cases} \mathcal{E}_\lambda(z_\epsilon) \leq c + \epsilon \\ \mathcal{E}_\lambda(z_\epsilon) < \mathcal{E}_\lambda(z) + \frac{\epsilon}{1+\|z_\epsilon\|_{\mathcal{W}_V}}\|z - z_\epsilon\|_{\mathcal{W}_V} \quad \text{for all} \quad z \in \overline{B}_\delta \quad z \neq z_\epsilon. \end{cases} \tag{2.19}$$

This implies that $z_\epsilon \in B_\delta$ since $\mathcal{E}_\lambda(z_\epsilon) \le c+\epsilon < \inf_{z\in\partial B_\delta} \mathcal{E}_\lambda(z)$. From these facts we have that z_ϵ is a local minimum of $\widetilde{\mathcal{E}}_\lambda(z) = \mathcal{E}_\lambda(z) + \frac{\epsilon}{1+\|z_\epsilon\|_{\mathcal{W}_V}}\|z-z_\epsilon\|_{\mathcal{W}_V}$. Now by taking $z = z_\epsilon + tw$ for $w \in B_1$ and sufficiently small $t>0$, from (2.19), we deduce

$$0 \le \frac{\widetilde{\mathcal{E}}_\lambda(z_\epsilon+tw)-\widetilde{\mathcal{E}}_\lambda(z_\epsilon)}{t} = \frac{\mathcal{E}_\lambda(z_\epsilon+tw)-\mathcal{E}_\lambda(z_\epsilon)}{t} + \frac{\epsilon}{1+\|z_\epsilon\|_{\mathcal{W}_V}}\|w\|_{\mathcal{W}_V}.$$

Therefore, letting $t \to 0+$, we get

$$\langle \mathcal{E}'_\lambda(z_\epsilon), w\rangle + \frac{\epsilon}{1+\|z_\epsilon\|_{\mathcal{W}_\mathcal{V}}}\|w\|_{\mathcal{W}_V} \ge 0.$$

Replacing w by $-w$ in the argument above, we have

$$-\langle \mathcal{E}'_\lambda(z_\epsilon), w\rangle + \frac{\epsilon}{1+\|z_\epsilon\|_{\mathcal{W}_V}}\|w\|_{\mathcal{W}_V} \ge 0.$$

Thus, one has

$$(1+\|z_\epsilon\|_{\mathcal{W}_V})\,|\langle \mathcal{E}'_\lambda(z_\epsilon), w\rangle| \le \epsilon\|w\|_{\mathcal{W}_V}$$

for any $w \in \overline{B}_1$. Hence we know

$$(1+\|z_\epsilon\|_{\mathcal{W}_V})\|\mathcal{E}'_\lambda(z_\epsilon)\|_{\mathcal{W}_V^*} \le \epsilon. \tag{2.20}$$

Using (2.19) and (2.20), we can choose a sequence $\{z_n\} \subset B_\delta$ such that

$$\begin{cases} \mathcal{E}_\lambda(z_n) \to c \quad \text{as} \quad n\to\infty, \\ (1+\|z_n\|_{\mathcal{W}_V})\|\mathcal{E}'_\lambda(z_n)\|_{\mathcal{W}_V^*} \to 0 \quad \text{as} \quad n \to \infty. \end{cases} \tag{2.21}$$

Thus, $\{z_n\}$ is a bounded Cerami sequence in $\mathcal{W}_V$. According to Lemma 2.15, $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ such that $\{z_{n_k}\} \to z_1$ in $\mathcal{W}_V$ as $k\to\infty$. This together with (2.21) yields that $\mathcal{E}_\lambda(z_1) = c$ and $\mathcal{E}'_\lambda(z_1) = 0$. Hence z_1 is a nontrivial solution of (1.1) with $\mathcal{E}_\lambda(z_1) < 0$ which differs from z_0. In view of Proposition 2.12, these solutions belong to L^∞-space. This completes the proof. □

3. Concluding remark

As we know, the study of quasilinear elliptic equations including nonhomogeneous operators has extensively been considered in light of the pure or applied mathematical theory to explain some concrete phenomena arising from nonlinear elasticity, fluid mechanics, generalized Newtonian fluids, plasticity theory,biophysics problems, Non-Newtonian fluids and plasma physics. In this paper, we use the variational methods to guarantee the existence of nontrivial solutions for Schrödinger type problems in case where the nonlinear term is concave-convex. As far as we are aware, the present paper is the first attempt to study the multiplicity of nontrivial weak solutions to Schrödinger type problems with the concave-convex nonlinearity in these situations. Also, we address to the readers for that a new research direction in strong relationship with several related applications is the study of Kirchhoff-type equations

$$\begin{cases} -\mathcal{K}\Big(\int_{\mathbb{R}^N} \varphi(|\nabla z|^2)\, dx\Big)\mathrm{div}(\varphi'(|\nabla z|^2)\nabla z) + V(x)|z|^{\alpha-2}z \\ \qquad\qquad\qquad = \lambda\rho(x)|z|^{r-2}z + h(x,z), \qquad\qquad \text{in}\, \mathbb{R}^N, \\ z(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $\mathcal{K} \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ is a Kirchhoff-type function and $\mathcal{K} : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfies the following conditions:

(M1) $\mathcal{K} \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ fulfils $\inf_{t\in\mathbb{R}_0^+} \mathcal{K}(t) \geq m_0 > 0$, where m_0 is a constant.

(M2) There is a positive constant $\theta \in [1, \frac{N}{N-ps})$ such that $\theta\mathfrak{K}(t) \geq \mathcal{K}(t)t$ for any $t \geq 0$, where $\mathfrak{K}(t) := \int_0^t \mathcal{K}(\tau)d\tau$.

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