Commun. Korean Math. Soc. **36** (2021), No. 4, pp. 651–669
https://doi.org/10.4134/CKMS.c200012
pISSN: 1225-1763 / eISSN: 2234-3024

SLICE REGULAR BESOV SPACES OF HYPERHOLOMORPHIC FUNCTIONS AND COMPOSITION OPERATORS

Sanjay Kumar and Khalid Manzoor

Abstract. In this paper, we investigate some basic results on the slice regular Besov spaces of hyperholomorphic functions on the unit ball $\mathbb{B}$. We also characterize the boundedness, compactness and find the essential norm estimates for composition operators between these spaces.

1. Introduction

The theory of slice hyperholomorphic functions has been developed systemically and have found wide range of applications, for example, in operator theroy, mathematical physics, in Schur analysis and to define some functional calculus. It is well known that there are several different types of definitions of regularity for functions in quaternions. For details on slice regular holomorphic functions one can refer to the books [6, 7].

Now, we recall some preliminaries about slice regular holomorphic functions.

Let $\mathbb{H}$ denote the noncommutative, associative, real algebra of quaternions with standard basis $\{1, i, j, k\}$, subject to the multiplication rules

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

i.e., $\mathbb{H}$ is the set of the quaternions

$$q = x_0 + x_1 i + x_2 j + x_3 k = Re(q) + Im(q)$$

with $Re(q) = x_0$ and $Im(q) = x_1 i + x_2 j + x_3 k$, where $x_l \in \mathbb{R}$ for $l = 1, 2, 3$. The conjugate of $q \in \mathbb{H}$ is then $\bar{q} = Re(q) - Im(q) = x_0 - (x_1 i + x_2 j + x_3 k)$ and its modulus is defined by $|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.

Received January 14, 2020; Revised May 31, 2021; Accepted June 10, 2021.
2010 *Mathematics Subject Classification.* 30G35, 30H10, 30H20.
Key words and phrases. Carleson measure, quaternions, Besov space, composition operators, boundedness, compactness, essential norm.
The work of the first author is supported by NBHM (DAE) (Grant No. 2/11/41/2017/R&D-II/3480).

By the symbol $\mathbb{S}$, we denote the two dimensional unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q = x_1 i + x_2 j + x_3 k : x_1^2 + x_2^2 + x_3^2 = 1\},$$

where $q^2 = -1$ for $q \in \mathbb{S}$ and for any $I \in \mathbb{S}$, we define

$$\mathbb{C}_I = \{x + Iy : x, y \in \mathbb{R}\}$$

which can be identified with a complex plane. Moreover

$$\mathbb{H} = \cup_{I \in \mathbb{S}} \mathbb{C}_I.$$

We can therefore calculate the multipicative inverse of each $q \neq 0$ as $q^{-1} = \frac{\bar{q}}{|q|^2}$. However, any $q \in \mathbb{H}$ can be expressed as $q = x + I_q y$, where $x, y \in \mathbb{R}$ and $I_q = \frac{Im(q)}{|Im(q)|}$ if $Im(q) \neq 0$, otherwise we take I_q arbitrarily such that $I_q^2 = -1$. Then I_q is an element of the unit 2-sphere of purly imaginary quaternions,

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}.$$

By $\mathbb{B}_I$, we denote the intersection $\mathbb{B} \cap \mathbb{C}_I$, where $B(0,1) = \mathbb{B} = \{q \in \mathbb{H} : |q| < 1\}$. The study of slice holomorphic functions is now an active area of research and lot of work is being done in this direction.

Definition 1.1 ([7, Definition 2.1.1])**.** Let Ω be an open set in $\mathbb{H}$. A real differentiable function $f : \Omega \to \mathbb{H}$ is said to be slice regular or slice hyperholomorphic if for every $I \in \mathbb{S}$, its restriction $f_I(x + Iy) = f(x + Iy)$ is holomorphic, i.e., it has continuous partial derivatives and satisfies

$$\frac{\partial}{\partial x} f_I(x + yI) + I \frac{\partial}{\partial y} f_I(x + yI) = 0$$

for all $x + yI \in \Omega_I$, where f_I denotes the restriction of f to $\Omega_I = \Omega \cap \mathbb{C}_I$. The set of slice regular functions on Ω denoted by $SR(\Omega)$ and the collection of all entire functions on $\mathbb{H}$ denoted by $R(\mathbb{H})$ is the right linear space on $\mathbb{H}$.

Splitting Lemma gives the relation between classical holomorphy and slice regularity.

Lemma 1.2 ([7, Lemma 2.1.4], Splitting Lemma)**.** *If $f \in SR(\Omega)$, then for any $I, J \in \mathbb{S}$, with $I \perp J$ there exist two holomorphic functions $F, G : \Omega_I = \Omega \cap \mathbb{C}_I \to \mathbb{C}_I$ such that*

$$f_I(z) = F(z) + G(z)J \quad \text{for any} \quad z = x + yI \in \Omega_I. \tag{1}$$

Definition 1.3 ([7, Definition 2.2.1])**.** Let Ω be an open set in $\mathbb{H}$. We say Ω is axially symmetric if for evey $x + yI \in \Omega$ with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, all the elements $x + y\mathbb{S} = \{x + yJ : J \in \mathbb{S}\}$ are contained in Ω and Ω is said to be slice domain (s-domain) if $\Omega \cap \mathbb{R}$ is non empty and Ω_I is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.

One of the most important properties of the slice regular functions is their Representation Formula which stated below.

Theorem 1.4 ([7, Theorem 2.2.4], Representation Formula). *Let f be a slice regular function on an axially symmetric s-domain $\Omega \in \mathbb{H}$. Let $J \in \mathbb{S}$ and let $x \pm yJ \in \Omega \cap \mathbb{C}_J$. Then*

$$f(x+yI) = \frac{1}{2}[(1-IJ)f(x+yJ)]+\frac{1}{2}[(1+IJ)f(x-yJ)] \quad \text{for any} \quad q = x+yI \in \Omega.$$

As we know pointwise product of functions does not preserve slice regularity, a new multiplication operation for regular functions is defined. In the special case of power series, the regular product (or $\star$-product) of $f(q) = \sum_{n=0}^{\infty} q^n a_n$ with $a_n = \frac{f^{(n)}(0)}{n!} \in \mathbb{H}$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$, where $b_n \in \mathbb{H}$ is given by

$$f \star g(q) = \sum_{n\geq 0} q^n \sum_{k=0}^{n} a_k b_{n-k}.$$

The notation $\star$-product coincides with the classical notation of product of series with coefficients in a ring. It is easy to check that the function $f \star g$ is slice hyperholomorphic. Let $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain and let $f, g : \Omega \to \mathbb{H}$ be slice regular functions. Let any $I, J \in \mathbb{S}$ with $I \perp J$. Then by Splitting Lemma there exit four holomorphic functions $F, G, H, K : \Omega \cap \mathbb{C}_I \to \mathbb{C}_I$ such that

$$f_I(z) = F(z) + G(z)J, \quad g_I(z) = H(z) + K(z)J \quad \text{for all} \quad z = x + yI \in \Omega_I.$$

Therefore $f_I \star g_I : \Omega \cap \mathbb{C}_I \to \mathbb{C}_I$ is defined by

$$f_I \star g_I(z) = [F(z)K(z) + G(z)\overline{(H(\bar{z}))}] + [F(z)H(z) - G(z)\overline{(K(\bar{z}))}]J. \tag{2}$$

Thus $f_I \star g_I$ is a holomorphic map and hence it admits an unique slice regular extension to Ω defined by $ext(f_I \star g_I)(q)$.

Definition 1.5 ([7]). Let $\Omega \in \mathbb{H}$ be an axially symmetric s-domain and let $f, g : \Omega \to \mathbb{H}$ be slice regular. Then the function defined by

$$f \star g(q) = ext(f_I \star g_I)(q)$$

as the extension of (2) is called the slice regular product of f and g.

2. Besov spaces

Let $\mathbb{D}$ be a unit disk in the complex plane $\mathbb{C}$ and dA denote the normalized area measure on $\mathbb{D}$. For $1 < p < \infty$, a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ is said to be in a Besov space $\mathfrak{B}_{p,\mathbb{C}}(\mathbb{D})$ if

$$\int_{\mathbb{D}} \left|(1 - |z|^2)f'(z)\right|^p d\lambda(z) < \infty,$$

where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ is the normalized area measure and Möbius invariant measure on $\mathbb{D}$. The space $\mathfrak{B}_{p,\mathbb{C}}$ is a Banach space under the norm

$$\|f\|_{\mathfrak{B}_{p,\mathbb{C}}} = |f(0)| + \left(\int_{\mathbb{D}} \left|(1 - |z|^2)f'(z)\right|^p d\lambda(z)\right)^{\frac{1}{p}}.$$

For definitions on $\mathbb{C}$-valued Besov spaces, see [17]. Next we define Besov spaces of quaternions holomorphic functions.

Definition 2.1 ([5])**.** Let $p > 1$ and let $I \in \mathbb{S}$. The quaternionic right linear space of slice regular functions f is said to be the quaternionic slice regular Besov space on the unit ball $\mathbb{B}$, if

$$\sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left|(1-|q|^2)\frac{\partial f}{\partial x_0}(q)\right|^p d\lambda_I(q) < \infty, \qquad q \in \mathbb{B}.$$

That is,

$$\mathfrak{B}_p = \{f \in SR(\mathbb{B}) : \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left|(1-|q|^2)\frac{\partial f}{\partial x_0}(q)\right|^p d\lambda_I(q) < \infty\},$$

where $d\lambda_I(q) = \frac{dA_I(q)}{(1-|q|^2)^2}$ is again the normalized differentiable of area in the plane and is Möbius invariant measure on $\mathbb{B}$. The space $\mathfrak{B}_p$ is a Banach space under the norm

$$\|f\|_{\mathfrak{B}_p} = |f(0)| + \left(\sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left|(1-|q|^2)\frac{\partial f}{\partial x_0}(q)\right|^p d\lambda_I(q)\right)^{\frac{1}{p}}.$$

By space $\mathfrak{B}_{p,I}$, $p > 1$, we mean the quaternionic right linear space of slice regular functions on the unit ball $\mathbb{B}$ such that

$$\int_{\mathbb{B}_I} \left|(1-|z|^2)Q_I[f]'(z)\right|^p d\lambda_I(z) < \infty,$$

and the norm of this space is given by

$$\|f\|_{\mathfrak{B}_{p,I}} = |f(0)| + \left(\int_{\mathbb{B}_I} \left|(1-|z|^2)Q_I[f]'(z)\right|^p d\lambda_I(z)\right)^{\frac{1}{p}},$$

where $Q_I[f]'(z) = \frac{\partial Q_I[f]}{\partial x_0}(z)$ is a holomorphic map of complex plane and $I \in \mathbb{S}$.

Remark 2.2 ([5]). Let $J \in \mathbb{S}$ be such that $J \perp I$. Then there exist holomorphic functions $f_1, f_2 : \mathbb{B}_I \to \mathbb{C}_I$ such that $Q_I[f] = f_1 + f_2 J$ and so $\frac{\partial f}{\partial x_0}(z) = f_1'(z) + f_2'(z)J$ for $z \in \mathbb{B}_I$. Then

$$|f_l'(z)|^p \leq \left|\frac{\partial f}{\partial x_0}(z)\right|^p \leq 2^{\max\{0,p-1\}} \left(|f_1'(z)|^p + |f_2'(z))|^p\right), \ l = 1, 2.$$

Also, $f \in \mathfrak{B}_{p,I}$ if and only if $f_1, f_2 \in \mathfrak{B}_{p,\mathbb{C}}$.

The proof of the following proposition is analogus to [5, Proposition 2.6].

Proposition 2.3. *Let $I \in \mathbb{S}$. Then $f \in \mathfrak{B}_{p,I}$, $p > 1$ if and only if $f \in \mathfrak{B}_p$. Moreover, the spaces $(\mathfrak{B}_{p,I}, \|.\|_{\mathfrak{B}_{p,I}})$ and $(\mathfrak{B}_p, \|.\|_{\mathfrak{B}_p})$ have equivalent norms. More precisely, one has*

$$\|f\|^p_{\mathfrak{B}_{p,I}} \leq \|f\|^p_{\mathfrak{B}_p} \leq 2^p \|f\|^p_{\mathfrak{B}_{p,I}}.$$

For all $z, w \in \mathbb{D}$, Bergman metric on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ is given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)},$$

where $\rho(z,w) = |\frac{z-w}{1-\bar{z}w}|$.

Definition 2.4 ([5])**.** For $I \in \mathbb{S}$ and all $z, w \in \mathbb{B}_I$, we define

$$\beta_I(z,w) = \frac{1}{2}\log\left(\frac{1+\frac{|z-w|}{|1-\bar{z}w|}}{1-\frac{|z-w|}{|1-\bar{z}w|}}\right).$$

Proposition 2.5. *For $1 < p, t < \infty$, with $\frac{1}{p} + \frac{1}{t} = 1$, let $f \in \mathfrak{B}_p$ and $I \in \mathbb{S}$ be fixed. Then for all $q, w \in \mathbb{B}_I$, there exists a constant $M_p > 0$ such that*

$$|f(q) - f(w)| \le 2M_p \|f\|_{\mathfrak{B}_p} \beta_I(q,w)^{\frac{1}{t}},$$

where

$$\beta_I(q,w) = \frac{1}{2}\log\left(\frac{1+\frac{|q-w|}{|1-\bar{q}w|}}{1-\frac{|q-w|}{|1-\bar{q}w|}}\right).$$

Proof. By Lemma 1.2, there exist two holomorphic functions $f_1, f_2 : \mathbb{B}_I \to \mathbb{C}_I$ such that $Q_I[f] = f_1 + f_2 J$, where $J \perp I$. Moreover, the functions $f_l \in \mathfrak{B}_{p,\mathbb{C}}$; $l = 1, 2$. Furthermore, $\|f_l\|^p_{\mathfrak{B}_{p,\mathbb{C}}} \le \|f\|^p_{\mathfrak{B}_{p,I}}$; $l = 1, 2$ and $p > 1$. Therefore, from [17, Theorem 9], it follows that for all $q, w \in \mathbb{B}_I$ in the plane $\mathbb{C}_I$, one has

$$\begin{aligned}
|f(q) - f(w)|^p &\le 2^{p-1}\left(|f_1(q) - f_1(w)|^p + |f_2(q) - f_2(w)|^p\right) \\
&\le 2^{p-1} M_p \left(\|f_1\|^p_{\mathfrak{B}_{p,\mathbb{C}}} \beta(q,w)^{\frac{p}{t}} + \|f_2\|^p_{\mathfrak{B}_{p,\mathbb{C}}} \beta(q,w)^{\frac{p}{t}}\right) \\
&\le 2^{p-1} 2 M_p \|f\|^p_{\mathfrak{B}_{p,I}} \beta_I(q,w)^{\frac{p}{t}} \\
&\le 2^p M_p \|f\|^p_{\mathfrak{B}_p} \beta_I(q,w)^{\frac{p}{t}}.
\end{aligned}$$

□

The following proposition on Besov spaces over the unit disk was proved in [17, Theorem 8] and for its proof on Bloch spaces of slice hyperholomorphic functions one can refer to [5, Theorem 2.19], so we omitted the proof.

Proposition 2.6. *Let $f \in \mathfrak{B}_p$, $p > 1$ and $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ be a sequence of quaternions such that*

$$f(q) = \sum_{n=0}^{\infty} q^n a_n \ for \ q \in \mathbb{B}.$$

Then there exists a constant $K_p > 0$ such that

$$|a_n|^p \le 2^p \frac{K_p}{n!} \|f\|^p_{\mathfrak{B}_p} \quad for\ any\ n \in \mathbb{N} \cup \{0\}.$$

Remark 2.7. Let $L^p(\mathbb{B}_I, d\lambda_I, \mathbb{H}),\ 1 \leq p < \infty$ denote the space of quaternionic valued equivalence classes of measurable functions $g : \mathbb{B}_I \to \mathbb{H}$ such that

$$\int_{\mathbb{B}_I} |g(w)|^p d\lambda_I(w) < \infty.$$

Furthermore, for any $J \in \mathbb{S}$ with $J \perp I$ and $g = g_1 + g_2 J$, where g_1, g_2 are holomorphic functions in plane $\mathbb{C}_I$. Then, $g \in L^p(\mathbb{B}_I, d\lambda_I, \mathbb{H})$ if and only if $g_l \in L^p(\mathbb{B}_I, d\lambda_I, \mathbb{C}_I),\ l = 1, 2$, the usual L^p-space of complex valued measurable functions on $\mathbb{B}_I$.

Now we define the bounded mean oscillation of the slice regular functions, see [9].

Definition 2.8. For any $z \in \mathbb{B}_I$, let $\Delta_I(z, r) = \{w \in \mathbb{B}_I : \beta_I(z, w) < r\} \subset \mathbb{B}_I$, for some $r > 0$, be the Euclidean disk. Let

$$f^*_{r,I}(z) = \frac{1}{2\pi} \int_{\Delta_I(z,r)} f(w) dA_I(w) \quad \text{ for } I \in \mathbb{S}.$$

A slice regular function f is said to be in $BMO(\mathbb{B}_I)$ if

$$\sup_{z \in \mathbb{B}_I} \frac{1}{2\pi} \int_{\Delta_I(z,r)} |f(w) - f^*_{r,I}(z)|^p dA_I(w) < \infty,$$

with norm defined by

$$\|f\|_{BMO(\mathbb{B}_I)} = \sup_{z \in \mathbb{B}_I} \left(\frac{1}{2\pi} \int_{\Delta_I(z,r)} |f(w) - f^*_{r,I}(z)|^p dA_I(w) \right)^{\frac{1}{p}}.$$

We say function $f \in BMO(\mathbb{B})$ if

$$\|f\|_{BMO(\mathbb{B})} := \sup_{I \in \mathbb{S}} \|f\|_{BMO(\mathbb{B}_I)} = \sup_{I \in \mathbb{S}} \Lambda_{r,I}(f) < \infty,$$

where

$$\Lambda_{r,I}(f)(z) = \sup_{I \in \mathbb{S}} \{|f(z) - f(w)| : w \in \Delta_I(z, r)\}.$$

The following propositions are essentially proved in [4].

Proposition 2.9. *Let $I, J \in \mathbb{S}$. Then $f \in BMO(\mathbb{B}_I)$ if and only if $f \in BMO(\mathbb{B}_J)$.*

Proof. Let $f \in SR(\mathbb{B})$ and choose $w = x + yJ \in \mathbb{B}_J$ and $z = x + yI \in \mathbb{B}_I$. Then by Representation Formula, we have

$$|f(w)| = \frac{1}{2} |(1 - JI) f(z) + (1 + JI) f(\bar{z})| \leq |f(z)| + |f(\bar{z})|.$$

Therefore

$$\begin{aligned}&\frac{1}{2\pi}\int_{\Delta_J(z,r)}|f(w)-f^*_{r,J}(z)|^p dA_J(w)\\ &\le 2^{\max\{p-1,0\}}\frac{1}{2\pi}\int_{\Delta_I(w,r)}|f(z)-f^*_{r,I}(w)|^p dA_I(z)\\ &\quad+2^{\max\{p-1,0\}}\frac{1}{2\pi}\int_{\Delta_I(w,r)}|f(\bar z)-f^*_{r,I}(\bar w)|^p dA_I(\bar z).\end{aligned}$$

On changing $\bar z\to z$ and $\bar w\to w$, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{\Delta_J(z,r)}\left|f(w)-f^*_{r,J}(z)\right|^p dA_J(w)\\ &\le 2^{\max\{p,1\}}\frac{1}{2\pi}\int_{\Delta_I(w,r)}\left|f(z)-f^*_{r,I}(w)\right|^p dA_I(z).\end{aligned}$$

Thus, we conclude that for any $f\in BMO(\mathbb{B}_I)$ implies $f\in BMO(\mathbb{B}_J)$. On interchanging the role of I and J, we get the remaining one. □

Proposition 2.10. *For $I\in\mathbb{S}$. Then $f\in BMO(\mathbb{B})$ if and only if $f\in BMO(\mathbb{B}_I)$.*

Proof. Since the direct part is obivious, so we only remains to prove the converse part. Suppose $f\in BMO(\mathbb{B}_I)$ for some arbitrary imaginary unit I in $\mathbb{S}$. Therefore by Representation Formula, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{\Delta_J(z,r)}\left|f(w)-f^*_{r,J}(z)\right|^p dA_J(w)\\ &\le 2^{p-1}\frac{1}{2\pi}\left(\int_{\Delta_I(w,r)}\left|f(z)-f^*_{r,I}(w)\right|^p dA_I(z)\right)\\ &\quad+2^{p-1}\frac{1}{2\pi}\left(\int_{\Delta_I(w,r)}\left|f(\bar z)-f^*_{r,I}(\bar w)\right|^p dA_I(\bar z)\right).\end{aligned}$$

On taking supremum over all $z\in\mathbb{B}_I$, we have

$$\begin{aligned}\|f\|_{BMO(\mathbb{B}_J)}&\le \sup_{z\in\Delta_I(w,r)}2^{p-1}\frac{1}{2\pi}\left(\int_{\Delta_I(w,r)}\left|f(z)-f^*_{r,I}(w)\right|^p dA_I(z)\right)\\ &\quad+\sup_{z\in\Delta_I(w,r)}2^{p-1}\frac{1}{2\pi}\left(\int_{\Delta_I(w,r)}\left|f(\bar z)-f^*_{r,I}(\bar w)\right|^p dA_I(\bar z)\right)\\ &\le 2^{p-1}2\|f\|_{BMO(\mathbb{B}_I)}\\ &<\infty.\end{aligned}$$

Since J is arbitrary, so we get the desired result. □

Corollary 2.11. *By previous proposition we have the following inequality*

$$\|f\|^p_{BMO(\mathbb{B}_I)}\le\|f\|^p_{BMO(\mathbb{B})}\le 2^p\|f\|^p_{BMO(\mathbb{B}_I)}.$$

Proposition 2.12. *Let $f \in SR(\mathbb{B})$. Then for $1 < p < \infty$, $f \in \mathfrak{B}_p$ if and only if $\Lambda_{r,I}(f) \in L^p(\mathbb{B}_I, d\lambda_I, \mathbb{H})$ for $I \in \mathbb{S}$.*

Proof. Suppose $f \in \mathfrak{B}_p$ implies $f \in \mathfrak{B}_{p,I}$. Let $J \in \mathbb{S}$ be such that $J \perp I$. By Splitting Lemma 1.2, we can restrict f on $\mathbb{B}_I$ with respect to J, as $Q_I[f](z) = f_1(z) + f_2(z)J$ for some holomorphic functions $f_1, f_2 \in \mathbb{C}_I$. If we decompose $\Lambda_{r,I}(f)$ on $\mathbb{B}_I$ as $\Lambda_{r,I}(f) = \Lambda_{r,1}(f_1) + \Lambda_{r,2}(f_2)J$ for some complex oscillation functions $\Lambda_{r,1}(f_1)$ and $\Lambda_{r,2}(f_2)$. Then one can see directly from the complex valued result (see [17, Theorem 6]) and Remark 2.7 that the functions $\Lambda_{r,l}(f_l)$; $l = 1, 2$ lie in the usual L^p-space of complex valued measurable functions on $\mathbb{B}_I$ if and only if $\Lambda_{r,I}(f) \in L^p(\mathbb{B}_I, d\lambda_I, \mathbb{H})$.

Conversely, assume $\Lambda_{r,I}(f) \in L^p(\mathbb{B}_I, d\lambda_I, \mathbb{H})$. So we can write

$$\begin{aligned}\Lambda_{r,1}(f_1) + \Lambda_{r,2}(f_2)J &= \Lambda_{r,I}(f)\\ &= \sup_{I\in\mathbb{S}} \sup\{|f_1(z) - f_1(w)| : w \in \Delta_I(z,r) \subset \mathbb{B}_I\}\\ &\quad + \sup_{I\in\mathbb{S}} \sup\{|f_2(z) - f_2(w)| : w \in \Delta_I(z,r) \subset \mathbb{B}_I\}.\end{aligned}$$

This implies that

$$\begin{aligned}\Lambda_{r,l}(f_l) &= \sup_{I\in\mathbb{S}} \sup\{|f_l(z) - f_l(w)| : w \in \Delta_I(z,r) \subset \mathbb{B}_I\}\\ &\in L^p(\mathbb{B}_I, d\lambda_I, \mathbb{C}_I) \quad \text{for} \quad l = 1, 2.\end{aligned}$$

Again by Splitting Lemma, we conclude that both f_1 and f_2 belong to complex Besov space $\mathfrak{B}_{p,\mathbb{C}}$ on $\mathbb{B}_I$ which is equivalent to $f \in \mathfrak{B}_{p,I}(\mathbb{B}_I)$ and so $f \in \mathfrak{B}_p(\mathbb{B})$. ☐

Proposition 2.13. *For $p > 1$, let $f \in SR(\mathbb{B})$. Then $f \in \mathfrak{B}_p$ if and only if*

$$BMO(f) \in L^p(\mathbb{B}_I, d\lambda_I, \mathbb{H}) \quad \textit{for} \quad I \in \mathbb{S}. \tag{3}$$

Proof. Let $f \in \mathfrak{B}_p$. Then $f \in \mathfrak{B}_{p,I}$. Let $J \in \mathbb{S}$ with $J \perp I$. According to Lemma 1.2, any $f \in SR(\mathbb{B})$ restricted to $\mathbb{B}_I$ decomposes as $Q_I[f](z) = f_1(z) + f_2(z)J$ for $z \in \mathbb{B}_I$ and holomorphic functions $f_1, f_2 \in \mathbb{B}_I$. Thus, the condition (3) holds if and only if

$$BMO(f_l) \in L^p(\mathbb{B}_I, d\lambda_I, \mathbb{C}_I) \text{ for } I \in \mathbb{S},\ l = 1, 2.$$

Now, by [17, Theorem 7], it follows that the above condition holds if and only if f_1, f_2 lie in the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}$ on $\mathbb{B}_I$ which is same as $f \in \mathfrak{B}_{p,I}$ and so $f \in \mathfrak{B}_p$. ☐

3. Composition operators on Besov spaces

3.1. Boundedness and compactness

In this section, we characterize boundedness and compactness of composition operators on Besov spaces of the slice hyperholomorphic functions. Composition operators are extensively studied on various holomorphic function spaces of different domains in $\mathbb{C}$ and $\mathbb{C}^n$. For a study of composition operators on spaces of holomorphic functions, one can refer to [8] and [14]. For composiion operators on Besov spaces, see [3]. A study of composition operators on Hardy spaces of slice holomorphic functions is initated in [12]. Recently, Carleson measures for Hardy and Bergman spaces in the quaternionic unit ball are characterized in [13]. For composition operator and weighted composition operator on spaces of slice holomorphic functions, see [11, 12]. In [4], Hankel operators are studied on Hardy spaces via Carleson measures in a quaternionic variables. In the theory of slice regularity, the composition of two slice regular functions is not a slice regular function, in general. Now we define slice regular composition operators C_Φ on $\mathfrak{B}_p$ for $1 < p < \infty$.

Definition 3.1. Let $\Phi : \mathbb{B} \to \mathbb{B}$ be a slice hyperholomorphic map such that $\Phi(\mathbb{B}_I) \subset \mathbb{B}_I$ for some $I \in \mathbb{S}$. The composition operator C_Φ on $\mathfrak{B}_p, 1 < p < \infty$ induced by Φ is defined by

$$(C_\Phi f)_I(z) = (f_I \circ \Phi_I)(z) = F \circ \Phi_I(z) + G \circ \Phi_I(z)J$$

for all $f \in \mathfrak{B}_p$ with $f_I(z) = F(z) + G(z)J$.

Definition 3.2 ([1,10])**.** The slice regular Möbius transformation σ_a for every $a \in \mathbb{B}$ is define as

$$\sigma_a(q) = (1 - qa)^{-*} * (a - q) \quad \text{for} \quad q \in \mathbb{B},$$

where $*$ is slice regular product.

The slice regular Möbius transformation σ_a satisfies the following conditions:

(i) $\sigma_a : \mathbb{B} \to \mathbb{B}$ is a bijective mapping;
(ii) for all $z \in \mathbb{B}_I,\ \sigma_a(z) = \frac{a-z}{1-\bar{a}z}$;
(iii) for all $q \in \mathbb{B},\ \sigma_a(a) = 0, \sigma_a(0) = a$ and $\sigma_a \circ \sigma_a(q) = q$.

The following theorem characterizes bounded composition operators on the slice regular Besov spaces $\mathfrak{B}_p$.

Theorem 3.3. *Let Φ be a slice hyperholomorphic map on $\mathbb{B}$ such that $\Phi(\mathbb{B}_I) \subset \mathbb{B}_I$ for some $I \in \mathbb{S}$. For all $q \in \mathbb{B}$ and $a \in \mathbb{B}_I$, let $\sigma_a(q) = (1 - qa)^*(a - q)$ be a slice regular Möbius transformation. Then the composition operator C_Φ is bounded on Besov space $\mathfrak{B}_p,\ 1 < p < \infty$ if and only if*

$$\|C_\Phi \sigma_a\|_{\mathfrak{B}_p} < \infty. \tag{4}$$

Proof. Since the slice regular Möbius transformation on $\mathbb{B}_I$ coincides with the usual one dimensional complex Möbius transformation, so assume $\sigma_a \in \mathfrak{B}_{p,I}$. Let $J \in \mathbb{S}$ with $J \perp I$. So we can write $\sigma_a = \sigma_{a,1} + \sigma_{a,2}J$ for each one dimensional complex Möbius transformation $\sigma_{a,l} \in \mathfrak{B}_{p,\mathbb{C}},\ l = 1, 2$.

Therefore

$$\begin{aligned}&\sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left|(1-|z|^2)\frac{\partial C_\Phi \sigma_a}{\partial x_0}(z)\right|^p d\lambda_I(z)\\ \le\ & 2^{p-1} \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left|(1-|z|^2)\frac{\partial C_\Phi \sigma_{a,1}}{\partial x_0}(z)\right|^p d\lambda_I(z)\end{aligned}$$

$$
\begin{aligned}
& +2^{p-1}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi\sigma_{a,2}}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&= 2^{p-1}(\|C_\Phi\sigma_{a,1}\|^p_{\mathfrak{B}_{p,\mathbb{C}}}+\|C_\Phi\sigma_{a,2}\|^p_{\mathfrak{B}_{p,\mathbb{C}}})\\
&\leq 2^p\|C_\Phi\sigma_a\|^p_{\mathfrak{B}_{p,I}}. \qquad (5)
\end{aligned}
$$

Now, let $q=x_0+Iy\in\mathbb{B}$ for $I\in\mathbb{S}$. Then by Theorem 1.4, it follows that

$$
\left|\frac{\partial C_\Phi\sigma_a}{\partial x_0}(q)\right| = \left|\frac{1}{2}(1-I_qI)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(z)+\frac{1}{2}(1+I_qI)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(\bar{z})\right|.
$$

As $|q|=|z|=|\bar{z}|$, on applying triangle inequality, we have

$$
\left|(1-|q|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(q)\right| \leq \left|(1-|z|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(z)\right| + \left|(1-|\bar{z}|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(\bar{z})\right|.
$$

On taking integral over $\mathbb{B}_I$ on both sides of the above inequality and for $p>1$, we have

$$
\begin{aligned}
&\sup_{q\in\mathbb{B}}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|q|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(q)\right|^p d\lambda_I(q)\\
&\leq \sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&\quad + \sup_{\bar{z}\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|\bar{z}|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(\bar{z})\right|^p d\lambda_I(\bar{z})\\
&\leq 2\sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(z)\right|^p d\lambda_I(z). \qquad (6)
\end{aligned}
$$

Thus, by using (5) in (6), we have

$$
\begin{aligned}
\sup_{q\in\mathbb{B}_I}\|C_\Phi\sigma_a\|^p_{\mathfrak{B}_p} &= \sup_{q\in\mathbb{B}}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|q|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(q)\right|^p d\lambda_I(q)\\
&\leq 2\sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi\sigma_a}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&\leq 2^{p+1}\sup_{z\in\mathbb{B}_I}\|C_\Phi\sigma_a\|^p_{\mathfrak{B}_{p,I}}.
\end{aligned}
$$

Since C_Φ is a bounded operator on the complex Besov space, we have $\|C_\Phi\sigma_a\|^p_{\mathfrak{B}_p}<\infty$. Now suppose condition (4) holds. Then by [2, Theorem 13], it holds if and only if C_Φ is a bounded operator on the complex Besov space which is equivalent to the boundedness of C_Φ on $\mathfrak{B}_{p,I}$ and so C_Φ is bounded on $\mathfrak{B}_p$. □

By using Splitting Lemma, Remark 2.2 and [16, Lemma 3.6], the proof of the following lemma follows easily.

Lemma 3.4. *For $p\geq 1$, let $\mathfrak{B}_p$ be a slice regular Besov space on the unit ball $\mathbb{B}$. Then the following condition holds*:

(1) *every slice regular bounded sequence* $\{f_n\}_{n\in\mathbb{N}}$ *in* $\mathfrak{B}_p$ *on compact sets is uniformly bounded*;
(2) *for any slice regular sequence* $\{f_n\}_{n\in\mathbb{N}}$ *in* $\mathfrak{B}_p$ *such that* $\|f_n\|_{\mathfrak{B}_p} \to 0$, $f_n - f_n(0) \to 0$ *uniformly on the compact sets.*

The next result is essential for the proof of Theorem 3.6.

Lemma 3.5 ([16, Lemma 3.7])**.** *Let* X, Y *be two Banach spaces of analytic functions on the unit disk* $\mathbb{D}$. *Suppose*

(1) *the point evaluation functionals on* X *are continuous*;
(2) *the closed unit ball in* X *is a compact subset of* X *in the topology of uniform convergence on compact sets*;
(3) $T : X \to Y$ *is continuous, where* X *and* Y *are equipped with the topology of uniform convergence on compact sets.*

Then T *is a compact operator if and only if given a bounded sequence* $\{f_n\}$ *in* X *such that* $f_n \to 0$ *uniformly on compact sets, then the sequence* $\{Tf_n\}$ *converges to zero in the norm of* Y.

The following theorem gives the characterization for compact composition operators.

Theorem 3.6. *For* $p > 1$, *let* $\mathfrak{B}_p$ *be a slice regular Besov space on the unit ball* $\mathbb{B}$. *Let* Φ *be a slice hyperholomorphic map on* $\mathbb{B}$ *such that* $\Phi(\mathbb{B}_I) \subset \mathbb{B}_I$ *for some* $I \in \mathbb{S}$. *Then* $C_\Phi : \mathfrak{B}_p \to \mathfrak{B}_p$ *is compact if and only if for any bounded sequence* $\{f_m\}_{m\in\mathbb{N}}$ *in* $\mathfrak{B}_p$ *with* $f_m \to 0$ *as* $m \to \infty$ *on compact sets,* $\|C_\Phi f_m\|_{\mathfrak{B}_p} \to 0$ *as* $m \to \infty$.

Proof. The proof of the theorem is established if we prove the condition of Lemma 3.5. As a consequence of Lemma 3.4, we see that conditions (1) and (3) hold. Now, it remains to prove the condition (2). For this, let $\{f_m\}$ be a slice regular bounded sequence in $\mathfrak{B}_p$. Then by Lemma 3.4, $\{f_m\}$ is uniformly bounded on the compact sets. Consider $\{f_{m_k}\}$ a subsequence of $\{f_m\}$ in $\mathfrak{B}_p$ such that $\{f_{m_k}\}$ converges uniformly to h on the compact sets, for some $h \in SR(\mathbb{B})$. Let $J \in \mathbb{S}$ with $J \perp I$. Then by Lemma 1.2, there exist holomorphic functions $f_{1,m_k}, f_{2,m_k} : \mathbb{B}_I \to \mathbb{C}_I$ such that $Q_I[f_{m_k}](z) = f_{1,m_k}(z) + f_{2,m_k}(z)J$ for some $z \in \mathbb{B}_I$. Furthermore, $f_{1,m_k} \to h_1$ and $f_{2,m_k} \to h_2$ uniformly on the compact sets, where $h_l \in \mathbb{C}_I, l = 1, 2$ with $Q_I[h] = h_1 + h_2 J$. From Remark 2.2, we conclude that f_{1,m_k} and f_{1,m_k} belong to the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_I)$. Thus, from [16, Lemma 3.8] and by applying Minkowski's inequality and Fatou's Theorem, for $p > 1$, we have

$$\left(\sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left| \left(\frac{\partial h}{\partial x_0}(z)\right)(1 - |z|^2)\right|^p d\lambda_I(z)\right)^{\frac{1}{p}}$$
$$\leq \left(2^{p-1} \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left|(h_1'(z))(1 - |z|^2)\right|^p d\lambda_I(z)\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&+\left(2^{p-1}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(h_2'(z))(1-|z|^2)\right|^p d\lambda_I(z)\right)^{\frac{1}{p}}\\
=\;& 2^{\frac{p-1}{p}}\left(\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\lim_{k\to\infty}\left|f_{1,m_k}'(z)(1-|z|^2)\right|^p d\lambda_I(z)\right)^{\frac{1}{p}}\\
&+2^{\frac{p-1}{p}}\left(\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\lim_{k\to\infty}\left|f_{2,m_k}'(z)(1-|z|^2)\right|^p d\lambda_I(z)\right)^{\frac{1}{p}}\\
\le\;& 2^{\frac{p-1}{p}}\lim_{k\to\infty}\inf\left(\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|f_{1,m_k}'(z)(1-|z|^2)\right|^p d\lambda_I(z)\right)^{\frac{1}{p}}\\
&+2^{\frac{p-1}{p}}\lim_{k\to\infty}\inf\left(\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|f_{2,m_k}'(z)(1-|z|^2)\right|^p d\lambda_I(z)\right)^{\frac{1}{p}}\\
=\;& 2^{\frac{p-1}{p}}\left(\lim_{k\to\infty}\inf\|f_{1,m_k}\|_{\mathfrak{B}_{p,\mathbb{C}}}+\lim_{k\to\infty}\inf\|f_{2,m_k}\|_{\mathfrak{B}_{p,\mathbb{C}}}\right)\\
\le\;& 2^{\frac{2p-1}{p}}\lim_{k\to\infty}\inf\left(\|f_{m_k}\|_{\mathfrak{B}_{p,I}}\right)\\
<\;& \infty.
\end{aligned}
$$

Therefore, Lemma 3.5 yields that $C_\Phi:\mathfrak{B}_p\to\mathfrak{B}_p$ is compact if and only if for any bounded sequence $\{f_m\}_{m\in\mathbb{N}}$ in $\mathfrak{B}_p$ such that $f_m\to 0$ uniformly on compact sets as $m\to\infty$ and so $|f_m(\Phi(0))|+\|C_\Phi f_m\|_{\mathfrak{B}_p}\to 0$ as $m\to\infty$. ☐

The next result is the immediate consequence of Theorem 3.6.

Corollary 3.7. *For* $1<p<\infty$, *let* Φ *be a slice hyperholomorphic map such that* $\Phi(\mathbb{B}_I)\subset\mathbb{B}_I$ *for some* $I\in\mathbb{S}$. *If* $\|\Phi\|_\infty<1$, *then* $C_\Phi:\mathfrak{B}_p\to\mathfrak{B}_p$ *is compact.*

Proof. Let $\{f_n\}$ be a bounded sequence in $\mathfrak{B}_p$. Then $f_n\in\mathfrak{B}_{p,I}$ such that $f_n\to 0$ uniformly on the compact subsets of $\mathbb{B}_I$ for $I\in\mathbb{S}$. Let $J\in\mathbb{S}$ be such that $J\perp I$. Let $f_{1,n},f_{2,n}:\mathbb{B}_I\to\mathbb{C}_I$ be holomorphic functions such that $Q_I[f](z)=f_{1,n}(z)+f_{2,n}(z)J$ for $z=x_0+Iy\in\mathbb{B}_I$. By Remark 2.2, we have $f_{1,n},\ f_{2,n}$ lie in the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}$ on $\mathbb{B}_I$, where $\mathbb{B}_I$ is identified with $\mathbb{B}_I\subset\mathbb{C}_I$. Therefore,

$$
\begin{aligned}
&\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi f_n}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
\le\;& 2^{p-1}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi f_{1,n}}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&+2^{p-1}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi f_{2,n}}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
=\;& 2^{p-1}(\|C_\Phi f_{1,n}\|^p_{\mathfrak{B}_{p,\mathbb{C}}}+\|C_\Phi f_{2,n}\|^p_{\mathfrak{B}_{p,\mathbb{C}}})\\
\le\;& 2^p\|C_\Phi f_n\|^p_{\mathfrak{B}_{p,I}}. \qquad (7)
\end{aligned}
$$

Therefore by Theorem 1.4 and the fact that $|q| = |\bar{z}| = |z|$, equation (7) and [15, Corollary 2.12], it follows that

$$
\begin{aligned}
&\sup_{q\in\mathbb{B}}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|q|^2)\frac{\partial C_\Phi f_n}{\partial x_0}(q)\right|^p d\lambda_I(q)\\
&\leq \sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi f_n}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&\quad+\sup_{\bar{z}\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|\bar{z}|^2)\frac{\partial C_\Phi f_n}{\partial x_0}(\bar{z})\right|^p d\lambda_I(\bar{z})\\
&\leq 2^p\sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial C_\Phi f_n}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&\leq 2^{p+1}\sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|\frac{\partial f_n}{\partial x_0}(\Phi(z))\right|^p\left|(1-|z|^2)\right|^p\cdot\left|\frac{\partial\Phi}{\partial x_0}(z)\right|^p d\lambda_I(z). \qquad (8)
\end{aligned}
$$

Suppose $\varepsilon > 0$ is given. Since $\overline{\Phi(\mathbb{B}_I)}$ is a compact subset of $\mathbb{B}_I$, there exists positive integer $N > 0$ such that if $n \geq N$, then $|\frac{\partial f_n}{\partial x_0}(\Phi(z))|^p < \varepsilon$ for all $z \in \mathbb{B}_I$. Therefore from equation (8), we have

$$\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|q|^2)\frac{\partial C_\Phi f_n}{\partial x_0}(q)\right|^p d\lambda_I(q) \leq 2^{p+1}\varepsilon\|\Phi\|^p_{\mathfrak{B}_p,I} < \varepsilon \;\; const.$$

Hence $\|C_\Phi f_n\|^p_{\mathfrak{B}_p} \to 0$ as $n \to \infty$ and so Lemma 3.5 yields that $C_\Phi : \mathfrak{B}_p \to \mathfrak{B}_p$ is compact. □

The following proposition gives the compactness between Besov and Bloch spaces of slice regular functions.

Proposition 3.8. *For $p > 1$, let Φ be a slice hyperholomorphic map on $\mathbb{B}$ such that $\Phi(\mathbb{B}_I) \subset \mathbb{B}_I$ for some $I \in \mathbb{S}$. Then $C_\Phi : \mathfrak{B}_p \to \mathcal{B}$ is compact if and only if*

$$\|C_\Phi\sigma_a\|_{\mathcal{B}} \to 0 \quad as \quad |a| \to 1, \qquad (9)$$

where $\sigma_a(q) = (1 - q\bar{a})^* * (a - q)$, $q \in \mathbb{B}$ and $\mathcal{B}$ is a slice regular Bloch space on the unit ball $\mathbb{B}$. Here $\star$ denotes the slice regular product.

Proof. Let $\{\sigma_a : a \in \mathbb{B}\}$ be a set in $\mathfrak{B}_p$ such that $\sigma_a - a \to 0$ as $|a| \to 1$. Suppose C_Φ is a compact operator. Then by Theorem 3.6, $\{\sigma_a\}$ is a bounded set in $\mathfrak{B}_p$. Therefore, $\|C_\Phi\sigma_a\|_{\mathcal{B}} = 0$. Suppose condition (9) holds. Let f_m be a bounded sequence in $\mathfrak{B}_{p,I}$ such that $f_m \to 0$ uniformly on the compact sets as $m \to \infty$. We claim $C_\Phi : \mathfrak{B}_p \to \mathcal{B}$ is compact. For this, take $J \in \mathbb{S}$ with $J \perp I$. Let $f_{1,m}, f_{2,m}$ be holomorphic functions such that $Q_I[f_m] = f_{1,m}(z) + f_{2,m}(z)J$ for some $z = x_0 + Iy \in \mathbb{B}_I$. By Remark 2.2, we have $f_{1,m}, f_{2,m}$ lie in the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_I)$. Therefore, from [16, Theorem 4.1] and as $\|f_l\|_{\mathcal{B}_{p,\mathbb{C}}} \leq$

$\|f\|_{\mathcal{B}_{p,I}}$, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{B}_I}\|C_\Phi f_m\|_{\mathcal{B}}\\
&= \sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\left\{(1-|z|^2)\left|\frac{\partial C_\Phi(f_{1,m}+f_{2,m}J)}{\partial x_0}(z)\right|\right\}\\
&= \sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\left\{(1-|z|^2)\left|\frac{\partial C_\Phi f_{1,m}}{\partial x_0}(z)+\frac{\partial C_\Phi f_{2,m}}{\partial x_0}(z)J\right|\right\}\\
&\le \sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\left\{(1-|z|^2)\left|\frac{\partial C_\Phi f_{1,m}}{\partial x_0}(z)\right|\right\}+\sup_{z\in\mathbb{B}_I}\left\{(1-|z|^2)\left|\frac{\partial C_\Phi f_{2,m}}{\partial x_0}(z)\right|\right\}\\
&\le 2\sup_{z\in\mathbb{B}_I}\sup_{I\in\mathbb{S}}\left\{(1-|z|^2)\left|\frac{\partial C_\Phi f_m}{\partial x_0}(z)\right|\right\}\\
&= 2\sup_{z\in\mathbb{B}_I}\left\{\frac{(1-|z|^2)}{(1-|\Phi(z)|^2)}\left|\frac{\partial \Phi}{\partial x_0}(z)\right|\sup_{I\in\mathbb{S}}(1-|\Phi(z)|^2)\left|\frac{\partial f_m}{\partial x_0}(\Phi(z))\right|\right\}\\
&\le 2\sup_{z\in\mathbb{B}_I}\left\{\frac{(1-|z|^2)}{(1-|\Phi(z)|^2)}\left|\frac{\partial \Phi}{\partial x_0}(z)\right|\right\}\|f_m\|_{\mathcal{B}_I}\\
&\le 2\sup_{z\in\mathbb{B}_I}\left\{\frac{(1-|z|^2)}{(1-|\Phi(z)|^2)}\left|\frac{\partial \Phi}{\partial x_0}(z)\right|\right\}\|f_m\|_{\mathfrak{B}_{p,I}}.
\end{aligned}$$

Since $\{f_m\}$ is bounded in $\mathfrak{B}_{p,I}$, so $\|C_\Phi f_m\|_{\mathfrak{B}_{p,I}}\to 0$ as $m\to\infty$. Thus, $\|C_\Phi f_m\|_{\mathcal{B}}\to 0$ as $m\to\infty$. Hence by Theorem 3.6, $C_\Phi:\mathfrak{B}_p\to\mathcal{B}$ is compact. □

4. Essential norm

In this section, we find some estimates for the essential norm of composition operators on the slice regular Besov space. Firstly, we define Carelson measure. For Carleson mesures for Hardy and Bergman spaces in the quaternionic unit ball (see [13]).

Definition 4.1. For $1<p<\infty$, let $\mathfrak{B}_p$ be a slice regular Besov space. Let μ be an $\mathbb{H}$-valued positive measure on $\mathbb{B}_I$. Then μ is said to be an $\mathbb{H}$-valued p-Carleson measure on $\mathbb{B}$ if there is a constant $M>0$ such that

$$\int_{\mathbb{B}_I}|f(q)|^p\,d\mu(q)\le M\|f\|^p_{\mathfrak{B}_p}$$

for all $f\in\mathfrak{B}_p(\mathbb{B})$.

Suppose Φ is a slice hyperholomorphic map on $\mathbb{B}$ such that $\Phi(\mathbb{B}_I)\subset\mathbb{B}_I$ for some $I\in\mathbb{S}$ and $\Phi'(q)(1-|q|^2)\in L^p(\mathbb{B}_I,d\lambda_I(q))$, where $d\lambda_I(q)=\frac{dA_I(q)}{(1-|q|^2)^2}$ is the normalized differentiable of area in the plane and is a Möbius invariant measure on $\mathbb{B}$. Then we define the measure μ_p on $\mathbb{B}$ by

$$\mu_p(E)=\int_{\Phi^{-1}(E)}|\Phi'(q)|^p(1-|q|^2)dA_I(q),$$

where E is a measurable subset of $\mathbb{B}$.

Theorem 4.2. *Let $f \in SR(\mathbb{B})$. Suppose $\mu = \mu_1 + \mu_2 J$ for some $I \in \mathbb{S}$. Then μ is an $\mathbb{H}$-valued p-carleson measure on the slice regular Besov space if and only if μ_1, μ_2 are p-Carleson measures on the complex Besov space $\mathfrak{B}_{p,\mathbb{C}}$ for $1 < p < \infty$ in $\mathbb{B}_I$.*

Proof. Let $J \in \mathbb{S}$ be such that $I \perp J$. Then for any $f \in \mathfrak{B}_{p,I}$ there exist holomorphic functions $f_1, f_2 : \mathbb{B}_I \to \mathbb{C}_I$ such that $Q_I[f] = f_1(z) + f_2(z)J$ for some $z = x_0 + Iy \in \mathbb{B}_I$. Now, μ is an $\mathbb{H}$-valued p-Carleson measure on $\mathfrak{B}_p$ if and only if μ is an $\mathbb{H}$-valued p-Carleson measure on $\mathfrak{B}_{p,I}$ if and only if

$$\int_{\mathbb{B}_I} |f(q)|^p \, d\mu(q) \le M \|f\|^p_{\mathfrak{B}_{p,I}}$$

if and only if

$$\int_{\mathbb{B}_I} |f_l(q)|^p \, d\mu_l(q) \le 2^p M \|f_l\|^p_{\mathfrak{B}_{p,\mathbb{C}}}$$

if and only if μ_l, for $l = 1, 2$, is a p-Carleson measure on $\mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_I)$. □

Now we give the definition of essential norm.

Definition 4.3. The essential norm of a continuous linear operator T between the normed linear spaces X and Y is its distance from the compact operator K, that is

$$\|T\|_e^{X\to Y} = \inf\{\|T - K\|^{X\to Y} : K \text{ is a compact operator}\},$$

where $\|\cdot\|^{X\to Y}$ denotes the operator norm and $\|\cdot\|_e^{X\to Y}$ is the essential norm.

Here, we give an essential norm estimate for composition operators on the slice regular Besov space $\mathfrak{B}_p$.

Theorem 4.4. *For $1 < p < \infty$ and $\alpha > -1$, let Φ be a slice hyperholomorphic map such that $\Phi(\mathbb{B}_I) \subset \mathbb{B}_I$ for some $I \in \mathbb{S}$. Suppose $C_\Phi : \mathfrak{B}_p \to \mathfrak{B}_p$ is bounded. Then there is an absolute constant $C_1 C_2 \ge 1$ such that*

$$\lim_{|a|\to 1} \sup \|C_\Phi \sigma_a\|^p_{\mathfrak{B}_p} \le \|C_\Phi\|_e \le 2^p C_1 C_2 \lim_{|a|\to 1} \sup \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left(\frac{(1-|a|^2}{|1-\bar{a}q|^2} \right)^p d\mu_p(\omega).$$

Proof. Let $f = \sum_{k=0}^\infty q^k a_k \in \mathfrak{B}_{p,I}$ for $I \in \mathbb{S}$. For $0 < r < 1$, denote $\mathbb{B}_r = \{z : |z| < r\}$ in the complex plane $\mathbb{C}_I$. Consider an operator $R_n f(q) = \sum_{k=n+1}^\infty q^k a_k$ for some integer n. Suppose $J \in \mathbb{S}$ with $J \perp I$. Then there exist holomorphic functions $f_1, f_2 : \mathbb{B}_I \to \mathbb{C}_I$ such that $Q_I[f] = f_1(z) + f_2(z)J$ for $z = x_0 + Iy \in \mathbb{B}_I$. By Remark 2.2, we have $f_l = \sum_{k=0}^\infty q^k a_{l,k} \in \mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_I)$, and $R_{l,n} f_l(q) = \sum_{k=n+1}^\infty q^k a_{l,k}$ for some integer n and $l = 1, 2$. Therefore, we have

$$\|C_\Phi\|_e \le \lim_{n\to\infty} \inf \|C_\Phi R_n\|^p_{\mathfrak{B}_p} \le \lim_{n\to\infty} \inf \sup_{\|f\|_{\mathfrak{B}_p}\le 1} \|(C_\Phi R_n) f\|^p_{\mathfrak{B}_p}.$$

Now, for any fixed $0 < r < 1$, we have

$$\begin{aligned}
&\|(C_\Phi R_n)f\|^p_{\mathfrak{B}_p}\\
&= \left(|R_{1,n}f_1(\Phi(0))|^p + \sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial(C_\Phi R_{1,n}f_1)}{\partial x_0}(z)\right|^p d\lambda_I(z)\right)\\
&\quad + \left(|R_{2,n}f_2(\Phi(0))|^p + \sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial(C_\Phi R_{2,n}f_2)}{\partial x_0}(z)\right|^p d\lambda_I(z)J\right).
\end{aligned}$$

Since $|R_{1,n}f_1(\Phi(0))|$ and $|R_{2,n}f_2(\Phi(0))|$ are bounded as $n\to\infty$ and so

$$\begin{aligned}
\|(C_\Phi R_n)f\|^p_{\mathfrak{B}_p} &= \sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial(C_\Phi R_{1,n}f_1)}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&\quad + \sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial(C_\Phi R_{2,n})f_2}{\partial x_0}(z)\right|^p d\lambda_I(z)J.
\end{aligned}$$

Let $\mu_p = \mu_{1,p} + \mu_{2,p}J$, where $\mu_{1,p}$ and $\mu_{2,p}$ are two p-Carleson measures on $\mathbb{B}_I$ with the values in $\mathbb{C}_I$. Therefore

$$\begin{aligned}
&\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial(C_\Phi R_n f)}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&\le 2^{p-1}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial(C_\Phi R_{1,n}f_1)}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&\quad +2^{p-1}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|(1-|z|^2)\frac{\partial(C_\Phi R_{2,n}f_2)}{\partial x_0}(z)\right|^p d\lambda_I(z)\\
&= 2^{p-1}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|\frac{\partial(R_{1,n}f_1)}{\partial x_0}(q)\right|^p d\mu_{1,p}(q)\\
&\quad +2^{p-1}\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I}\left|\frac{\partial(R_{2,n}f_2)}{\partial x_0}(q)\right|^p d\mu_{2,p}(q)\\
&\le 2^p\|R_n f\|^p_{\mathfrak{B}_{p,I}}\\
&= 2^p\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_I\setminus\mathbb{B}_r}\left|\frac{\partial(R_n f)}{\partial x_0}(q)\right|^p d\mu_p(q)\\
&\quad +2^p\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_r}\left|\frac{\partial(R_n f)}{\partial x_0}(q)\right|^p d\mu_p(q)\\
&= I_1 + I_2.
\end{aligned}$$

Since $C_\Phi : \mathfrak{B}_p \to \mathfrak{B}_p$ is bounded so the measure μ_p is a p-Carleson measure. From the proof of Proposition 3 in [8], we see that for given $\varepsilon > 0$ and n large enough that $\left|\frac{\partial(R_n f)}{\partial x_0}(q)\right| \le \varepsilon\|\frac{\partial f}{\partial x_0}\|_{A^p_{p-2}}$. Thus

$$I_1 = 2^p\sup_{I\in\mathbb{S}}\int_{\mathbb{B}_r}\left|\frac{\partial(R_n f)}{\partial x_0}(q)\right|^p d\mu_p(q) \le 2^p\varepsilon^p\|f\|^p_{\mathfrak{B}_p}.$$

Therefore for a fixed r, we have

$$I_1 = 2^p \sup_{I\in\mathbb{S}} \sup_{||f||_{\mathfrak{B}_p}\le 1} \int_{\mathbb{B}_r} \left|\frac{\partial(R_n f)}{\partial x_0}(q)\right|^p d\mu_p(q) \to 0 \text{ as } n\to\infty.$$

On the other hand, if $\mu_{p,r}$ is the restriction of measure μ_p to the set $\mathbb{B}_I \setminus \mathbb{B}_r$, then

$$I_2 = 2^p \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I\setminus\mathbb{B}_r} \left|\frac{\partial(R_n f)}{\partial x_0}(q)\right|^p d\mu_{p,r}(q) \le 2^p C_1 C_2 \|\mu_p\|_r^*$$

for some absolute constants C_1, C_2 and $\|\mu_p\|_r^* = \sup_{|a|\ge r} \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left|\frac{\partial\sigma_a(q)}{\partial x_0}\right|^p d\mu_p(q)$.

Therefore,

$$\lim_{n\to\infty} \inf \sup_{\|f\|_{\mathfrak{B}_p}\le 1} \|(C_\Phi R_n)f\|_{\mathfrak{B}_p} \le \lim_{n\to\infty} \inf 2^p C_1 C_2 \|\mu_p\|_r^*.$$

Hence

$$\|C_\Phi\|_e^p \le 2^p C_1 C_2 \|\mu_p\|_r^*.$$

Taking $r \to 1$, we have

$$\begin{aligned}\|C_\Phi\|_e^p \le \lim_{r\to 1} 2^p C_1 C_2 \|\mu_p\|_r^* \;&\le\; 2^p C_1 C_2 \lim_{|a|\to 1} \sup \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left|\frac{\partial\sigma_a(q)}{\partial x_0}\right|^p d\mu_p(q)\\ &=\; 2^p C_1 C_2 \lim_{|a|\to 1} \sup \sup_{I\in\mathbb{S}} \int_{\mathbb{B}_I} \left(\frac{(1-|a|^2}{|1-\bar{a}q|^2}\right)^p d\mu_p(q)\end{aligned}$$

which is the desired upper bound.

For lower estimate, let $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ be the complex Möbius transformation on $\mathbb{B}_I$, associated with a. Clearly σ_a is bounded in $\mathfrak{B}_{p,I}$. Also $\sigma_a - a \to 0$ as $|a| \to 1$ uniformly on the compact subsets of $\mathbb{B}_I$ and $|\sigma_a(z) - a| = |z|\frac{1-|a|^2}{|1-\bar{a}z|}$. Furthermore, $\|K(\sigma_a - a)\|_{\mathfrak{B}_{p,I}} \to 0$ as $|a| \to 1$ for some compact operator K on $\mathfrak{B}_{p,I}$. Thus $\|K(\sigma_a)\|_{\mathfrak{B}_{p,I}} \to 0$ as $|a| \to 1$. Therefore,

$$\begin{aligned}\|C_\Phi\|_e^p \;&\ge\; \|C_\Phi - K\|_{\mathfrak{B}_p}^p \ge \|C_\Phi - K\|_{\mathfrak{B}_{p,I}}^p\\ &\ge\; \lim_{|a|\to 1} \sup \|(C_\Phi - K)\sigma_a\|_{\mathfrak{B}_{p,I}}^p\\ &\ge\; \lim_{|a|\to 1} \sup \|C_\Phi\sigma_a\|_{\mathfrak{B}_{p,I}}^p - \lim_{|a|\to 1} \sup \|K\sigma_a\|_{\mathfrak{B}_{p,I}}^p\\ &=\; \lim_{|a|\to 1} \sup \|C_\Phi\sigma_a\|_{\mathfrak{B}_{p,I}}^p.\end{aligned}$$

Hence the desired result. □

Acknowledgements. The authors would like to thank the referee for his\her helpful comments and valuable suggestions for improving this manuscript.

References

[1] D. Alpay, F. Colombo, and I. Sabadini, *Slice hyperholomorphic Schur analysis*, Operator Theory: Advances and Applications, 256, Birkhäuser/Springer, Cham, 2016. https://doi.org/10.1007/978-3-319-42514-6

[2] J. Arazy, S. D. Fisher, and J. Peetre, *Möbius invariant function spaces*, J. Reine Angew. Math. **363** (1985), 110–145. https://doi.org/10.1007/BFb0078341

[3] N. Arcozzi, R. Rochberg, and E. Sawyer, *Carleson measures for analytic Besov spaces*, Rev. Mat. Iberoamericana **18** (2002), no. 2, 443–510. https://doi.org/10.4171/RMI/326

[4] N. Arcozzi and G. Sarfatti, *From Hankel operators to Carleson measures in a quaternionic variable*, Proc. Edinb. Math. Soc. (2) **60** (2017), no. 3, 565–585. https://doi.org/10.1017/S0013091516000626

[5] C. M. P. Castillo Villalba, F. Colombo, J. Gantner, and J. O. González-Cervantes, *Bloch, Besov and Dirichlet spaces of slice hyperholomorphic functions*, Complex Anal. Oper. Theory **9** (2015), no. 2, 479–517. https://doi.org/10.1007/s11785-014-0380-4

[6] F. Colombo, I. Sabadini, and D. C. Struppa, *Noncommutative functional calculus*, Progress in Mathematics, 289, Birkhäuser/Springer Basel AG, Basel, 2011. https://doi.org/10.1007/978-3-0348-0110-2

[7] ———, *Entire slice regular functions*, Springer Briefs in Mathematics, Springer, Cham, 2016. https://doi.org/10.1007/978-3-319-49265-0

[8] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[9] J. Gantner, J. O. González-Cervantes, and T. Janssens, *BMO- and VMO-spaces of slice hyperholomorphic functions*, Math. Nachr. **290** (2017), no. 14-15, 2259–2279. https://doi.org/10.1002/mana.201600379

[10] G. Gentili, C. Stoppato, and D. C. Struppa, *Regular functions of a quaternionic variable*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013. https://doi.org/10.1007/978-3-642-33871-7

[11] P. Lian and Y. Liang, *Weighted composition operator on quaternionic Fock space*, Banach J. Math. Anal. **15** (2021), no. 1, Paper No. 7, 20 pp. https://doi.org/10.1007/s43037-020-00087-6

[12] G. Ren and X. Wang, *Slice regular composition operators*, Complex Var. Elliptic Equ. **61** (2016), no. 5, 682–711. https://doi.org/10.1080/17476933.2015.1113270

[13] I. Sabadini and A. Saracco, *Carleson measures for Hardy and Bergman spaces in the quaternionic unit ball*, J. Lond. Math. Soc. (2) **95** (2017), no. 3, 853–874. https://doi.org/10.1112/jlms.12035

[14] J. H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4612-0887-7

[15] M. Tjani, *Compact composition operators on some Moebius invariant Banach spaces*, ProQuest LLC, Ann Arbor, MI, 1996.

[16] ———, *Compact composition operators on Besov spaces*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4683–4698. https://doi.org/10.1090/S0002-9947-03-03354-3

[17] K. H. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 139, Marcel Dekker, Inc., New York, 1990.

Sanjay Kumar
Department of Mathematics
Central University of Jammu
Rahya-Suchani (Bagla)-181 143, Jammu (J&K), India
Email address: sanjaykmath@gmail.com, sanjay.math@cujammu.ac.in

Khalid Manzoor
Department of Mathematics
Cluster University
Jammu (J&K), India
Email address: khalidcuj14@gmail.com