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Fuzzy Deterministic Relations

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Abstract A fuzzy relation between $X$and Y as fuzzy subset of $X\times Y$ was proposed by Zadeh. Subsequently, several researchers have applied the notion of fuzzy subsets to various branches of mathematics and computer sciences. Murali an Nemitz have studied fuzzy relations connected with fuzzy equivalence relations and fuzzy functions. Ounalli and Jaoua defined a fuzzy difunctional relation on a set. difunctional relations are versatile mathematical tool, which can be used in software design and in database theory. Their work have revealed the usefulness of difunctional relations in program specification and in defining program correctness. The main goal of this paper is to define a fuzzy deterministic relation on a set, characterize the fuzzy deterministic relation as its level subsets and investigate some properties in connection with fuzzy deterministic relation. In particular we prove that a fuzzy relation R is fuzzy deterministic iff R is a fuzzy function.

Key Words : G−reflexivity, fuzzy equivalence relation, fuzzy difunctional relation. fuzzy deterministic relation, fuzzy fuction.

요 약 X와 Y사이의 퍼지 관계를 곱집합 $X\times Y$의 퍼지 부분집합으로 Zadeh에 의해 처음으로 소개된 이후 퍼지집합에 대한 개념은 자연과학 및 컴퓨터과학에서 많은 연구성과가 이루어져 왔다. 그 결과 Muralli와 Nemitz는 동치관계 및 함수와 관련하여 퍼지관계를 연구하였고, Ounalli와 Jaoua는 중요한 수학적 도구로서 퍼지 다이펑션날 관계를 정의하여 소프트디자인과 데이터베이스 이론에서 중요한 역할을 하는 것으로 증명되었으며, 또한 프로그램 표식과 프로그램 정확도를 정의하는데 유용한 것으로 밝혀졌다. 본 논문에서는 한 집합 위에 퍼지 디터미니스틱 관계를 정의하여 퍼지 디터미니스틱 관계를 레벨 부분집합으로 특성화 하였고, 퍼지 디터미니스틱 관계와 관련하여 일부 성질을 증명하였다. 특히, 퍼지 디터미니스틱 관계와 퍼지 함수가 동치임을, 퍼지 함수가 퍼지 다이펑션날 관계가 동치임을 증명하였다.

주제어 : 반사성, 퍼지 동치관계, 퍼지 다이펑션날 관계, 퍼지 디터미니스틱 관계, 퍼지 함수

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1. Introduction

Just as the notion of fuzzy subsets of set generalises that of crisp subsets, the concept of ordinary relation between two elements lends itself to the generalisation of fuzzy relations on a set.

A fuzzy relation between $X$and Y as a fuzzy subset of $X \times Y$ was proposed by Zadeh[1]. Subsequently many researchers have studied fuzzy relations connected with fuzzy equivalences and fuzzy functions in various contexts.

Murali[2] defined and discussed properties of fuzzy equivalence on a set and studied the cuts of fuzzy equivalence relations.

Ounalli and Jaoua[3] characterized in a simple manner the fuzzy difunctional relations and showed that the most of the properties that characterize crip difunctional relations also hold for fuzzy difunctional relations. In [4], Sung et al proved that there exists a relationship between fuzzy equivalence relations and fuzzy difunctional relations. In this thesis, We attempts a few elementary observations concerning fuzzy difunctional relations and fuzzy deterministic relations. As a result, We characterize the fuzzy deterministic relations in the frame of fuzzy relations and investigate some of their properties.

In this paper, we assume that all fuzzy relations considered here are defined a fized universe K.

2. Preliminaries

We review some definitions that will be needed in the seguel. For detail we refer to [2,3,5]

Definition 2.1 The scalars set of a fuzzy relation R, written $\Phi(R)$ is defined as follow:

$\Phi(R)=\{a \neq 0 \mid \exists (x,y) \in K \times K, R(x,y)=\alpha\}$.

Definition 2.2 Let R be a fuzzy relation and $\alpha \in \Phi(R)$. The α cut relative to R, written R_α is a relation such that for all $x,y \in K$:

$$R_\alpha(x,y)=\begin{cases}1, \text{if } R(x,y) \geq \alpha \\ 0, otherwise.\end{cases}$$

Definition 2.3 Let R be a relation on K. An element of R is denoted (x,y), where x is an argument and y is an image of x by R. The image set of $x \in K$, written

xR is defined by $xR=\{z \mid (x,z) \in R\}$.

Definition 2.4 The sup-min product $R \circ S$ for two fuzzy relations R,S on a set K is defined by

$RS(x,y)=\bigvee_{t \in K}(S(x,t) \wedge R(t,y)).x,y \in K.$

Definition 2.5 Let R be a fuzzy relation on a set K. Then R is fuzzy reflexive on K if $R(x,x)=1$ for $x \in K$; R is fuzzy symmetric on K if $R(x,y)=R(y,x)$ for all $x,y \in K$; and R is fuzzy transitive on K if $RR \subseteq R$. We say that R is a fuzzy equivalence relation on K if R is fuzzy reflexive, fuzzy symmetric and fuzzy transitive on K.

Definition 2.6 A fuzzy relation R on a set K is $G-$reflexive if for $x \neq y$ in K.

(1) $R(x,x)>0$, and

(2) $R(x,y) \leq \delta(R)$.where $\delta(R)=\bigwedge_{t \in K} R(t,t)$.

A $G-$reflexive and transitive fuzzy relation on K is a $G-$preorder on K. A symmetric $G-$preorder on K is a $G-$equivalence on K.

Definition 2.7 Let R and S be two fuzzy relations. We say that

(i) R is more deterministic than S if and only if $R^{-1}R \subseteq S^{-1}S$,

(ii) R is fuzzy deterministic if and only if it is more deterministic than the

identity I, i.e., $R^{-1}R \subseteq I$.

Definition 2.8 R is fuzzy difunctional if and only if it satisfies condition $RR^{-1}R \subseteq R$.

Definition 2.9 A fuzzy function is a fuzzy relation R such that for all $\alpha \in \Phi(R)$. R_α is a crisp function.

3. Main Results

In this section, we deal with a few elementary observations concerning fuzzy deterministic relations.

Theorem 3.1 Let R be a fuzzy reflexive relation. If R is fuzzy difunctional, then R is a fuzzy equivalence relation.

Proof. First, we show that R is fuzzy symmetric:

$R(x,y) = RR^{-1}R(x,y)$
$= \vee z \in K(R(x,z) \wedge (\vee w \in KR(w,z) \wedge R(w,y)))$
$\geq R(x,x) \wedge (\vee w \in K(R(w,z) \wedge R(w,y)))$
$= \vee w \in K(R(w,z) \wedge R(w,y))$
$\geq R(y,x) \wedge R(y,y)$, as R is reflexive
$= R(y,x)$.

Hence $R(y,x) \geq R(x,y)$ for all $x,y \in K$.
Similarly, interchanging the roles of x and y, we get that $R(x,y) \geq R(y,x)$ for all $x,y \in K$. Hence we have $R = R^{-1}$.

Next, we show below that R is transitive:

$R(x,y) = RR^{-1}R(x,y)$
$= \vee z \in K(R(x,z) \wedge (\vee w \in K(R(w,z) \wedge R(w,y)))$
$\geq R(x,x) \wedge (\vee w \in K(R(w,z) \wedge R(w,y)))$
$= \vee w \in K(R(w,x) \wedge R(w,y))$
$= \vee w \in K(R(x,w) \wedge R(w,y))$
$= RR(x,y)$.
Thus $RR \subseteq R$. Therefore, R is fuzzy difunctional.

Theorem 3.2 If R is fuzzy symmetric and fuzzy transitive, then R is fuzzy difunctional.

Proof. $RR^{-1}R = RRR \subseteq RR \subseteq R$. Which yields R is fuzzy difunctional.

Theorem 3.3 If R is fuzzy deterministic, then R is fuzzy difuntional.

Proof. $RR^{-1}R = R(R^{-1}R) \subseteq RI = R$, and so R is fuzzy difunctional.

Theorem 3.4 If R is fuzzy deterministic, then αR is fuzzy deterministic for all $\alpha \in \Phi(R)$.

Proof. $(\alpha R)^{-1}(\alpha R) = (\alpha R^{-1})(\alpha R) = \alpha(R^{-1}R)$
$\subseteq \alpha I \subseteq I$.

Which yields αR is fuzzy deterministic.

Theorem 3.5 Let R be a fuzzy symmetric relation. Then R is fuzzy deteministic iff R^{-1} is fuzzy deterministic.

Proof. Assume R is fuzzy deterministic. Then $(R^{-1})^{-1}R^{-1} = RR^{-1} = R^{-1}R \subseteq I$, and so R^{-1} is fuzzy deterministic. Conversely, assume R^{-1} is fuzzy deterministic, as seen in above argument, $R = (R^{-1})^{-1}$ is fuzzy deterministic.

Theorem 3.6 Let R,S be fuzzy deterministic. Then RS is fuzzy deterministic.

Proof. $(RS)^{-1}(RS) = (S^{-1}R^{-1})(RS)$

$= S^{-1}(R^{-1}R)S \subset S^{-1}S \subset I$, and so RS is fuzzy deterministic.

Theorem 3.7 Let R,S be fuzzy deterministic. Then $R \cap S$ is fuzzy deterministic.
Proof. $(R \cap S)^{-1}(R \cap S) = (R^{-1} \cap S^{-1})(R \cap S)$
$\subseteq R^{-1}R \cap S^{-1}S^{-1} \subset I$

, which yields $R \cap S$ is fuzzy deterministic,

Theorem 3.8 R is fuzzy deterministic if and only if R_α is deterministic for all $\alpha \in \Phi(R)$.

Proof. Suppose $xR_\alpha \cap yR_\alpha \neq \varnothing, x,y \in K$. Then there exists $w \in K$ such that $w \in xR_\alpha \cap yR_\alpha$, and so, $R(x,w) \geq \alpha$ and $R(y,w) \geq \alpha$. Since R is deterministic, we have $R^{-1}R \subseteq I$.

On the other hand,

$$\begin{aligned} R^{-1}R(x,y) &= \bigvee_{z \in K} [R(x,z) \wedge R^{-1}(z,y)] \\ &= \bigvee_{z \in K} [R(x,z) \wedge R(y,z)] \\ &\geq R(x,w) \wedge R(y,w) \\ &\geq \alpha \wedge \alpha \\ &= \alpha. \end{aligned}$$

This entails $I(x,y) \geq \alpha > 0$, and so $x = y$.

Conversely, assume that R_α is deterministic for all $\alpha \in \Phi(R)$. Then. we also prove that R is fuzzy deterministic i.e., $R^{-1}R(x,y) \leq I(x,y)$ for all $x,y \in U$. If $x = y$, then $I(x,y) = 1$. The equality is obvious. Hence we may assume that $x \neq y$. Then $I(x,y) = 0$, and we claim that $R^{-1}R(x,y) = 0$. Suppose not, then $R^{-1}R(x,y) > 0$. This means that there exists $z \in K$ such that $R(x,z) > 0$ and $R(y,z) > 0$. Now, letting $R(x,z) = \alpha_1, R(y,z) = \alpha_2$ and $\alpha = \alpha_1 \wedge \alpha_2$. Then we have $R(x,z) \geq \alpha$ and $R(y,z) \geq \alpha$, this means $(x,z) \in R_\alpha$, and $(y,z) \in R_\alpha$, which implies $z \in xR_\alpha$ and $z \in yR_\alpha$. Thus we have $xR_\alpha \cap yR_\alpha \neq \varnothing$. Since R_α is deterministic. we obtain that $x = y$. This contradicts, Therefore R is fuzzy deterministic.

Theorem 3.9 If R and S are fuzzy deterministic, then RS^{-1} is fuzzy difunctional.

Proof.
$$\begin{aligned} RS^{-1}(RS^{-1})^{-1}(RS^{-1}) &= RS^{-1}(SR^{-1})(RS^{-1}) \\ &= R(S^{-1}S)(R^{-1}R)S^{-1} \\ &\subseteq RS^{-1}. \end{aligned}$$

Which yields RS^{-1} is fuzzy difunctional.

Theorem 3.10 If R is fuzzy deterministic, then $R^{-1}R^{-1}$ is fuzzy difunctional.

Proof.
$$\begin{aligned} (R^{-1}R^{-1})(R^{-1}R^{-1})^{-1}(R^{-1}R^{-1}) &= (R^{-1}R^{-1})(RR)(R^{-1}R^{-1}) \\ &= R^{-1}(R^{-1}R)RR^{-1}R^{-1} \\ &\subseteq R^{-1}RR^{-1}R^{-1} \\ &= (R^{-1}R)R^{-1}R^{-1} \\ &\subseteq R^{-1}R^{-1} \\ &= R^{-1}R^{-1}. \end{aligned}$$

Which yields $R^{-1}R^{-1}$ is fuzzy difunctional.

Theorem 3.11 Let R be fuzzy reflexive and fuzzy symmetric. Then R is fuzzy deterministic if and only if R is a fuzzy function.

Proof. Assume that R is fuzzy deterministic, then R_α is deterministic for all $\alpha \in \Phi(R)$. We prove that R is a fuzzy function. It suffices to show that R_α is a function. Let $x \in K$ be given. Since R_α is reflexive, then $(x,x) \in R_\alpha$. Let $(x,y_1) \in R_\alpha$ and $(x,y_2) \in R_\alpha$. Since R_α is symmetric, then we have $(y_1,x) \in R_\alpha$ and $(y_2,x) \in R_\alpha$, which implies $x \in y_1R_\alpha$ and $x \in y_2R_\alpha$. Hence we have $y_1R_\alpha \cap y_2R_\alpha \neq \varnothing$, because R_α is deterministic, we obtain that $y_1 = y_2$ and R_α is a function.

Conversely, suppose that R is a fuzzy function. Then R_α is a ordinary function. We must prove that R is a fuzzy deterministic. It suffices to show that R_α is deterministic. Assume $xR \cap yR \neq \varnothing, x,y \in K$. Then there exists $z \in K$ such that $z \in xR_\alpha \cap yR_\alpha$, and so $(x,z) \in R_\alpha$ and $(y,z) \in R_\alpha$. Since R is symmetric, we have

$(z,x) \in R_\alpha$ and $(z,y) \in R_\alpha$. Also, since R is a function, we get that $x=y$. Therefore, R_α is deterministic.

Theorem 3.12 Let R be fuzzy reflexive and R_α be anti-symmetric for all $\alpha \in \Phi(R)$. Then R is a fuzzy function if and only if R is fuzzy difunctional.

Proof. Assume that R is a fuzzy function, then R_α is a ordinary function for all $\alpha \in \Phi(R)$. We prove that R is fuzzy difunctional. It suffices to show that R_α is difunctional. Suppose $xR_\alpha \cap yR_\alpha \neq \varnothing$, $x,y \in K$. Let $z_1 \in xR_\alpha$. Then $(x,z_1) \in R_\alpha$. $xR_\alpha \cap yR_\alpha \neq \varnothing$ meas that there exists $z_2 \in K$ such that $z_2 \in xR_\alpha \cap yR_\alpha$, which implies $(x,z_2) \in R_\alpha$ and $(y,z_2) \in R_\alpha$. Since R_α is an ordinary function, from $(x,z_1) \in R_\alpha$ and $(x,z_2) \in R_\alpha$, we obtain $z_1 = z_2$. Hence we have $xR_\alpha \subseteq yR_\alpha$. Similarly, if $z_1 \in yR_\alpha$, then $z_1 \in xR_\alpha$. This means $yR_\alpha \subseteq xR_\alpha$. Therefore, R_α is difunctional for all $\alpha \in \Phi(R)$. Conversely, assume that R is fuzzy difunctional, then R_α is difunctional for all $\alpha \in \Phi(R)$. Hence we must prove that R is fuzzy function. It suffices to show that R_α is a function. Let $x \in K$ be given. Since R_α is reflexive, then $(x,x) \in R_\alpha$. Next, let $(x,y_1) \in R_\alpha$ and $(x,y_2) \in R_\alpha$. Then $y_1 \in xR_\alpha$ and $y_2 \in xR_\alpha$. Since R_α is reflexive, $y_1 \in y_1R_\alpha$ and $y_2 \in y_2R_\alpha$, and so $xR_\alpha \cap y_1R_\alpha \neq \varnothing$ and $xR_\alpha \cap y_2R_\alpha \neq \varnothing$. This implies $xR_\alpha = y_1R_\alpha$ and $xR_\alpha = y_2R_\alpha$, and so $y_1R_\alpha = y_2R_\alpha$. Thus, we have $y_1 \in y_2R_\alpha$ and $y_2 \in y_1R_\alpha$, this means that $(y_1,y_2) \in R_\alpha$ and $(y_2,y_1) \in R_\alpha$. Since R_α is anti-symmetric, we obtain that $y_1 = y_2$. Therefore, R_α is a function.

Theorem 3.13 Let R be fuzzy reflexive and R_α be anti-symmetric for all $\alpha \in \Phi(R)$. Then R is fuzzy difunctional if and only if R is fuzzy deterministic.

Proof. Suppose that R is fuzzy difunctional, then R_α is difunctional for all $\alpha \in \Phi(R)$. We must prove that R is fuzzy deterministic. It suffices to show that R_α is deterministic. Assume that R_α is difucntional and $xR_\alpha \cap y_2R_\alpha \neq \varnothing$, $x,y \in K$. Then $xR_\alpha = yR_\alpha$. Since R is reflexive, R_α is reflexive. Also, since R_α is anti-symmetric, then we have $x=y$. Hence R_α is deterministic. Conversely, let R is fuzzy deterministic.

Then $R^{-1}R \subseteq I$, $R(R^{-1}R) \subseteq RI$ and $RR^{-1}R \subseteq R$.

Thus R is fuzzy difunctional.

Theorem 3.14 If a fuzzy relation R on X is G-reflexive, then the reflexive closure R^* of R is G-reflexive.

Proof. We note that $R^* = R \cup I$. Now let x be any element in X, then $R^*(x,x) = (R \cup I)(x,x) = R(x,x) > 0$. Assume that $x \neq y$ in X. Then $R^*(x,x) = (R \cup I)(x,y) = R(x,y) \leq \delta(R)$, and so R^* is G-reflexive.

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