KYUNGPOOK Math. J. 61(2021), 613-629
https://doi.org/10.5666/KMJ.2021.61.3.613
pISSN 1225-6951 eISSN 0454-8124

Fixed Point Theorems for Mixed Monotone Vector Operators with Application to Systems of Nonlinear Boundary Value Problems

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Abstract. In this paper, we present and prove new existence and uniqueness fixed point theorems for vector operators having a mixed monotone property in partially ordered product Banach spaces. Our results extend and improve existing works on τ-φ-concave operators in the scalar case. As an application, we study the existence and uniqueness of positive solutions for systems of nonlinear Neumann boundary value problems.

1. Introduction

In [25], C. B. Zhai and X. M. Cao introduced the concept of τ-φ-concave operators $A : P \longrightarrow P$, where P is a cone in a Banach space, and proved the existence and uniqueness of fixed points for such operators. They did so without requiring upper and lower solutions, compactness or continuity conditions. Krasnoselskii [15] studied u_0-concave operators with $u_0 \succ \theta$. In [27], Zhang and Zhai used the fixed point theorem for increasing α-concave operators to obtain the existence and uniqueness of positive solutions for Neumann boundary value problems. The same authors in [26], proved new fixed point theorems for mixed monotone operators, and then they established some criterions for the local existenceuniqueness of positive solutions to some boundary value problems.

Indeed, there has been much attention focused on problems of positive solutions for diverse nonlinear boundary value problems (See, for instance, [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24]). However, most of these works studied the scalar case. Therefore, motivated by some papers, for example [25, 26]

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* Corresponding Author.
Received April 11, 2020; revised November 18, 2020; accepted November 23, 2020.
2020 Mathematics Subject Classification: 54H25, 47H10, 54E50.
Key words and phrases: fixed point in product cone, mixed monotone vector operators, systems of boundary value problems.

and the references therein, we propose in the present work to extend a fixed point theorem and its application to the vector case. In other words, we construct a fixed point theorems for a vector operator, and then we apply it to systems of nonlinear Neumman boundary value problems of the following type

$$\begin{cases} -x''(t)+\theta^2 x(t)=\lambda f(t,x(t),x(t),y(t)),\ 0<t<1,\\ -y''(t)+\omega^2 y(t)=\beta g(t,x(t),y(t),y(t)),\ 0<t<1,\\ x'(0)=x'(1)=1,\ y'(0)=y'(1)=0, \end{cases} \tag{1.1}$$

in order to obtain existence and uniqueness of the positive solution.

Let $(E,\|.\|)$ be a real Banach space which is partially ordered by a cone $P\subset E$, i.e., $x\preceq y$ if and only if $y-x\in P$. If $x\preceq y$ and $x\neq y$, then we denote $x\prec y$ or $y\succ x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P\subset E$ is a cone if it satisfies $(i) x\in P,\lambda\geq 0\Rightarrow \lambda x\in P$, $(ii) x\in P,-x\in P\Rightarrow x=\theta$. A cone P is said to be solid if its interior $\overset{\circ}{P}$ is non-empty. P is called normal if there exists a constant $N>0$ such that, for all $x,y\in E,\theta\preceq x\preceq y$ implies $\|x\|\leq N\|y\|$; in this case N is called the normality constant of P.

For all $x,y\in E$, the notation $x\sim y$ means that there exist $\lambda>0$ and $\mu>0$ such that $\lambda x\preceq y\preceq \mu x$. Clearly $\sim$ is an equivalence relation. Given $h\succ\theta$ (i.e., $h\in P$ and $h\neq\theta$), we denote by P_h the set $P_h=\{x\in E: x\sim h\}$. It is easy to see that for $h\in P$, $P_h\subset P$ is convex and $\lambda P_h=P_h$ for all $\lambda>0$. If $\overset{\circ}{P}\neq\varnothing$ and $h\in\overset{\circ}{P}$, it is clear that $P_h=\overset{\circ}{P}$. Let us give the definition of mixed monotone operators with three variables as it is known in the literature.

Definition 1.1.([3]) Let $(X,\preceq)$ be a partially ordered set and $A: X\times X\times X\longrightarrow X$. Then the trivariate operator A is said to *have the mixed monotone property* if $A(.,u,y)$ and $A(x,u,.)$ are monotone non-decreasing, and $A(x,.,y)$ is monotone non-increasing, i.e., for any $x,u,y\in X$

$$\begin{aligned} x_1,x_2\in X,\ x_1\preceq x_2 &\Longrightarrow A(x_1,u,y)\preceq A(x_2,u,y),\\ u_1,u_2\in X,\ u_1\preceq u_2 &\Longrightarrow A(x,u_1,y)\succeq A(x,u_2,y),\\ y_1,y_2\in X,\ y_1\preceq y_2 &\Longrightarrow A(x,u,y_1)\preceq A(x,u,y_2). \end{aligned}$$

The organization of this paper can be described as follows. In section 2, after introducing the definition of cooperative and competitive mixed monotone vector operators, we present two fixed point theorems corresponding to these two cases. We prove the first result and leave the second to the reader, since the steps of the proof will be analogous. In section 3, we give some applications of the results obtained in section 2 on the existence and uniqueness of solutions of system (1.1). Our results will be illustrated by concrete examples.

2. Fixed Point Theorems

Inspired by the works [25] and [26], we present in this section our fixed point theorems for a system of two operators with three variables, which can be written as a vector operator. In other words, if $(X, \preceq)$ is a partially ordered set and, if $A_1, A_2 : X \times X \times X \longrightarrow X$ are two operators, then we define the vector operator $\Phi = (A_1, A_2) : X \times X \times X \times X \longrightarrow X \times X$, noted $\Phi = (A_1, A_2)$, by

$$\Phi(x, y, u, v) = (A_1(x, u, y), A_2(x, v, y)),\ \forall x, y, u, v \in X. \tag{2.1}$$

Then, we introduce the following definition.

Definition 2.1. Let $(X, \preceq)$ be a partially ordered set. Let $A_1, A_2 : X \times X \times X \longrightarrow X$ be two operators and $\Phi = (A_1, A_2)$ be given as in (2.1).

(i) We say that the operator $\Phi = (A_1, A_2)$ is a cooperative mixed monotone vector operator if A_1, A_2 are mixed monotone as in Definition 1.1.

(ii) We say that $\Phi = (A_1, A_2)$ is a competitive mixed monotone vector operator if $A_1(., u, y)$, $A_2(x, u, .)$ are monotone non-decreasing, and $A_1(x, ., y)$, $A_1(x, u, .)$, $A_2(x, ., y)$, $A_2(., u, y)$ are monotone non-increasing.

2.1. Cooperative mixed monotone vector operator

Lemma 2.2. *Let E be a real Banach space and P be a cone in E. Consider two operators $A_1, A_2 : P \times P \times P \longrightarrow P$ such that $\Phi = (A_1, A_2)$ satisfies the following conditions:*

(C_1) *$\Phi = (A_1, A_2)$ is cooperative mixed monotone, and there exist $h, k \in P$ with $h \neq \theta, k \neq \theta$ such that*

$$A_1(h, h, k) \in P_h \text{ and } A_2(h, k, k) \in P_k; \tag{2.2}$$

(C_2) *There exist positive-valued functions τ_1, τ_2 on interval (a, b), φ_1, φ_2 on $(a, b) \times (a, b) \times P \times P \times P$ and $\psi_1, \psi_2 : (a, b) \times (a, b) \times (0, 1]$ such that*

(i) *$\tau_1, \tau_2 : (a, b) \longrightarrow (0, 1)$ are surjections.*

(ii) *For any $x, u \in P_h$, for any $y, v \in P_k$, for any $t, s \in (a, b)$ and any $\varepsilon \in (0, 1]$*

$$\inf_{x,u\in[\varepsilon h, \frac{1}{\varepsilon} h], y\in[\varepsilon k, \frac{1}{\varepsilon} k]} \varphi_1(t, s, x, u, y) = \psi_1(t, s, \varepsilon) > \min\{\tau_1(t), \tau_2(s)\},$$
$$\inf_{x\in[\varepsilon h, \frac{1}{\varepsilon} h], v,y\in[\varepsilon k, \frac{1}{\varepsilon} k]} \varphi_2(t, s, x, v, y) = \psi_2(t, s, \varepsilon) > \min\{\tau_1(t), \tau_2(s)\} \tag{2.3}$$

and

$$\begin{aligned} A_1\left(\tau_1(t)x, \frac{1}{\tau_1(t)}u, \tau_2(s)y\right) &\succeq \varphi_1(t, s, x, u, y)A_1(x, u, y), \\ A_2\left(\tau_1(t)x, \frac{1}{\tau_2(s)}v, \tau_2(s)y\right) &\succeq \varphi_2(t, s, x, v, y)A_2(x, v, y). \end{aligned} \tag{2.4}$$

Then $A_1 : P_h \times P_h \times P_k \longrightarrow P_h$, $A_2 : P_h \times P_k \times P_k \longrightarrow P_k$. *Moreover, there exist* $x_0, u_0 \in P_h$, $y_0, v_0 \in P_k$ *and* $r \in (0,1)$ *such that*

$$\begin{cases} ru_0 \preceq x_0 \preceq u_0, \\ rv_0 \preceq y_0 \preceq v_0 \end{cases} \quad \text{and} \quad \begin{cases} x_0 \preceq A_1(x_0, u_0, y_0) \preceq A_1(u_0, x_0, v_0) \preceq u_0, \\ y_0 \preceq A_2(x_0, v_0, y_0) \preceq A_2(u_0, y_0, v_0) \preceq v_0. \end{cases} \tag{2.5}$$

Proof. For any $x, u \in P_h$ and $y, v \in P_k$ there exists $\lambda_* \in (0,1)$ such that

$$\lambda_* h \preceq x, u \preceq \frac{1}{\lambda_*} h \quad \text{and} \quad \lambda_* k \preceq y, v \preceq \frac{1}{\lambda_*} k.$$

It follows from $(C_2)(i)$ that there exist $t_*, s_* \in (a,b)$ such that $\tau_1(t_*) = \lambda_*$ and $\tau_2(s_*) = \lambda_*$, which gives

$$\tau_1(t_*) h \preceq x, u \preceq \frac{1}{\tau_1(t_*)} h \quad \text{and} \quad \tau_2(s_*) k \preceq y, v \preceq \frac{1}{\tau_2(s_*)} k.$$

Then, by the mixed monotone properties of operators A_1, A_2 and condition $(C_2)(ii)$, we have

$$\begin{aligned} A_1(x,u,y) &\succeq A_1\left(\tau_1(t_*)h, \frac{1}{\tau_1(t_*)}h, \tau_2(s_*)k\right) \\ &\succeq \varphi_1(t_*, s_*, h, h, k) A_1(h,h,k) \end{aligned}$$

and

$$\begin{aligned} A_1(x,u,y) &\preceq A_1\left(\frac{1}{\tau_1(t_*)}h, \tau_1(t_*)h, \frac{1}{\tau_2(s_*)}k\right) \\ &\preceq \frac{1}{\varphi_1(t_*, s_*, \frac{1}{\tau_1(t_*)}h, \tau_1(t_*)h, \frac{1}{\tau_2(s_*)}k)} A_1(h,h,k). \end{aligned}$$

From $A_1(h,h,k) \in P_h$, we have $A_1(x,u,y) \in P_h$ and hence, $A_1 : P_h \times P_h \times P_k \longrightarrow P_h$. Analogously we obtain $A_2 : P_h \times P_k \times P_k \longrightarrow P_k$.

Now, since $A_1(h,h,k) \in P_h, A_2(h,k,k) \in P_k$, there exists $\lambda_0 \in (0,1)$ such that

$$\lambda_0 h \preceq A_1(h,h,k) \preceq \frac{1}{\lambda_0} h \quad \text{and} \quad \lambda_0 k \preceq A_2(h,k,k) \preceq \frac{1}{\lambda_0} k.$$

It follows from $(C_2)(i)$ that there exist $t_0, s_0 \in (a,b)$ such that $\tau_1(t_0) = \lambda_0$ and $\tau_2(s_0) = \lambda_0$, which gives

$$\tau_1(t_0) h \preceq A_1(h,h,k) \preceq \frac{1}{\tau_1(t_0)} h \quad \text{and} \quad \tau_2(s_0) k \preceq A_2(h,k,k) \frac{1}{\tau_2(s_0)} k. \tag{2.6}$$

Set $\varepsilon_0 = \min\{\tau_1(t_0), \tau_2(s_0)\}$. Then we have $\psi_i(t_0, s_0, \varepsilon_0^n) \leq \psi_i(t_0, s_0, \varepsilon_0^{n-1})$, for all $n = 1, 2, ...$ and $i = 1, 2$. By $(C_2)(ii)$, we can choose a positive integer m such that

$$\prod_{i=1}^{m} \left[\frac{\psi_1(t_0, s_0, \varepsilon_0^i)}{\varepsilon_0}\right] \geq \frac{1}{\varepsilon_0} \text{ and } \prod_{i=1}^{m} \left[\frac{\psi_2(t_0, s_0, \varepsilon_0^i)}{\varepsilon_0}\right] \geq \frac{1}{\varepsilon_0}. \tag{2.7}$$

Put $x_0 = \varepsilon_0^m h$, $u_0 = \frac{1}{\varepsilon_0^m} h$, $v_0 = \frac{1}{\varepsilon_0^m} k$ and $y_0 = \varepsilon_0^m k$. It is clear that $x_0, u_0 \in P_h$ with $x_0 = \varepsilon_0^{2m} u_0 < u_0$ and $y_0, v_0 \in P_k$ with $y_0 = \varepsilon_0^{2m} v_0 < v_0$. Furtheremore, for any $r \in (0, \varepsilon_0^{2m}) \subset (0,1)$, $x_0 \succeq r u_0$ and $y_0 \succeq r v_0$. Also, by the mixed monotone properties, $A_1(x_0, u_0, y_0) \preceq A_1(u_0, x_0, v_0)$ and $A_2(x_0, v_0, y_0) \preceq A_2(u_0, y_0, v_0)$. Moreover, combining $(C_2)(ii)$ with (2.6) and (2.7), we have on the one hand,

$$\begin{aligned}
A_1(x_0,u_0,y_0) &= A_1\left(\tau_1(t_0)[\tau_1(t_0)]^{m-1}h, \frac{1}{\tau_1(t_0)}\frac{1}{[\tau_1(t_0)]^{m-1}}h, \tau_2(s_0)[\tau_2(s_0)]^{m-1}k\right)\\
&\succeq \varphi_1\left(t_0, s_0, [\tau_1(t_0)]^{m-1}h, \frac{1}{[\tau_1(t_0)]^{m-1}}h, [\tau_2(s_0)]^{m-1}k\right)\\
&\quad A_1\left([\tau_1(t_0)]^{m-1}h, \frac{1}{[\tau_1(t_0)]^{m-1}}h, [\tau_2(s_0)]^{m-1}k\right)\\
&\succeq \psi_1(t_0, s_0, \varepsilon_0^{m-1}) A_1\left([\tau_1(t_0)]^{m-1}h, \frac{1}{[\tau_1(t_0)]^{m-1}}h, [\tau_2(s_0)]^{m-1}k\right)\\
&\succeq \psi_1(t_0, s_0, \varepsilon_0^{m-1}) ... \psi_1(t_0, s_0, 1) A_1(h, h, k)\\
&\succeq \psi_1(t_0, s_0, \varepsilon_0^{m-1}) ... \psi_1(t_0, s_0, 1) \tau_1(t_0) h\\
&\succeq \psi_1(t_0, s_0, \varepsilon_0^{m}) ... \psi_1(t_0, s_0, \varepsilon_0) \varepsilon_0 h\\
&\succeq \varepsilon_0^m h = x_0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A_1(u_0,x_0,v_0) &= A_1\left(\frac{1}{[\tau_1(t_0)]^m}h, [\tau_1(t_0)]^m h, \frac{1}{[\tau_2(s_0)]^m}k\right)\\
&\preceq \frac{1}{\varphi_1\left(t_0, s_0, \frac{1}{[\tau_1(t_0)]^m}h, [\tau_1(t_0)]^m h, \frac{1}{[\tau_2(s_0)]^m}k\right)}\\
&\quad A_1\left(\frac{1}{[\tau_1(t_0)]^{m-1}}h, [\tau_1(t_0)]^{m-1}h, \frac{1}{[\tau_2(s_0)]^{m-1}}k\right)\\
&\preceq \frac{1}{\psi_1(t_0, s_0, \varepsilon_0^m)} A_1\left(\frac{1}{[\tau_1(t_0)]^{m-1}}h, [\tau_1(t_0)]^{m-1}h, \frac{1}{[\tau_2(s_0)]^{m-1}}k\right)\\
&\preceq \frac{1}{\psi_1(t_0, s_0, \varepsilon_0^m)} \cdots \frac{1}{\psi_1(t_0, s_0, \varepsilon_0)} A_1(h, h, k)\\
&\preceq \frac{1}{\psi_1(t_0, s_0, \varepsilon_0^m)} \cdots \frac{1}{\psi_1(t_0, s_0, \varepsilon_0)} \frac{1}{\tau_1(t_0)} h\\
&\preceq \frac{1}{\varepsilon_0^m} h = u_0.
\end{aligned}$$

in a similar way, we obtain

$$A_2(x_0, v_0, y_0) \succeq y_0 \quad \text{and} \quad A_2(u_0, y_0, v_0) \preceq v_0. \qquad \square$$

Theorem 2.3. *Let P be a normal cone in a Banach space E. Consider two operators $A_1, A_2 : P \times P \times P \longrightarrow P$ such that $(C_1), (C_2)$ in Lemma 2.2 hold. Then* *the operator* $\Phi = (A_1, A_2) : P_h \times P_k \times P_h \times P_k \longrightarrow P_h \times P_k$, *defined by (2.1), has a unique fixed point* $(x^*, y^*) \in P_h \times P_k$, *that is,* $\Phi(x^*, y^*, x^*, y^*) = (x^*, y^*)$, *or equivalently* $A_1(x^*, x^*, y^*) = x^*$ *and* $A_2(x^*, y^*, y^*) = y^*$. *Moreover, for any initial* $x_0, u_0 \in P_h$ *and* $y_0, v_0 \in P_k$, *constructing successively the sequences*

$$\begin{array}{ll} x_n = A_1(x_{n-1}, u_{n-1}, y_{n-1}), & y_n = A_2(x_{n-1}, v_{n-1}, y_{n-1}), \\ u_n = A_1(u_{n-1}, x_{n-1}, v_{n-1}), & v_n = A_2(u_{n-1}, y_{n-1}, v_{n-1}), \end{array} n = 1, 2, ..., \tag{2.8}$$

we have $\|x_n - x^*\| \longrightarrow 0$, $\|u_n - x^*\| \longrightarrow 0$ *and* $\|y_n - y^*\| \longrightarrow 0$, $\|v_n - y^*\| \longrightarrow 0$ *(as* $n \longrightarrow \infty$*).*

Proof. Let $x_0, u_0 \in P_h$, $y_0, v_0 \in P_k$ and $r \in (0, 1)$ be as obtained in Lemma 2.2 . By constructing successively the sequences as in (2.8), and using Lemma 2.2 combined with the mixed monotone property of the operators A_1, A_2, we obtain

$$\begin{aligned} x_0 \preceq x_1 \preceq ... \preceq x_n \preceq ... \preceq u_n \preceq ... \preceq u_1 \preceq u_0, \\ y_0 \preceq y_1 \preceq ... \preceq y_n \preceq ... \preceq v_n \preceq ... \preceq v_1 \preceq v_0. \end{aligned}$$

In addition, we have

$$\begin{aligned} x_n \succeq x_0 \succeq r u_0 \succeq r u_n, \\ y_n \succeq y_0 \succeq r v_0 \succeq r v_n, \end{aligned} \quad n = 1, 2, ...$$

We put

$$r_n = \sup\{r > 0 : x_n \succeq r u_n \text{ and } y_n \succeq r v_n\},\ n = 1, 2, ...$$

Then, $x_n \succeq r_n u_n$ and $y_n \succeq r_n v_n$, $n = 1, 2, ...$, and therefore

$$\begin{aligned} x_{n+1} \succeq x_n \succeq r_n u_n \succeq r_n u_{n+1}, \\ y_{n+1} \succeq y_n \succeq r_n v_n \succeq r_n v_{n+1}, \end{aligned} \quad n = 1, 2, ...$$

Hence, $r_{n+1} \geq r_n$, that is, $\{r_n\}$ is an increasing convergent sequence with $\{r_n\} \subset (0, 1]$. Set $r^* = \lim_{n \longrightarrow \infty} r_n$. We claim that $r^* = 1$, otherwise, $0 < r_n \leq r^* < 1$. By $(C_2)(i)$, there exist $t^*, s^* \in (a, b)$ such that $\tau_1(t^*) = r^*$ and $\tau_2(s^*) = r^*$. We distinguish two cases.

First case. There exists n_0 such that $r_{n_0} = r^*$. Thus, for all $n \geq n_0$ we have $r_n = r^*$ and

$$\begin{aligned} x_{n+1} = A_1(x_n, u_n, y_n) &\succeq A_1\left(r_n u_n, \frac{1}{r_n} x_n, r_n v_n\right) \\ &= A_1\left(\tau_1(t^*) u_n, \frac{1}{\tau_1(t^*)} x_n, \tau_2(s^*) v_n\right) \\ &\succeq \varphi_1(t^*, s^*, u_n, x_n, v_n) A_1(u_n, x_n, v_n) \\ &\succeq \psi_1\left(t^*, s^*, \varepsilon_0^m\right) u_{n+1}, \end{aligned}$$

Analogously we have

$$y_{n+1} \succeq \psi_2(t^*, s^*, \varepsilon_0^m)v_{n+1}.$$

Which means that

$$\begin{aligned} r_{n+1} = r^* &\geq \min\{\psi_1(t^*, s^*, \varepsilon_0^m), \psi_2(t^*, s^*, \varepsilon^m)\} \\ &> \min\{\tau_1(t^*), \tau_2(s^*)\} = r^*. \end{aligned}$$

This is a contradiction.
Second case. For all integer n, $r_n < r^* < 1$, then $0 < \dfrac{r_n}{r^*} < 1$. By $(C_2)(i)$, there exist $\alpha_n, \beta_n \in (a, b)$ such that $\tau_1(\alpha_n) = \frac{r_n}{r^*} = \tau_2(\beta_n)$. In this case we have

$$\begin{aligned} x_{n+1} &= A_1(x_n, u_n, y_n) \\ &\succeq A_1\left(r_n u_n, \frac{1}{r_n}x_n, r_n v_n\right) \\ &= A_1\left(\frac{r_n}{r^*}r^* u_n, \frac{r^*}{r_n}\frac{1}{r^*}x_n, \frac{r_n}{r^*}r^* v_n\right) \\ &= A_1\left(\tau_1(\alpha_n)r^* u_n, \frac{1}{\tau_1(\alpha_n)}\frac{1}{r^*}x_n, \tau_2(\beta_n)r^* v_n\right) \\ &\succeq \varphi_1\left(\alpha_n, \beta_n, r^* u_n, \frac{1}{r^*}x_n, r^* v_n\right) A_1\left(r^* u_n, \frac{1}{r^*}x_n, r^* v_n\right) \\ &\succeq \psi_1(\alpha_n, \beta_n, (r^*\varepsilon_0)^m)\psi_1(t^*, s^*, \varepsilon_0^m)u_{n+1}. \end{aligned}$$

Analogously we obtain

$$y_{n+1} \succeq \psi_2(\alpha_n, \beta_n, (r^*\varepsilon_0)^m)\psi_2(t^*, s^*, \varepsilon_0^m)v_{n+1}.$$

It follows that

$$\begin{aligned} r_{n+1} &\geq \min\{\psi_1(\alpha_n, \beta_n, (r^*\varepsilon_0)^m)\psi_1(t^*, s^*, \varepsilon_0^m), \psi_2(\alpha_n, \beta_n, (r^*\varepsilon_0)^m)\psi_2(t^*, s^*, \varepsilon_0^m)\} \\ &\geq \min\{\tau_1(\alpha_n)\psi_1(t^*, s^*, \varepsilon_0^m), \tau_2(\beta_n)\psi_2(t^*, s^*, \varepsilon_0^m)\} \\ &= \min\{\frac{r_n}{r^*}\psi_1(t^*, s^*, \varepsilon_0^m), \frac{r_n}{r^*}\psi_2(t^*, s^*, \varepsilon_0^m)\}. \end{aligned}$$

If $n \longrightarrow \infty$, we get

$$\begin{aligned} r^* &\geq \min\{\psi_1(t^*, s^*, \varepsilon_0^m), \psi_2(t^*, s^*, \varepsilon_0^m)\} \\ &> \min\{\tau_1(t^*), \tau_2(s^*)\} = r^*. \end{aligned}$$

This is also a contradiction.

Now, by the same reasoning as in [4, Theore 2.2] and [26, Lemma 2.1] we obtain $\lim_{n\to\infty} x_n = \lim_{n\to\infty} u_n = x^*$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} v_n = y^*$. Hence

$$\begin{aligned} x_{n+1} = A_1(x_n, u_n, y_n) \preceq A_1(x^*, x^*, y^*) \preceq A_1(u_n, x_n, v_n) = u_{n+1}, \\ y_{n+1} = A_2(x_n, v_n, y_n) \preceq A_2(x^*, y^*, y^*) \preceq A_2(u_n, y_n, v_n) = v_{n+1}. \end{aligned}$$

If $n \longrightarrow \infty$, we get $x^* = A_1(x^*, x^*, y^*)$ and $y^* = A_2(x^*, y^*, y^*)$. That is, (x^*, y^*) is a fixed point of Φ in $P_h \times P_k$.

Now, for any $x_0, u_0 \in P_h$ and $y_0, v_0 \in P_k$, we can choose a small number $\lambda_1 \in (0,1)$ such that

$$\lambda_1 h \preceq x_0, u_0 \preceq \frac{1}{\lambda_1} h, \quad \lambda_1 k \preceq y_0, v_0 \preceq \frac{1}{\lambda_1} k.$$

From $(C_2)(i)$, there exist $t_1, s_1 \in (a,b)$ such that $\tau_1(t_1) = \lambda_1 = \tau_2(s_1)$, and hence

$$\tau_1(t_1) h \preceq x_0, u_0 \preceq \frac{1}{\tau_1(t_1)} h, \quad \tau_2(s_1) k \preceq y_0, v_0 \preceq \frac{1}{\tau_2(s_1)} k.$$

Similarly to Lemma 2.2, set $\varepsilon_1 = \min\{[\tau_1(t_1)], [\tau_2(s_1)]\}$ and choose a sufficiently large integer m such that

$$\prod_{i=1}^{m} \left[\frac{\psi_1(t_1, s_1, \varepsilon_1^i)}{\varepsilon_1} \right] \geq \frac{1}{\varepsilon_1} \text{ and } \prod_{i=1}^{m} \left[\frac{\psi_2(t_1, s_1, \varepsilon_1^i)}{\varepsilon_1} \right] \geq \frac{1}{\varepsilon_1}.$$

Put $\overline{x}_0 = \varepsilon_1^m h$, $\overline{u}_0 = \frac{1}{\varepsilon_1^m} h$, $\overline{v}_0 = \frac{1}{\varepsilon_1^m} k$ and $\overline{y}_0 = \varepsilon_1^m k$. Then, $\overline{x}_0, \overline{u}_0 \in P_h$ and $\overline{y}_0, \overline{v}_0 \in P_k$ with $\overline{x}_0 < x_0, u_0 < \overline{u}_0$ and $\overline{y}_0 < y_0, v_0 < \overline{v}_0$. Construct the sequences

$$\begin{array}{ll} \overline{x}_n = A_1(\overline{x}_{n-1}, \overline{u}_{n-1}, \overline{y}_{n-1}), & \overline{y}_n = A_2(\overline{x}_{n-1}, \overline{v}_{n-1}, \overline{y}_{n-1}), \\ \overline{u}_n = A_1(\overline{u}_{n-1}, \overline{x}_{n-1}, \overline{v}_{n-1}), & \overline{v}_n = A_2(\overline{u}_{n-1}, \overline{y}_{n-1}, \overline{v}_{n-1}), \end{array} n = 1, 2, \ldots$$

Therefore, there exist $(u^*, v^*) \in P_h \times P_k$ such that $\Phi(u^*, v^*, u^*, v^*) = (u^*, v^*)$ and $\lim_{n\to\infty} \overline{x}_n = \lim_{n\to\infty} \overline{u}_n = u^*$, $\lim_{n\to\infty} \overline{y}_n = \lim_{n\to\infty} \overline{v}_n = v^*$. By the uniqueness of fixed points of operator Φ in $P_h \times P_k$, we have $x^* = u^*$ and $y^* = v^*$. Moreover, by induction, $\overline{x}_n \preceq x_n, u_n \preceq \overline{u}_n$ and $\overline{y}_n \preceq y_n, v_n \preceq \overline{v}_n$, for $n = 1, 2, \ldots$ Finally, by the normality of the cone P we get $\lim_{n\to\infty} x_n = \lim_{n\to\infty} u_n = x^*$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} v_n = v^*$. □

2.2. Competitive Mixed Monotone Vector Operator

We will give below, another result of existence and uniqueness of a fixed point concerning competitive mixed monotone vector operators. Similarly to the case of cooperative mixed monotone vector operators, we will have a lemma, then the existence theorem. The steps of the proofs are not very far from those of the previous case, for that we leave them to the reader.

Lemma 2.4. *Let E be a real Banach space and P be a cone in E. Consider two operators $A_1, A_2 : P \times P \times P \longrightarrow P$ such that $\Phi = (A_1, A_2)$ satisfies the following conditions:*

$(C_1)'$ *$\Phi = (A_1, A_2)$ is competitive mixed monotone, and there exist $h, k \in P$ with $h \neq \theta, k \neq \theta$ such that*

$$A_1(h, h, k) \in P_h \text{ and } A_2(h, k, k) \in P_k;$$

$(C_2)'$ *There exist positive-valued functions* τ *on interval* (a,b), φ_1, φ_2 *on* $(a,b) \times P \times P \times P$ *and* $\psi_1, \psi_2 : (a,b) \times (0,1]$ *such that*

(i) $\tau : (a,b) \longrightarrow (0,1)$ *is surjection.*

(ii) *For any* $x, u \in P_h$, *for any* $y, v \in P_k$, *for any* $t \in (a,b)$ *and any* $\varepsilon \in (0,1]$

$$\inf_{x,u \in [\varepsilon h, \frac{1}{\varepsilon} h], y \in [\varepsilon k, \frac{1}{\varepsilon} k]} \varphi_1(t,x,u,y) = \psi_1(t,\varepsilon) > \tau(t),$$
$$\inf_{x \in [\varepsilon h, \frac{1}{\varepsilon} h], v,y \in [\varepsilon k, \frac{1}{\varepsilon} k]} \varphi_2(t,x,v,y) = \psi_2(t,\varepsilon) > \tau(t)$$

and

$$A_1\left(\tau(t)x, \frac{1}{\tau(t)}u, \frac{1}{\tau(t)}y\right) \succeq \varphi_1(t,x,u,y) A_1(x,u,y),$$
$$A_2\left(\frac{1}{\tau(t)}x, \frac{1}{\tau(t)}v, \tau(t)y\right) \succeq \varphi_2(t,x,v,y) A_2(x,v,y).$$

Then $A_1 : P_h \times P_h \times P_k \longrightarrow P_h$, $A_2 : P_h \times P_k \times P_k \longrightarrow P_k$. *Moreover, there exist* $x_0, u_0 \in P_h$, $y_0, v_0 \in P_k$ *and* $r \in (0,1)$ *such that*

$$\begin{cases} ru_0 \preceq x_0 \preceq u_0, \\ rv_0 \preceq y_0 \preceq v_0 \end{cases} \quad \text{and} \quad \begin{cases} x_0 \preceq A_1(x_0,u_0,v_0) \preceq A_1(u_0,x_0,y_0) \preceq u_0, \\ y_0 \preceq A_2(u_0,v_0,y_0) \preceq A_2(x_0,y_0,v_0) \preceq v_0. \end{cases}$$

Theorem 2.5. *Let* P *be a normal cone in a Banach space* E. *Consider two operators* $A_1, A_2 : P \times P \times P \longrightarrow P$ *such that* $(C_1)'$ *and* $(C_2)'$ *of Lemma 2.4 hold. Then, the operator* $\Phi = (A_1, A_2) : P_h \times P_k \times P_h \times P_k \longrightarrow P_h \times P_k$ *defined by (2.1) has a unique fixed point* $(x^*, y^*) \in P_h \times P_k$, *that is,* $\Phi(x^*, y^*, x^*, y^*) = (x^*, y^*)$, *or equivalently* $A_1(x^*, x^*, y^*) = x^*$ *and* $A_2(x^*, y^*, y^*) = y^*$. *Moreover, for any initial* $x_0, u_0 \in P_h$ *and* $y_0, v_0 \in P_k$, *constructing successively the sequences*

$$\begin{aligned} x_n &= A_1(x_{n-1}, u_{n-1}, v_{n-1}), \quad & y_n &= A_2(u_{n-1}, v_{n-1}, y_{n-1}), \\ u_n &= A_1(u_{n-1}, x_{n-1}, y_{n-1}), \quad & v_n &= A_2(x_{n-1}, y_{n-1}, v_{n-1}), \end{aligned} \quad n = 1, 2, ...,$$

we have $\|x_n - x^*\| \longrightarrow 0$, $\|u_n - x^*\| \longrightarrow 0$ *and* $\|y_n - y^*\| \longrightarrow 0$, $\|v_n - y^*\| \longrightarrow 0$ *(as* $n \longrightarrow \infty$*).*

3. Applications

In this section, we study the existence and uniqueness of the solution to a system of nonlinear boundary value problems (SNBVPs for short), as applications to the fixed point theorems in the previous section.

Consider the following systems of NBVPs

$$\begin{cases} -x''(t) + \theta^2 x(t) = \lambda f(t, x(t), x(t), y(t)), \ 0 < t < 1, \\ -y''(t) + \omega^2 y(t) = \beta g(t, x(t), y(t), y(t)), \ 0 < t < 1, \\ x'(0) = x'(1) = 1, \ y'(0) = y'(1) = 0, \end{cases} \tag{3.1}$$

where θ and ω are positive constants, λ and β are positive parameters, f and g are continuous functions.

Note that the existence results of the scalar version of the above systems, namely nonlinear boundary value problems (NBVPs for short), was studied by many researchers (see, e.g., [2, 12, 22, 23, 24]) by using fixed piont theorem in cone.

Let $C[0,1]$ be the Banach space equipped with the sup norm. Set

$$P = \{x \in C[0,1], x(t) \geq 0, t \in [0,1]\}.$$

It is easy to show that P is a normal cone in $C[0,1]$ of which the normality constant is 1. By a positive solution of (3.1) we means a couple of functions $(x, y) \in C^2[0,1] \times C^2[0,1]$, with $x(t)$ and $y(t)$ are positive on $(0,1)$, such that (x, y) satisfies the system of differential equations and the boundary conditions in (3.1). It is well known that the Green's function $G_m(t,s)$ for the boundary problem

$$\begin{cases} -x''(t) + m^2 x(t) = 0, \ 0 < t < 1, \\ x'(0) = x'(1) = 1 \end{cases} \tag{3.2}$$

is

$$G_m(t,s) = \frac{1}{\rho_m} \begin{cases} \psi_m(s)\psi_m(1-t), \ 0 \leq s \leq t \leq 1, \\ \psi_m(t)\psi_m(1-s), \ 0 \leq t \leq s \leq 1, \end{cases} \tag{3.3}$$

where $\rho_m = \frac{1}{2}m(e^m - e^{-m})$, $\psi_m(t) = \frac{1}{2}(e^{mt} + e^{-mt})$. In addition, $\psi_m(t)$ is increasing in $t \in [0,1]$ and $0 < G_m(t,s) \leq G_m(t,t), \ 0 \leq t, s \leq 1$. Also, we have the following lemma.

Lemma 3.1.([24, Lemma 2.1]) *Let $G_m(t,s)$ be the Green's function for the boundary value problem (3.2). Then*

$$G_m(t,s) \geq C\psi_m(t)\psi_m(1-t)G_m(t_0,s), \ t, t_0, s \in [0,1],$$

where $C = \frac{1}{\psi_m^2(1)}$.

In the sequel, we will need the following notations.

For $t \in [0,1]$, let

$$h(t) = \psi_\theta(t)\psi_\theta(1-t) \quad \text{and} \quad k(t) = \psi_\omega(t)\psi_\omega(1-t), \tag{3.4}$$

where constants θ, ω replace m in the BVP (3.2). Then it is easy to check that

$$h_0 = \min_{t\in[0,1]} h(t) = \frac{1}{4}(e^\theta + e^{-\theta} + 2), \quad h^0 = \max_{t\in[0,1]} h(t) = \frac{1}{2}(e^\theta + e^{-\theta}),$$
$$k_0 = \min_{t\in[0,1]} k(t) = \frac{1}{4}(e^\omega + e^{-\omega} + 2), \quad k^0 = \max_{t\in[0,1]} k(t) = \frac{1}{2}(e^\omega + e^{-\omega}).$$

Now, we are able to formulate and prove the main results in this section. The following theorems give sufficient conditions so that SNBVP (3.1) has a unique positive solution.

Theorem 3.2. *Let $f, g : [0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ be functions satisfying*

(H_1) *$f, g : [0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ are continuous functions such that for all $t \in [0,1]$, the functions $f(t,.,u,y)$, $f(t,x,u,.)$, $g(t,.,u,y)$, $g(t,x,u,.)$ are nondecreasing and $f(t,x,.,y)$, $g(t,x,.,y)$ are nonincreasing.*

(H_2) *There exist positive-value functions τ_i on $(0,1)$, ϕ_i on $(0,1)\times(0,1)\times[0,\infty)\times[0,\infty)\times[0,\infty)$ $(i = 1,2)$ such that*

(i) $\tau_1, \tau_2 : (0,1) \to (0,1)$ *are surjections.*

(ii) For all $x, u, y \in (0,\infty)$, for all $t \in [0,1]$ and all $\gamma, \nu \in (0,1)$

$$\begin{aligned} f\left(t, \tau_1(\gamma)x, \frac{1}{\tau_1(\gamma)}u, \tau_2(\nu)y\right) &\geq \phi_1(\gamma,\nu,x,u,y) f(t,x,u,y), \\ g\left(t, \tau_1(\gamma)x, \frac{1}{\tau_2(\nu)}u, \tau_2(\nu)y\right) &\geq \phi_2(\gamma,\nu,x,u,y) g(t,x,u,y). \end{aligned} \tag{3.5}$$

Moreover, for any $\varepsilon \in (0,1)$

$$\begin{aligned} \inf_{x,u\in[\varepsilon h_0,\frac{1}{\varepsilon}h^0],y\in[\varepsilon k_0,\frac{1}{\varepsilon}k^0]} \phi_1(\gamma,\nu,x,u,y) &> \min\{\tau_1(\gamma),\tau_2(\nu)\}, \\ \inf_{x\in[\varepsilon h_0,\frac{1}{\varepsilon}h^0],u,y\in[\varepsilon k_0,\frac{1}{\varepsilon}k^0]} \varphi_2(\gamma,\nu,x,u,y) &> \min\{\tau_1(\gamma),\tau_2(\nu)\}. \end{aligned} \tag{3.6}$$

(H_3) *There exist continuous functions $a_i, b_i : [0,1] \to \mathbb{R}$ $(i = 1,2)$ and numbers $p, p', q, q' \in \mathbb{R}$ such that*

$$\liminf_{|(x,y)|\to+\infty,u\to 0^+} \frac{f(t,x,u,y)}{|(x,y)|^p} = a_1(t), \quad \liminf_{|(x,y)|\to+\infty,u\to 0^+} \frac{g(t,x,u,y)}{|(x,y)|^{p'}} = b_1(t)$$

uniformly in $t \in [0,1]$ (with $|(x,y)| = |x|+|y|$is the usual norm in $\mathbb{R}\times\mathbb{R}$) and

$$\limsup_{|(x,y)|\to 0^+,u\to+\infty} \frac{f(t,x,u,y)}{|(x,y)|^q} = a_2(t), \quad \limsup_{(x,y)\to 0^+,u\to+\infty} \frac{g(t,x,u,y)}{|(x,y)|^{q'}} = b_2(t)$$

uniformly in $t \in [0,1]$. Moreover,

$$\overset{\circ}{A} \neq \varnothing \quad \text{and} \quad \overset{\circ}{B} \neq \varnothing, \tag{3.7}$$

where $A = \{s \in [0,1] : a_1(s) \neq 0\}$ and $B = \{s \in [0,1] : b_1(s) \neq 0\}$.

Then, the SNBVP (3.1) has a unique positive solution (x^*_λ, y^*_β) in $P_h \times P_k$.

Proof. We are going to prove that all hypotheses of Theorem 2.3 are verified for adequate vector operator. First, it is a standard result that (x, y) is a solution of the SNBVP (3.1) if, and only if

$$x(t) = \lambda \int_0^1 G_\theta(t,s) f(s, x(s), x(s), y(s)) ds,$$
$$y(t) = \beta \int_0^1 G_\omega(t,s) g(s, x(s), y(s), y(s)) ds,$$

where $G_\theta(t, s)$ and $G_\omega(t, s)$ are the Green's functions as in (3.2). Define

$$A_{1,\lambda}(x,u,y)(t) = \lambda \int_0^1 G_\theta(t,s) f(s, x(s), u(s), y(s)) ds,$$
$$A_{2,\beta}(x,v,y)(t) = \beta \int_0^1 G_\omega(t,s) g(s, x(s), v(s), y(s)) ds,$$

for any $x, y, u, v \in P$ and set $\Phi_{(\lambda,\beta)}(x, y, u, v) = (A_{1,\lambda}(x, u, y), A_{2,\beta}(x, v, y))$. Then, (x_λ, y_β) is a solution of SNBVP (3.1) if, and only if $\Phi_{(\lambda,\beta)}(x_\lambda, y_\beta, x_\lambda, y_\beta) = (x_\lambda, y_\beta)$.

By (H_1), it is easy to see that $A_{1,\lambda}, A_{2,\beta} : P \times P \times P \to P$ and that $A_{1,\lambda}, A_{2,\beta}$ are mixed monotone operators.

On the one hand, since $G_\theta(t, s) > 0$, for all $t, s \in [0, 1]$, using (3.7) we have for $t_1 \in [0, 1]$ fix, $\int_0^1 G_\theta(t_1, s) a_1(s) ds > 0$. It follows that for any $\varepsilon > 0$ verifying $\int_0^1 G_\theta(t_1, s)(a_1(s) - \varepsilon) ds > 0$, there exist numbers δ, M with $0 < \delta < M$ such that

$$f(t,x,u,y) \geq (a(t) - \varepsilon)(x+y)^p, \forall (x,y) : (x+y) \geq M, \forall u \leq \delta, \forall t \in [0,1].$$

Choose $\alpha \in (0, 1)$ satisfying $\frac{1}{\alpha}(h_0 + k_0) \geq M$ and $\alpha h^0 \leq \delta$. It follows that there exist $\gamma, \nu \in (0, 1)$ such that $\tau_1(\gamma) = \alpha$ and $\tau_2(\nu) = \alpha$. Then for all $t \in [0, 1]$

$$\begin{aligned}
&A_{1,\lambda}(h,h,k)(t) \\
&= \lambda \int_0^1 G_\theta(t,s) f(s, h(s), h(s), k(s)) ds \\
&= \lambda \int_0^1 G_\theta(t,s) f\left(s, \tau_1(\gamma)\frac{1}{\tau_1(\gamma)}h(s), \frac{1}{\tau_1(\gamma)}\tau_1(\gamma)h(s), \tau_2(\nu)\frac{1}{\tau_2(\nu)}k(s)\right) ds \\
&\geq \lambda \int_0^1 G_\theta(t,s) \phi_1\left(\gamma, \nu, \frac{1}{\tau_1(\gamma)}h(s), \tau_1(\gamma)h(s), \frac{1}{\tau_2(\nu)}k(s)\right)(a_1(s) - \varepsilon)\left[\frac{1}{\alpha}(h(s)+k(s))\right]^p ds \\
&\geq \lambda \int_0^1 C\psi_\theta(t)\psi_\theta(1-t) G_\theta(t_1,s)(a_1(s)-\varepsilon)\alpha^{1-p}\left[h(s)+k(s)\right]^p ds \\
&\geq \lambda C h(t) \alpha^{1-p}(h_0+k_0)^p \int_0^1 G_\theta(t_1,s)(a_1(s)-\varepsilon) ds.
\end{aligned}$$

Thus

$$A_{1,\lambda}(h,h,k) \succeq \left(\lambda C \alpha^{1-p}(h_0+k_0)^p \int_0^1 G_\theta(t_1,s)(a_1(s)-\varepsilon) ds\right) h.$$

On the other hand, for any $\varepsilon' > 0$ there exist numbers δ', M' with $0 < \delta' < M'$ such that

$$f(t,x,u,y) \leq (a_2(t)+\varepsilon')(x+y)^q, \forall (x,y) : (x+y) \leq \delta', \forall u \geq M', \forall t \in [0,1].$$

Choose $\alpha' \in (0,1)$ satisfying $\frac{1}{\alpha'}h_0 \geq M'$, $\alpha'(h^0+k^0) \leq \delta'$. It follows that there exist $\gamma', \nu' \in (0,1)$ such that $\tau_1(\gamma') = \alpha'$ and $\tau_2(\nu') = \alpha'$. Then for all $t \in [0,1]$

$$\begin{aligned}
&A_{1,\lambda}(h,h,k)(t)\\
&= \lambda \int_0^1 G_\theta(t,s) f(s,h(s),h(s),k(s))ds\\
&\leq \lambda \int_0^1 G_\theta(t,t) f\left(s, \frac{1}{\tau_1(\gamma')}\tau_1(\gamma')h(s), \tau_1(\gamma')\frac{1}{\tau_1(\gamma')}h(s), \frac{1}{\tau_2(\nu')}\tau_2(\nu')k(s)\right) ds\\
&\leq \lambda \int_0^1 G_\theta(t,t) \frac{1}{\phi_1(\gamma',\nu',h(s),h(s),k(s))}(a_2(s)+\varepsilon')\left[\alpha'(h(s)+k(s)\right]^q ds\\
&\leq \lambda \frac{1}{\rho_\theta} h(t)(\alpha')^{q-1}(h^0+k^0)^q \int_0^1 (a_2(s)+\varepsilon')ds.
\end{aligned}$$

Thus

$$A_{1,\lambda}(h,h,k) \preceq \left(\lambda \frac{1}{\rho_\theta}(\alpha')^{q-1}(h^0+k^0)^q \int_0^1 (a_2(s)+\varepsilon')ds\right) h.$$

Consequently, $A_{1,\lambda}(h,h,k) \in P_h$. Similarly, we get $A_{2,\beta}(h,k,k) \in P_k$. The verification of (C_1) in Lemma 2.2 is completed.

Next, we prove that (C_2) holds. Let $x,u \in P_h$, $y \in P_k$ and $\gamma,\nu \in (0,1)$. Set

$$\begin{aligned}
a(x,u,y) &= \min\{\inf_{s\in[0,1]} x(s), \inf_{s\in[0,1]} u(s), \inf_{s\in[0,1]} y(s)\},\\
b(x,u,y) &= \max\{\sup_{s\in[0,1]} x(s), \sup_{s\in[0,1]} u(s), \sup_{s\in[0,1]} y(s)\},
\end{aligned}$$

and define

$$\varphi_1(\gamma,\nu,x,u,y) = \inf_{\alpha,\eta,\mu\in[a(x,u,y),b(x,u,y)]} \phi_1(\gamma,\nu,\alpha,\eta,\mu).$$

Then, the first inequality of (2.3) in (C_2) is verified and

$$\begin{aligned}
&A_{1,\lambda}\left(\tau_1(\gamma)x, \frac{1}{\tau_1(\gamma)}u, \tau_2(\nu)y\right)(t)\\
&= \lambda \int_0^1 G_\theta(t,s) f(s,\tau_1(\gamma)x(s), \frac{1}{\tau_1(\gamma)}u(s), \tau_2(\nu)y(s))ds\\
&\geq \lambda \int_0^1 G_\theta(t,s)\phi_1(\gamma,\nu,x(s),u(s),y(s)) f(s,x(s),u(s),y(s))ds\\
&\geq \varphi_1(\gamma,\nu,x,u,y)\lambda \int_0^1 G_\theta(t,s) f(s,x(s),u(s),y(s))ds.
\end{aligned}$$

Which means that

$$A_{1,\lambda}\left(\tau_1(\gamma)x, \frac{1}{\tau_1(\gamma)}u, \tau_2(\nu)y\right) \succeq \varphi_1(\gamma,\nu,x,u,y)A_{1,\lambda}(x,u,y).$$

Analogously, we do the same reasoning for $A_{2,\beta}$. This complete the proof. $\square$

Remark 3.3. Note that in [26, Theorem 3.1], the authors suppose a condition on their function f, which is equivalent in our case to $f(t,h_0,h^0,k_0) > 0$ and $g(t,h_0,k^0,k_0) > 0$, for all $t \in [0,1]$. But, our condition (3.7) in Theorem 3.2 is less restrictive.

Example 3.4. Let $a, b : [0,1] \longrightarrow (0,+\infty)$ be continuous functions. For any positive numbers c, c', d, d' with $3c' \geq c > c'$ and $2d' \geq d > d'$, consider system (3.1) by setting

$$f(t,x,u,y) = a(t)(x+y)\frac{(x+y)^2+c}{(x+y)^2+c'} \quad \text{and} \quad g(t,x,u,y) = b(t)(x+y)\frac{(x+y)^3+d}{(x+y)^3+d'},$$

for all $x, u, y \in [0,+\infty)$ and all $t \in [0,1]$. Then, we have for any surjective functions $\tau_1, \tau_2 : (0,1) \longrightarrow (0,1)$, and $\gamma, \nu \in (0,1)$

$$\phi_1(\gamma,\nu,x,u,y) = \min\{\tau_1(\gamma),\tau_2(\nu)\}\frac{(\tau_1(\gamma)x+\tau_2(\nu)y)^2+c}{(\tau_1(\gamma)x+\tau_2(\nu)y)^2+c'} \; \frac{(x+y)^2+c'}{(x+y)^2+c} \quad \text{and}$$
$$\phi_2(\gamma,\nu,x,u,y) = \min\{\tau_1(\gamma),\tau_2(\nu)\}\frac{(\tau_1(\gamma)x+\tau_2(\nu)y)^2+d}{(\tau_1(\gamma)x+\tau_2(\nu)y)^2+d'} \; \frac{(x+y)^2+d'}{(x+y)^2+d}.$$

Thus, all hypotheses of Theorem 3.2 are verified. Therefore, system (3.1) with the above functions has a unique solution in $P_h \times P_k$, where functions h and k are given by (3.4).

As an application of Theorem 2.5, we give the following result. The proof in this case will be similar to that of Theorem 3.2. Since this is almost verbal, we leave it to the reader.

Theorem 3.5. *Let* $f, g : [0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ *be functions satisfying*

$(H_1)'$ $f, g : [0,1] \times [0,\infty) \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ *are continuous functions such that for all* $t \in [0,1]$, *the functions* $f(t,.,u,y)$, $g(t,x,u,.)$ *are nondecreasing and* $f(t,x,.,y)$, $f(t,x,u,.)$, $g(t,.,u,y)$,$g(t,x,.,y)$ *are nonincreasing.*

$(H_2)'$ *There exist positive-value function* τ *on* $(0,1)$, *positive-value functions* ϕ_1 *and* ϕ_2 *on* $(0,1) \times [0,\infty) \times [0,\infty) \times [0,\infty)$ *such that*

(i) $\tau : (0,1) \to (0,1)$ *is surjection.*

(ii) *For all $x, u, y \in (0, \infty)$, for all $t \in [0,1]$ and all $\gamma \in (0,1)$*

$$f\left(t, \tau(\gamma)x, \frac{1}{\tau(\gamma)}u, \frac{1}{\tau(\gamma)}y\right) \geq \phi_1(\gamma, x, u, y) f(t, x, u, y),$$

$$g\left(t, \frac{1}{\tau(\gamma)}x, \frac{1}{\tau(\gamma)}u, \tau(\gamma)y\right) \geq \phi_2(\gamma, x, u, y) g(t, x, u, y).$$

Moreover, for any $\varepsilon \in (0,1)$

$$\inf_{x,u \in [\varepsilon h_0, \frac{1}{\varepsilon}h^0], y \in [\varepsilon k_0, \frac{1}{\varepsilon}k^0]} \phi_1(\gamma, x, u, y) > \tau(\gamma),$$

$$\inf_{x \in [\varepsilon h_0, \frac{1}{\varepsilon}h^0], u, y \in [\varepsilon k_0, \frac{1}{\varepsilon}k^0]} \varphi_2(\gamma, x, u, y) > \tau(\gamma).$$

$(H_3)'$ *There exist continuous functions $a_i, b_i : [0,1] \to \mathbb{R}$ $(i = 1,2)$ and numbers $p, p', q, q' \in \mathbb{R}$ such that*

$$\liminf_{(u,y)\to(0^+,0^+)} \liminf_{u,y\neq 0, x\to+\infty} \frac{f(t,x,u,y)}{x^p} = a_1(t),$$

$$\liminf_{(x,u)\to(0^+,0^+)} \liminf_{x,u\neq 0, y\to+\infty} \frac{g(t,x,u,y)}{y^{p'}} = b_1(t)$$

uniformly in $t \in [0,1]$ and

$$\limsup_{x\to 0^+} \limsup_{x\neq 0, (u,y)\to(+\infty,+\infty)} \frac{f(t,x,u,y)}{x^q} = a_2(t),$$

$$\limsup_{y\to 0^+} \limsup_{y\neq 0, (x,u)\to(+\infty,+\infty)} \frac{g(t,x,u,y)}{y^{q'}} = b_2(t)$$

uniformly in $t \in [0,1]$. Moreover,

$$\overset{\circ}{A} \neq \varnothing \quad \text{and} \quad \overset{\circ}{B} \neq \varnothing,$$

where $A = \{s \in [0,1] : a_1(s) \neq 0\}$ and $B = \{s \in [0,1] : b_1(s) \neq 0\}$.

Then, the SNBVP (3.1) has a unique positive solution (x^*_λ, y^*_β) in $P_h \times P_k$.

Example 3.6. Let $a, b : [0,1] \longrightarrow (0, +\infty)$ be continuous functions. For any positive numbers c, c', d, d' with $3c' \geq c > c'$ and $2d' \geq d > d'$, consider system (3.1) by setting

$$f(t,x,u,y) = a(t)x\frac{x^2y + c}{x^2y + c'} \quad \text{and} \quad g(t,x,u,y) = b(t)y\frac{y^3x^2 + d}{y^3x^2 + d'},$$

for all $x, u, y \in [0, +\infty)$ and all $t \in [0,1]$. Then, we have for any surjective functions $\tau : (0,1) \longrightarrow (0,1)$, and $\gamma \in (0,1)$

$$\phi_1(\gamma, x, u, y) = \tau(\gamma)\frac{\tau(\gamma)x^2y + c}{\tau(\gamma)x^2y + c'} \quad \frac{x^2y + c'}{x^2y + c} \quad \text{and}$$

$$\phi_2(\gamma, x, u, y) = \tau(\gamma)\frac{\tau(\gamma)y^3x^2 + d}{\tau(\gamma)y^3x^2 + d'} \quad \frac{y^3x^2 + d'}{y^3x^2 + d}.$$

Thus, all hypotheses of Theorem 3.5 are verified. Therefore, system (3.1) with the above functions has a unique solution in $P_h \times P_k$, where functions h and k are given by (3.4).

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