KYUNGPOOK Math. J. 61(2021), 661-669
https://doi.org/10.5666/KMJ.2021.61.3.661
pISSN 1225-6951 eISSN 0454-8124

Complexity Issues of Perfect Roman Domination in Graphs

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Abstract. For a simple, undirected graph $G = (V, E)$, a perfect Roman dominating function (PRDF) $f : V \to \{0, 1, 2\}$ has the property that, every vertex u with $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The weight of a PRDF is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of a PRDF is called the perfect Roman domination number, denoted by $\gamma_R^P(G)$. Given a graph G and a positive integer k, the PRDF problem is to check whether G has a perfect Roman dominating function of weight at most k. In this paper, we first investigate the complexity of PRDF problem for some subclasses of bipartite graphs namely, star convex bipartite graphs and comb convex bipartite graphs. Then we show that PRDF problem is linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs, a subclass of split graphs.

1. Introduction

Let $G = (V, E)$ be a simple, undirected and connected graph with no isolated vertices. For a vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v) = \{u \mid u \in V, (u, v) \in E\}$ and the *closed neighborhood* of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. The *degree* $deg(v)$ of a vertex v is $|N_G(v)|$. The *maximum degree* of a graph G, denoted by Δ and *minimum degree* of a graph, denoted by δ are the maximum and minimum degree of its vertices. An *induced subgraph* is a graph formed from a subset D of vertices of G and all of the edges in G connecting pairs of vertices in that subset, denoted by $\langle D \rangle$. A *clique* is a subset of vertices of G such that every two distinct vertices in the subset are adjacent. An *independent set* is a set of vertices in which no two vertices are adjacent. A vertex v of G is said to be a *pendant vertex* if $deg(v) = 1$. A vertex v is called *isolated vertex* if $deg(v) = 0$. A bipartite graph $G = (X, Y, E)$ is called *tree convex* if there exists a tree $T = (X, F)$ such that, for each y in Y, the neighbors of y induce a subtree in T. When T is a star (comb), G is called *star* (comb) *convex bipartite* graph [6]. For undefined terminology and

* Corresponding Author.
Received November 6, 2019; revised October 12, 2020; accepted November 23, 2020.
2010 Mathematics Subject Classification: 05C69, 68Q25.
Key words and phrases: Roman domination, Perfect Roman domination, NP-complete.

notations we refer the reader to [3].

A vertex v in G dominates the vertices of its closed neighborhood. A set of vertices $S \subseteq V$ is a *dominating set* (DS) in G if for every vertex $u \in V \setminus S$, there exists at least one vertex $v \in S$ such that $(u, v) \in E$, i.e., $N_G[S] = V$. A vertex $u \in V \setminus S$ is said to be *undominated* if $N_G(u) \cap S = \emptyset$. The *domination number* is the minimum cardinality of a dominating set in G and is denoted by $\gamma(G)$ [15].

Roman domination has been introduced by Cockayne et al. in [4]. A function $f : V \to \{0, 1, 2\}$ is a *Roman Dominating Function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. In this case, we say that u is *Roman dominated* by v or v *Roman dominates* u. The weight of an RDF is the value $f(V) = \sum_{u \in V} f(u)$. The *Roman domination number* is the minimum weight of an RDF on G and is denoted by $\gamma_R(G)$. The literature on Roman domination in graphs has been surveyed in [10, 12].

The concept of perfect Roman domination was introduced in 2018 by Henning et al. in [8]. A function $f : V \to \{0, 1, 2\}$ is a *perfect Roman Dominating Function* (PRDF) on G, if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The weight of a PRDF is the value $f(V) = \sum_{u \in V} f(u)$. The *perfect Roman domination number* is the minimum weight of a PRDF on G and is denoted by $\gamma_R^P(G)$. The perfect Roman domination has been studied in [14, 9].

Given a graph G and a positive integer k, the PRDF problem is to check whether G has a perfect Roman dominating function of weight at most k. Banerjee et al. in [14] proved that the PRDF problem is NP-complete for planar graphs, bipartite graphs and chordal graphs. In this paper we strengthen the result for bipartite graph by showing that this problem remains NP-complete for two subclasses of bipartite graphs, i.e., star convex and comb convex bipartite graphs.

2. Complexity Results

In this section, we show that the PRDF problem is NP-complete for star convex bipartite graphs and comb convex bipartite graphs by giving a polynomial time reduction from a well-known NP-complete problem, Exact-3-Cover ($X3C$)[7], which is defined as follows.

EXACT-3-COVER (X3C)
INSTANCE : A finite set X with $| X | = 3q$ and a collection C of three-element subsets of X.
QUESTION : Is there a subcollection C' of C such that every element of X appears in exactly one member of C'?

The decision version of perfect Roman dominating function problem is defined as follows.

PERFECT ROMAN DOMINATING FUNCTION PROBLEM (PRDFP)
INSTANCE : A simple, undirected graph $G = (V, E)$ and a positive integer $k \leq | V |$.
QUESTION : Does G have a perfect Roman dominating function of weight

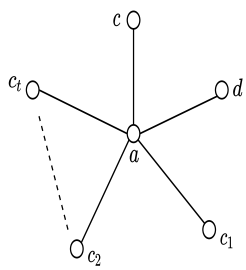

Figure 1: Star Graph

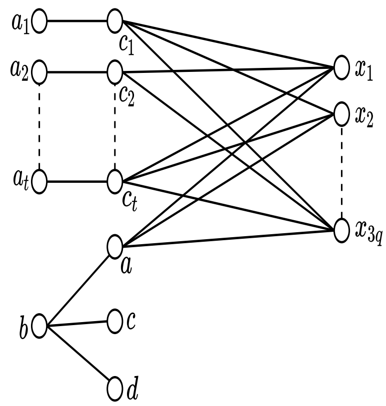

Figure 2: Construction of a star convex bipartite graph from an instance of X3C

at most k?

Theorem 2.1. *PRDFP is NP-complete for star convex bipartite graphs.*

Proof. Given any function $f : V \to \{0, 1, 2\}$ and a positive integer $k \leq |V|$, we can check in polynomial time whether $f(V) \leq k$ and for every vertex $u \in V$ with $f(u) = 0$, there is exactly one vertex $v \in N_G(u)$ such that $f(v) = 2$. So the PRDFP is a member of *NP*. We transform an instance of $X3C$, where $X = \{ x_1, x_2, . . ., x_{3q} \}$ and $C = \{ c_1, c_2, . . ., c_t \}$, to an instance of PRDFP as follows.

Create vertices x_i for each $x_i \in X$, c_i, a_i for each $c_i \in C$ and also create vertices a, b, c and d. Add edges a_ic_i for each c_i and c_jx_i if $x_i \in c_j$. Next, add edges x_ia for each x_i, ba, bc and bd. Let $A = \{c_i : 1 \leq i \leq t\} \cup \{a, c, d\}$, $B = \{x_i : 1 \leq i \leq 3q\}$ $\{a_i : 1 \leq i \leq t\} \cup \{b\}$. The set A induces a star with vertex a as central vertex as shown in the Figure 1. From the Figure 2, it is clear that the graph constructed is a star convex bipartite graph since the neighbors of each element in B induce a subtree of star, where $|V| = 2t + 3q + 4$ and $|E| = 3q + 4t + 3$. Next, we show that $X3C$ has a solution if and only if G has a PRDF with weight at most $2t + 2$. Let $k = 2t + 2$.

Suppose C' is a solution for $X3C$ with $|C'| = q$. We construct a perfect Roman dominating function f on G as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in C' \text{ or } v = a \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

Clearly, $f(V) \leq 2q + 2 = k$.

Conversely, suppose that G has a perfect Roman dominating function g with weight at most k. Clearly, $g(a) + g(a_1) + g(a_2) + g(a_3) \geq 2$. Without loss of generality, let $g(a) = 2$ and $g(a_1) = g(a_2) = g(a_3) = 0$. Since $(a, c_j) \in E$, it follows that each vertex c_j may be assigned the value 0.

Claim 1.1. *If $g(V) = k$ then for each $x_i \in X$, $g(x_i) = 0$.*

Proof. (Proof by contradiction) Assume $g(V) = k$ and there exist some x_i's such that $g(x_i) \neq 0$. Let $m = |\{x_i : g(x_i) \neq 0\}|$. The number of x_i's with $g(x_i) = 0$ is $3q-m$. Since g is a PRDF, each x_i with $g(x_i) = 0$ should have exactly one neighbor c_j with $g(c_j) = 2$. So the number of c_j's required with $g(c_j) = 2$ is $\lceil \frac{3q-m}{3} \rceil$. Hence $g(V) = 2 + m + 2\lceil \frac{3q-m}{3} \rceil = 2 + 2q + \frac{m}{3}$, which is greater than k, a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. □

Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 2 such that $\left(\bigcup_{g(c_i)=2} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 2\}$ is an exact cover for C. □

Theorem 2.2. *PRDFP is NP-complete for comb convex bipartite graphs.*

Proof. Clearly, PRDFP is a member of *NP*. We transform an instance of $X3C$, where $X = \{ x_1, x_2, \ldots, x_{3q} \}$ and $C = \{ c_1, c_2, \ldots, c_t \}$, to an instance of PRDFP as follows.

Create vertices x_i for each $x_i \in X$, c_i, a_i, c'_i for each $c_i \in C$ and also create vertices a, a' and b. Add edges $a_i c_i$ for each c_i and $c_j x_i$ if $x_i \in c_j$. Next add edges $c'_j b$ for each c'_j, ba and ba'. Also add edges by joining each c'_j to every x_i. Let $A = \{a, a'\} \cup \{c_i, c'_i : 1 \leq i \leq t\}$ and $B = V \setminus A$. The set A induces a comb with elements $\{c'_i : 1 \leq i \leq t \} \cup \{a'\}$ as backbone and $\{c_i : 1 \leq i \leq t \} \cup \{a\}$ as teeth as shown in the Figure 3. From the Figure 4, it is clear that the graph constructed is a comb convex bipartite graph since the neighbors of each element in B induce a subtree of the comb, where $|V| = 3t + 3q + 3$ and $|E| = 3qt + 5t + 2$. Next we show that, $X3C$ has a solution if and only if G has a PRDF with weight at most $2t + 2$.

Suppose C' is a solution for $X3C$ with $|C'| = q$. We construct a perfect Roman dominating function f on G as follows.

$$f(v) = \begin{cases} 2, & \text{if } v \in \{c_i : c_i \in C'\} \cup \{a_i : c_i \notin C'\} \text{ or } v = b \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Clearly, $f(V) \leq 2t + 2$.

Conversely, suppose that G has a perfect Roman dominating function g with weight at most $2t + 2$. Clearly, for each i, $g(a_i) + g(c_i) \geq 2$, these make the size at least $2t$, and $g(b) + g(a) + g(a') \geq 2$. Without loss of generality, $g(b) = 2$, $g(a) = 0$, $g(a') = 0$, $g(x_i) = 0$ where $1 \leq i \leq 3q$ and $g(c'_j) = 0$ where $1 \leq j \leq t$. Since g is a perfect Roman dominating function with weight $2t + 2$ or less, the c_i vertices with $g(c_i) = 2$ should be Roman dominating over all the x_j vertices in G. Then $C' = \{c_i : g(c_i) = 2\}$ is an exact cover for C; because if some vertex x_i is not

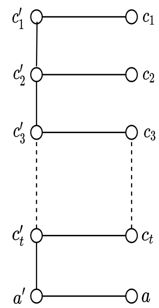

Figure 3: Comb Graph

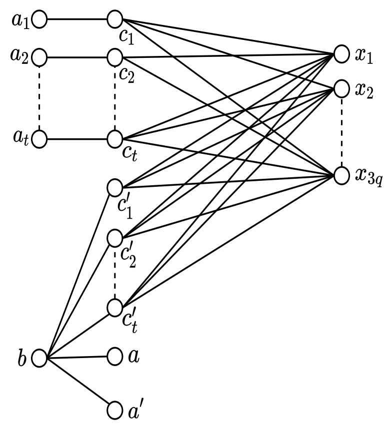

Figure 4: Construction of a comb convex bipartite graph from an instance of X3C

covered exactly once in C', the vertex x_i would not be Roman dominated exactly once in G and g would not be a PRDF. □

Now, the following result is immediate from Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *PRDFP is NP-complete for tree convex bipartite graphs.*

3. Threshold Graphs

In this section, we determine the perfect Roman domination number of threshold graph.

Definition 3.1. A graph $G = (V, E)$ is called a threshold graph if there is a real number T and a real number $w(v)$ for every $v \in V$ such that a set $S \subseteq V$ is independent if and only if $\sum_{v \in S} w(S) \leq T$.

Although several characterizations defined for threshold graphs, We use the following characterization of threshold graphs given in [11] to prove that the perfect Roman domination number can be computed in linear time for threshold graphs.

A graph $G = (V, E)$ is a threshold graph if and only if it is a split graph and, for split partition (C, I) of V where C is a clique and I is an independent set, there is an ordering $\{x_1, x_2, \ldots, x_p\}$ of vertices of C such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$, and there is an ordering $\{y_1, y_2, \ldots, y_q\}$ of the vertices of I such that $N_G(y_1) \supseteq N_G(y_2) \supseteq N_G(y_3) \supseteq \ldots \supseteq N_G(y_q)$. If $|C| = 0$ then, weight 1 is assigned for each vertex, clearly, $\gamma_R^P(G) = |V|$. If $|C| > 0$ then the following theorem holds.

Theorem 3.2. *Let G be a threshold graph. Then $\gamma_R^P(G) = k+1$, where k is the number of connected components in G.*

Proof. Let G be a threshold graph with p clique vertices such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$. Now, define a function $f : V \to \{0, 1, 2\}$ as follows.

$$f(v) = \begin{cases} 1, & \text{if } deg(v) = 0 \\ 2, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

Clearly, f is a PRDF and $\gamma_R^P(G) \leq k+1$. From the definition of PRDF, it follows that $\gamma_R^P(G) \geq k+1$. Therefore $\gamma_R^P(G) = k+1$. □

Now, the following result is immediate from Theorem 3.1.

Theorem 3.3. *PRDF problem can be solvable in linear time for threshold graphs.*

Proof. Since the ordering of the vertices of the clique and the number of connected components in a threshold graph can be determined in linear time [2, 11], the result follows. □

4. Chain Graphs

In this section, we propose a method to compute the perfect Roman domination number of a chain graph in linear time. A bipartite graph $G = (X, Y, E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a *chain*, that is, the vertices of X can be linearly ordered, say $x_1, x_2, ..., x_p$, such that $N_G(x_1) \subseteq N_G(x_2) \subseteq ... \subseteq N_G(x_p)$. If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain. An ordering $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq ... \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq ... \supseteq N_G(y_q)$. Every chain graph admits a chain ordering [5]. The following proposition is stated in [4].

Proposition 4.1. *Let $G = K_{m_1,\ldots,m_n}$ be the complete n-bipartite graph with $m_1 \leq m_2 \leq ... \leq m_n$.*

(a) *If $m_1 \geq 3$ then $\gamma_R(G) = 4$.*

(b) *If $m_1 = 2$ then $\gamma_R(G) = 3$.*

(c) *If $m_1 = 1$ then $\gamma_R(G) = 2$.*

If $G(X, Y, E)$ is a complete bipartite graph then, clearly, $\gamma_R(G) = \gamma_R^P(G)$ i.e., $\gamma_R^P(G)$ is obtained directly from Proposition 4.1. Otherwise, the following theorem holds.

Theorem 4.2. *Let $G(X,Y,E)(\ncong K_{r,s})$ be a chain graph. Then,*

$$\gamma_R^P(G) = \begin{cases} 3, & \text{if } |X| = 2 \text{ or } |Y| = 2 \\ 4, & \text{otherwise} \end{cases} \tag{4.1}$$

Proof. If $G \cong K_1$ then $\gamma_R^P(G) = 1$. Otherwise, let $G(X,Y,E)$ be a chain graph with $|X| = p$ and $|Y| = q$. Now, define a function $f : V \to \{0,1,2\}$ as follows.

Case (1) : $|X| = 2$ and $|Y| = 2$ then $f(v) = \begin{cases} 2, & \text{if } v = y_1 \\ 1, & \text{if } v = y_2 \\ 0, & \text{otherwise} \end{cases}$

Case (2) : $|X| = 2$ and $|Y| \neq 2$ then $f(v) = \begin{cases} 2, & \text{if } v = x_2 \\ 1, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$

Case (3) : $|X| \neq 2$ and $|Y| = 2$ then same condition holds as in Case (1).

Clearly, f is a PRDF and $\gamma_R^P(G) \leq 3$. From the definition of PRDF, it follows that $\gamma_R^P(G) \geq 3$. Therefore $\gamma_R^P(G) = 3$.

Case (4) : $|X| \neq 2$ and $|Y| \neq 2$ then $f(v) = \begin{cases} 2, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases}$

Clearly, f is a PRDF and $\gamma_R^P(G) \leq 4$. Since $p \geq 2$ and $q \geq 2$, in any PRDF of G, $f(X) \geq 2$ and $f(Y) \geq 2$. Therefore $\gamma_R^P(G) \geq 4$. Hence $\gamma_R^P(G) = 4$. □

If the chain graph G is disconnected with k connected components $G_1, G_2, \ldots, G_k$ then it is easy to verify that $\gamma_R^P(G) = \sum_{i=1}^{k} \gamma_R^P(G_i)$.

Now, the following result is immediate from Theorem 4.2.

Theorem 4.3. *PRDF problem can be solvable in linear time for chain graphs.*

Proof. Since the chain ordering and the connected components can be computed in linear time [2, 13], the result follows. □

5. Bounded Tree-width Graphs

Let G be a graph, T be a tree and v be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T . The pair $(T, v\ )$ is called a tree-decomposition of G if it satisfies the following three conditions: (i) $V(G) = \bigcup_{t \in V(T)} V_t$, (ii) for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t, (iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_1, t_2, $t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3. The width of $(T, v\ )$ is the number $max\{|V_t| - 1 : t \in T\}$, and the tree-width $tw(G)$ of G is the minimum width of any tree-decomposition of G. By Courcelle's Thoerem, it is well known that every graph problem that can be described by counting monadic second-order logic (CMSOL) can be solved in linear-time in graphs of bounded tree-width, given a tree decomposition as input [1]. We show that PRDF problem can be expressed in CMSOL.

Theorem 5.1.([*Courcelle's Theorem*][1]) *Let P be a graph property expressible in CMSOL and k be a constant. Then, for any graph G of tree-width at most k, it can be checked in linear-time whether G has property P.*

Theorem 5.2. *Given a graph G and a positive integer k, PRDF can be expressed in CMSOL.*

Proof. Let $f = (V_0, V_1, V_2)$ be a function $f : V \to \{0, 1, 2\}$ on a graph G, where $V_i = \{v|f(v) = i\}$ for $i \in \{0, 1, 2\}$. The CMSOL formula for the PRDF problem is expressed as follows.

$Perfect\_Rom\_Dom(V) = (f(V) \leq k) \wedge \exists V_0, V_1, V_2, \forall p((p \in V_1) \vee (p \in V_2) \vee (p \in V_0 \wedge \exists r(r \in V_2 \wedge adj(p, r)) \wedge \neg(\exists s, s \in V_2 \wedge s \neq r \wedge adj(p, s))),$

where $adj(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G. □

Now, the following result is immediate from Theorem 5.1 and Theorem 5.2.

Theorem 5.3. *PRDF problem can be solvable in linear time for bounded tree-width graphs.*

6. Conclusion

In this paper, we have shown that the decision problem associated with $\gamma_R^P(G)$ is NP-complete for some subclasses of bipartite graphs. Next, we have shown that PRDF problem is linear time solvable for bounded tree-width graphs, threshold graphs and chain graphs. Investigating the algorithmic complexity of this problem for other subclasses of bipartite graphs and chordal graphs remains open.

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