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A Relationship between the Second Largest Eigenvalue and Local Valency of an Edge-regular Graph

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Abstract. For a distance-regular graph with valency k, second largest eigenvalue r and diameter D, it is known that $r \geq \min\{\frac{\lambda+\sqrt{\lambda^2+4k}}{2}, a_3\}$ if $D = 3$ and $r \geq \frac{\lambda+\sqrt{\lambda^2+4k}}{2}$ if $D \geq 4$, where $\lambda = a_1$. This result can be generalized to the class of edge-regular graphs. For an edge-regular graph with parameters (v, k, λ) and diameter $D \geq 4$, we compare $\frac{\lambda+\sqrt{\lambda^2+4k}}{2}$ with the local valency λ to find a relationship between the second largest eigenvalue and the local valency. For an edge-regular graph with diameter 3, we look at the number $\frac{\lambda-\bar{\mu}+\sqrt{(\lambda-\bar{\mu})^2+4(k-\bar{\mu})}}{2}$, where $\bar{\mu} = \frac{k(k-1-\lambda)}{v-k-1}$, and compare this number with the local valency λ to give a relationship between the second largest eigenvalue and the local valency. Also, we apply these relationships to distance-regular graphs.

1. Introduction

In 2010, Koolen and Park [4] gave a lower bound on the second largest eigenvalue of a distance-regular graph with diameter 3 in terms of valency k and intersection numbers a_1 and a_3.

Theorem 1.1. *(cf. [4, Lemma 6]) Let Γ be a distance-regular graph with valency k and diameter 3. Then the second largest eigenvalue r of Γ satisfies*

$$r \geq \min\left\{\frac{\lambda+\sqrt{\lambda^2+4k}}{2}, a_3\right\},$$

where $\lambda = a_1$.

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In 2011, Koolen, Park and Yu [6] generalized this theorem to the class of distance-regular graphs with diameter at least 4. We note that in [6, Theorem 3.1], they assumed that the valency k is at least three, but it is also true for $k = 2$.

Theorem 1.2. *(cf. [6, Theorem 3.1]) Let* Γ *be a distance-regular graph with valency* k*, diameter* $D \geq 4$*. Then the second largest eigenvalue* r *of* Γ *satisfies*

$$r \geq \frac{\lambda + \sqrt{\lambda^2 + 4k}}{2},$$

where $\lambda = a_1$*.*

The proof of Theorem 1.2 also works for edge-regular graphs with diameter $D \geq 4$. And for edge-regular graphs Γ with diameter 3, the proof of Theorem 1.1 works if we replace a_3 by $\bar{a}_3(x) = \frac{1}{|\Gamma_3(x)|} \sum\limits_{y \in \Gamma_3(x)} a_3(x, y)$, where x is a vertex of Γ.

In this paper, we will try to give a lower bound on the second largest eigenvalue r of an edge-regular graph with parameters (v, k, λ) in terms of λ. In order to do so, we will compare $\frac{\lambda + \sqrt{\lambda^2+4k}}{2}$ with the local valency λ for edge-regular graphs with diameter $D \geq 4$. Since a lower bound on $\frac{\lambda + \sqrt{\lambda^2+4k}}{2}$ does not give an immediate lower bound on the second largest eigenvalue of an edge-regular graph with diameter 3, we will consider the number $\frac{\lambda - \bar{\mu} + \sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})}}{2}$, where $\bar{\mu} = \frac{k(k-1-\lambda)}{v-k-1}$. Once we have a relationship between r and λ for edge-regular graphs with diameter $D \geq 3$, we apply it to the class of distance-regular graphs with diameter $D \geq 3$. Then we obtain that for a distance-regular graph with diameter $D \geq 4$, the second largest eigenvalue is at least $\lambda + \sqrt{2}$. For a distance-regular graph with diameter 3, we can show that the second largest eigenvalue is larger than $\lambda + 1$ if the number v of vertices is large compared to λk.

2. Definitions and Preliminaries

All the graphs considered in this paper are finite, undirected and simple. The reader is referred to [1] for more information. Let Γ be a connected graph with vertex set $V(\Gamma)$. The *distance* $d_\Gamma(x, y)$ between two vertices $x, y \in V(\Gamma)$ is the length of a shortest path between x and y in Γ. The *diameter* $D = D(\Gamma)$ of Γ is the maximum distance between any two vertices of Γ. For each $x \in V(\Gamma)$, let $\Gamma_i(x)$ be the set of vertices of Γ at distance i from x $(0 \leq i \leq D)$. In addition, define $\Gamma_{-1}(x) = \emptyset$ and $\Gamma_{D+1}(x) = \emptyset$. For the sake of simplicity, let $\Gamma(x) = \Gamma_1(x)$ and we denote $x \sim y$ if two vertices x and y are adjacent. In particular, Γ is *regular* with *valency* k if $k = |\Gamma(x)|$ holds for all $x \in V(\Gamma)$. The graph Γ is called *edge-regular* with parameters (v, k, λ) if it has v vertices, is regular with valency k and satisfies that any two adjacent vertices of Γ have λ commnon neighbors. Note that for any vertex x of an edge-regular graph with parameters (v, k, λ), the subgraph induced on $\Gamma(x)$ is a regular graph with valency λ.

For a connected graph Γ with diameter D, we choose two vertices x, y at distance $i = d_\Gamma(x,y)$, and consider the numbers $c_i(x,y) = |\Gamma_{i-1}(x) \cap \Gamma(y)|$, $a_i(x,y) = |\Gamma_i(x) \cap \Gamma(y)|$ and $b_i(x,y) = |\Gamma_{i+1}(x) \cap \Gamma(y)|$ $(0 \le i \le D)$. We say that the *intersection number* c_i (a_i and b_i, respectively) exists if the number $c_i(x,y)$ ($a_i(x,y)$ and $b_i(x,y)$, respectively) does depend only on $i = d_\Gamma(x,y)$ not on the choice of x and y with $d_\Gamma(x,y) = i$. Set $c_0 = b_D = 0$ and observe $a_0 = 0$ and $c_1 = 1$. A connected graph Γ with diameter D is called a *distance-regular graph* if there exist intersection numbers c_i, a_i, b_i for all $i = 0, 1, \ldots, D$. Note that a distance-regular graph is edge-regular with parameters (v, b_0, a_1).

For any connected graph Γ with diameter D, the *distance-i graph* Γ_i $(0 \le i \le D)$ is the graph whose vertices are those of Γ and edges are the 2-subsets of vertices at mutual distance i in Γ. In particular, $\Gamma_1 = \Gamma$. An *antipodal* graph is a connected graph Γ with diameter $D > 1$ for which its distance-D graph Γ_D is a disjoint union of complete graphs. A graph Γ is called *bipartite* if it has no odd cycle. (If Γ is a distance-regular graph with diameter D and bipartite, then $a_1 = a_2 = \ldots = a_D = 0$.)

For a connected graph Γ with diameter D, the *adjacency matrix* $A = A(\Gamma)$ is the matrix whose rows and columns are indexed by $V(\Gamma)$, where the (x,y)-entry is 1 whenever $x \sim y$ and 0 otherwise. The *eigenvalues* of Γ are the eigenvalues of $A(\Gamma)$. For a partition $\Pi = \{P_1, P_2, \ldots, P_\ell\}$ of the vertex set $V(\Gamma)$, we look at the numbers β_{ij} $(1 \le i, j, \le \ell)$, where vertices in P_i have averagely β_{ij} neighbors in P_j. Then the *quotient matrix* $Q = Q(\Pi)$ corresponding to the partition Π is the $\ell \times \ell$ matrix whose (i,j)-entry is β_{ij}. Note that the eigenvalues of the quotient matrix Q interlace the eigenvalues of Γ (see [2, Corollary 2.5.4]).

3. Edge-regular Graphs with Diameter at Least 4

Recall that the same proof of Theorem 1.2 also works for any edge-regular graph Γ with parameters (v, k, λ) and diameter $D \ge 4$, and hence the second largest eigenvalue r of Γ is at least $\frac{\lambda + \sqrt{\lambda^2 + 4k}}{2}$.

In this section, for an edge-regular graph Γ with parameters (v, k, λ), second largest eigenvalue r and diameter $D \ge 4$, we compare $\frac{\lambda + \sqrt{\lambda^2 + 4k}}{2}$ with the local valency λ to find a relationship between r and λ. Note that if $k = 2$, then Γ is an n-gon for $n \ge 8$ and $r > \lambda + 1$.

Lemma 3.1. *Let Γ be an edge-regular graph with parameters (v, k, λ). Then for any positive integer t, $\frac{\lambda + \sqrt{\lambda^2 + 4k}}{2} > \lambda + t$ if and only if $\lambda < \frac{1}{t}k - t$.*

Proof. Let t be a positive integer. Clearly, $\frac{\lambda + \sqrt{\lambda^2 + 4k}}{2} > \lambda + t$ is equivalent to $\sqrt{\lambda^2 + 4k} > \lambda + 2t$. Since $\lambda + 2t > 0$, we know that $\sqrt{\lambda^2 + 4k} > \lambda + 2t$ is equivalent to $\lambda^2 + 4k > (\lambda + 2t)^2 = \lambda^2 + 4t\lambda + 4t^2$. As $\lambda^2 + 4k > \lambda^2 + 4t\lambda + 4t^2$ is equivalent to $t\lambda < k - t^2$, we conclude that $\frac{\lambda + \sqrt{\lambda^2 + 4k}}{2} > \lambda + t$ if and only if $\lambda < \frac{1}{t}k - t$. This finishes the proof. □

Remark 3.2. (*i*) As $\lambda \geq 0$, the condition $\lambda < \frac{k}{t} - t$ is meaningful when $k > t^2$.

(*ii*) For $t = 1$, we have that $\frac{\lambda+\sqrt{\lambda^2+4k}}{2} > \lambda + 1$ if and only if $\lambda < k - 1$. And $\lambda < k - 1$ is true except when the graph is a complete graph. (It also can be obtained from an easy calculation, $\frac{\lambda+\sqrt{\lambda^2+4k}}{2} = \frac{\lambda+\sqrt{\lambda^2+4(\lambda+1+b_1)}}{2} = \frac{\lambda+\sqrt{(\lambda+2)^2+4b_1}}{2} \geq \lambda + 1$ with equality holds if and only if $b_1 = 0$, where $b_1 = k - \lambda - 1$.)

(*iii*) For $t = 2$, we have that $\frac{\lambda+\sqrt{\lambda^2+4k}}{2} > \lambda + 2$ if and only if $\lambda < \frac{1}{2}k - 2$ (and $k > 4$). In Theorem 3.4, we will also consider the case $\lambda \geq \frac{1}{2}k - 2$ for distance-regular graphs with diameter $D \geq 4$.

We combine Theorem 1.2 and Lemma 3.1, and then we obtain the following result.

Theorem 3.3. *Let Γ be an edge-regular graph with parameters (v, k, λ), second largest eigenvalue r and diameter $D \geq 4$. For any positive integer t, if $\lambda < \frac{1}{t}k - t$, then $r > \lambda + t$.*

Proof. Since $D \geq 4$, Theorem 1.2 implies that $r \geq \frac{\lambda+\sqrt{\lambda^2+4k}}{2}$. Assume that $\lambda < \frac{1}{t}k - t$, then Lemma 3.1 says that $\frac{\lambda+\sqrt{\lambda^2+4k}}{2} > \lambda + t$. Thus, we obtain that $r > \lambda + t$. This finishes the proof. □

We apply this result to the class of distance-regular graphs with diameter $D \geq 4$. Then we obtaind the following result.

Theorem 3.4. *Let Γ be a distance-regular graph with valency $k \geq 2$, intersection number $a_1 = \lambda$, second largest eigenvalue r and diameter $D \geq 4$. Then $r \geq \lambda + \sqrt{2}$.*

Proof. If $k = 2$, then Γ is an n-gon for $n \geq 8$ and $r \geq \sqrt{2} = \lambda + \sqrt{2}$ (as $\lambda = 0$). So, we may assume that $k \geq 3$.

If $\lambda \geq \frac{1}{2}k - 1$, then by [5, Theorem 16], we know that Γ is the flag graph of a regular generalized D-gon of order (s, s) for some $s \geq 2$, and the second largest eigenvalue r of Γ satisfies $r \geq \lambda + \sqrt{2s} \geq \lambda + 2$ (see, [1, Section 6.5] or [3]).

If $\frac{1}{2}k - 2 \leq \lambda < \frac{1}{2}k - 1$, then Γ satisfies either (k is even and $\lambda = \frac{1}{2}k - 2$) or ($k$ is odd and $\lambda = \frac{1}{2}k - \frac{3}{2}$). The first case implies that $r \geq \lambda + 2$ as $\frac{\lambda+\sqrt{\lambda^2+4k}}{2} \geq \lambda + 2$. And the second case implies that $r > \lambda + \sqrt{3}$ as $\frac{\lambda+\sqrt{\lambda^2+4k}}{2} > \lambda + \sqrt{3}$.

If $\lambda < \frac{1}{2}k - 2$, then by Theorem 3.3, we know that $r > \lambda + 2$. This finishes the proof. □

Remark 3.5. (*i*) In Theorem 3.4, $r = \lambda + \sqrt{2}$ holds only for the 8-gon.

(*ii*) The flag graph of a regular generalized 4-gon of order $(2, 2)$ has second largest eigenvalue $r = 3 = 1+2 = \lambda+2$. And some antipodal distance-regular graphs with diameter 4 satisfy that k is even, $\lambda = \frac{1}{2}k - 2$ and $r = \frac{\lambda+\sqrt{\lambda^2+4k}}{2} = \lambda + 2$ (see, [1, p.421]).

4. Edge-regular Graphs With Diameter 3

Recall that for an edge-regular graph Γ with parameters (v, k, λ) and diameter 3, if we replace a_3 by $\bar{a}_3(x)$ and follow the proof of Theorem 1.1, then we obtain that the second largest eigenvalue of Γ is at least $\min\{\frac{\lambda+\sqrt{\lambda^2+4k}}{2}, \bar{a}_3(x)\}$, where x is a vertex of Γ and $\bar{a}_3(x) = \frac{1}{|\Gamma_3(x)|} \sum_{y \in \Gamma_3(x)} a_3(x, y)$. If $\bar{a}_3(x) \geq \frac{\lambda+\sqrt{\lambda^2+4k}}{2}$, then we find a result similar to Lemma 3.3. But it is not true in general for edge-regular graphs with diameter 3.

In this section, for an edge-regular graph Γ with parameters (v, k, λ), second largest eigenvalue r and diameter 3, we compare $\frac{\lambda-\bar{\mu}+\sqrt{(\lambda-\bar{\mu})^2+4(k-\bar{\mu})}}{2}$ with the local valency λ to find a relationship between r and λ. Note that if $k = 2$, then Γ is an n-gon for $n \in \{6, 7\}$ and $r \geq \lambda + 1$.

Lemma 4.1. *Let Γ be an edge-regular graph with parameters (v, k, λ), second largest eigenvalue r and diameter 3. Then $r \geq \frac{\lambda-\bar{\mu}+\sqrt{(\lambda-\bar{\mu})^2+4(k-\bar{\mu})}}{2}$, where $\bar{\mu} = \frac{k(k-1-\lambda)}{v-k-1}$.*

Proof. Let x be a vertex of Γ. Consider a partition $P = \{\{x\}, \Gamma_1(x), \Gamma_2(x) \cup \Gamma_3(x)\}$ of the set of vertices of Γ. As there are $v - k - 1$ vertices in $\Gamma_2(x) \cup \Gamma_3(x)$, we know that vertices in $\Gamma_2(x) \cup \Gamma_3(x)$ have averagely $\bar{\mu} = \frac{k(k-1-\lambda)}{v-k-1}$ neighbors in $\Gamma(x)$. Then one can easily see that the following matrix Q is the quotient matrix corresponding to the partition P:

$$Q = \begin{pmatrix} 0 & k & 0 \\ 1 & \lambda & k-1-\lambda \\ 0 & \bar{\mu} & k-\bar{\mu} \end{pmatrix}.$$

Note that the matrix Q has eigenvalues

$$k > \frac{\lambda - \bar{\mu} + \sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})}}{2} > \frac{\lambda - \bar{\mu} - \sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})}}{2}.$$

Thus, we know that the second largest eigenvalue r of Γ is at least the second largest eigenvalue of Q (see for example [2, Corollary 2.5.4]). This finishes the proof. □

In [6, Proposition 3.2], it was shown that for a distance-regular graph with second largest eigenvalue r, intersection numbers $a_1 = \lambda$, $c_2 = \mu$ and diameter 3, $r > \lambda + 1 - \mu$ holds. We generalize this to the class of edge-regular graphs (with diameter 3).

Lemma 4.2. *Let Γ be an edge-regular graph with parameters (v, k, λ), second largest eigenvalue r and diameter 3. Then $r > \lambda + 1 - \bar{\mu}$, where $\bar{\mu} = \frac{k(k-1-\lambda)}{v-k-1}$.*

Proof. Note that $r > 0$ (see for example Lemma 4.1). If $\lambda - \bar{\mu} < -2$, then $\lambda + 1 - \bar{\mu} < -1 < r$ holds. So, we may assume that $\lambda - \bar{\mu} \geq -2$.

Since $k > \lambda + 1$, we have $k - \bar{\mu} > \lambda + 1 - \bar{\mu}$, and this implies that $(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu}) > (\lambda - \bar{\mu})^2 + 4(\lambda + 1 - \bar{\mu}) = (\lambda + 2 - \bar{\mu})^2$. As $\lambda + 2 - \bar{\mu} \geq 0$, we obtain that $\sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})} > \lambda + 2 - \bar{\mu}$, and hence $\frac{\lambda - \bar{\mu} + \sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})}}{2} > \lambda + 1 - \bar{\mu}$ holds. Thus, by Lemma 4.1, we know that $r > \lambda + 1 - \bar{\mu}$. This finishes the proof. □

Remark 4.3. (i) Lemma 4.2 is also true for edge-regular graphs with diameter at least 4. But we have a better bound for edge-regular graphs with diameter at least 4. (see for example Lemma 3.3)

(ii) For distance-regular graphs with diameter 3, $c_2 = \frac{k(k-1-\lambda)}{k_2} > \frac{k(k-1-\lambda)}{k_2+k_3} = \bar{\mu}$ holds. Thus, Lemma 4.2 slightly strengthens a result of [6, Proposition 3.2].

Lemma 4.4. *Let Γ be an edge-regular graph with parameters (v, k, λ) and diameter 3. Let s be an integer satisfying $k > s\lambda + s^2$. If $v > (\lambda + 1 + s)\frac{k-\lambda-1}{k-s\lambda-s^2}k + k + 1$, then $\frac{\lambda - \bar{\mu} + \sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})}}{2} > \lambda + s$, where $\bar{\mu} = \frac{k(k-1-\lambda)}{v-k-1}$.*

Proof. From the assumption $v > (\lambda + 1 + s)\frac{k-\lambda-1}{k-s\lambda-s^2}k + k + 1$, we have that $v - k - 1 > (\lambda + 1 + s)\frac{k-\lambda-1}{k-s\lambda-s^2}k$. Since $v - k - 1 > 0$ and $k - s\lambda - s^2 > 0$, we obtain that $k - s\lambda - s^2 > (\lambda + 1 + s)\frac{k(k-\lambda-1)}{v-k-1} = (\lambda + 1 + s)\bar{\mu}$.

Multiply by 4 and then we obtain that $4k - 4s\lambda - 4s^2 > 4\lambda\bar{\mu} + 4\bar{\mu} + 4s\bar{\mu}$.

Add $\lambda^2 + \bar{\mu}^2$ to both sides. Then we have that $\lambda^2 + \bar{\mu}^2 + 4k - 4s\lambda - 4s^2 > \lambda^2 + \bar{\mu}^2 + 4\lambda\bar{\mu} + 4\bar{\mu} + 4s\bar{\mu}$, i.e., $(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu}) > (\lambda + \bar{\mu})^2 + 4s(\lambda + \bar{\mu}) + (2s)^2 = (\lambda + \bar{\mu} + 2s)^2$ holds. Since $\sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})} > \sqrt{(\lambda + \bar{\mu} + 2s)^2} = |\lambda + \bar{\mu} + 2s| \geq \lambda + \bar{\mu} + 2s$, we obtain that $\frac{\lambda - \bar{\mu} + \sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})}}{2} > \lambda + s$. This finishes the proof. □

In the following theorem, we consider the case $s = 1$. And we find that the second largest eigenvalue of an edge-regular graph with parameters (v, k, λ) and diameter 3 is larger than $\lambda + 1$ when v is large compared to λk.

Theorem 4.5. *Let Γ be an edge-regular graph with parameters (v, k, λ), second largest eigenvalue r and diameter 3. If $v > (\lambda + 3)k + 1$, then $r > \lambda + 1$.*

Proof. Note that $k > \lambda + 1$ holds (as the diameter of Γ is 3). Set $s = 1$ in Lemma 4.4. Then Lemma 4.4 says that $v > (\lambda + 3)k + 1$ implies that $\frac{\lambda - \bar{\mu} + \sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})}}{2} > \lambda + 1$. By Lemma 4.1, we obtain that $r \geq \frac{\lambda - \bar{\mu} + \sqrt{(\lambda - \bar{\mu})^2 + 4(k - \bar{\mu})}}{2} > \lambda + 1$. This finishes the proof. □

We apply this result to the class of distance-regular graphs with intersection number $a_1 = \lambda = 0$ and diameter 3. Then we obtaind the following result.

Theorem 4.6. *Let Γ be a distance-regular graph with valency $k \geq 2$, intersection number $a_1 = \lambda = 0$, second largest eigenvalue r and diameter 3. Then $r \geq \lambda + 1$.*

Proof. If the graph Γ has more than $3k+1$ vertices, then by Theorem 4.5, we know that $r > \lambda + 1$. So, we assume that Γ has at most $3k+1$ vertices. Then by [7, Theorem 1], we know that Γ is either a 7-gon, a Taylor graph or a bipartite graph. Note that a 7-gon satisfies $r > 1 = \lambda + 1$ and that a Taylor graph with $\lambda = 0$ satisfies $r = 1 = \lambda + 1$. So, we may assume that Γ is bipartite. Then Γ satisfies that $r \geq \sqrt{k - c_2} \geq 1 = \lambda + 1$, and $r = 1$ if and only if Γ is a Taylor graph. This finishes the proof. □

Remark 4.7. In Theorem 4.6, $r = \lambda + 1$ with $\lambda = 0$ holds only for a Taylor graph, for example, the 6-gon and the 3-cube.

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