KYUNGPOOK Math. J. 61(2021), 631-644
https://doi.org/10.5666/KMJ.2021.61.3.631
pISSN 1225-6951 eISSN 0454-8124


# Harnack Estimate for Positive Solutions to a Nonlinear Equation Under Geometric Flow

Ghodratallah Fasihi-Ramandi and Shahroud Azami*
*Department of Pure Mathematics, Faculty of Science Imam Khomeini International University, Qazvin, Iran*
*e-mail* : fasihi@sci.ikiu.ac.ir *and* azami@sci.ikiu.ac.ir

Abstract. In the present paper, we obtain gradient estimates for positive solutions to the following nonlinear parabolic equation under general geometric flow on complete noncompact manifolds

$$\frac{\partial u}{\partial t} = \triangle u + a(x,t)u^p + b(x,t)u^q$$

where, $0 < p, q < 1$ are real constants and $a(x,t)$ and $b(x,t)$ are functions which are $C^2$ in the $x$-variable and $C^1$ in the $t$-variable. We shall get an interesting Harnack inequality as an application.

## 1. Introduction and Main Results

Gradient estimates for nonlinear partial differential equations are of classical interest, and have been extensively studied, leading to many important results, especially in the area of geometric analysis. They were developed by Li and Yau [6] as a method to study the heat equation. Hamilton applied this method to Ricci flow on manifolds with scalar curvature [4]. Since then, there has been a lot of work on gradient estimates for solutions of differential equations under geometric flows, see, for instance [5, 7]. Extending some of this work, Sun [9] studied gradient estimates for positive solutions of the heat equation under the geometric flow. Also, the differential Harnack estimates plays an important role in solving the Poincaré conjecture and the geometrization conjecture [8].

In the present paper, we study the following nonlinear parabolic equation under general geometric flow on complete noncompact manifolds $M$,

$$\frac{\partial u}{\partial t} = \triangle u + a(x,t)u^p + b(x,t)u^q \tag{1.1}$$

where, $0 < p, q < 1$ are real constants and $a(x,t)$ and $b(x,t)$ are functions which are

* Corresponding Author.
Received August 28, 2019; revised June 16, 2020; accepted August 18, 2020.
2010 Mathematics Subject Classification: 53C21; 53C44; 58J35.
Key words and phrases: Geometric Flow, Harnack Estimate, Nonlinear Parabolic Equations.

$C^2$ in the $x$-variable and $C^1$ in the $t$-variable. Before presenting our main results about the equation, we motivate its consideration as a topic of study. If $a(x,t)$ and $b(x,t)$ are identically zero, then (1.1) is the heat equation. In bio-mathematics, the following equation

$$\frac{\partial u}{\partial t} = \triangle u + a(x,t)u^p, \qquad p > 0,$$

could be used to model population dynamics. Similar equations arise in the study of the conformal deformation of scalar curvature on a manifold (See [10], equation (1.4)).

Let $(M, g(t))$ be a smooth 1-parameter family of complete Riemannian metrics on a manifold $M$ evolving by equation

$$\frac{\partial g_{ij}}{\partial t} = 2s_{ij} \tag{1.2}$$

for $t$ in some time interval $[0,T]$, where $s_{ij}$ are componnents of a symmmetric (0,2)-tensor $s$. Notice that

- if $s_{ij} = -R_{ij}$ then geometric flow (1.2) called Ricci flow,
- if $s_{ij} = -\frac{1}{2}Rg_{ij}$ then geometric flow (1.2) called Yamabe flow,
- if $s_{ij} = -(R_{ij} + \rho R g_{ij})$ then geometric flow (1.2) called Ricci-Bourguignon flow,
- if $s_{ij} = -R_{ij} + \alpha\nabla\phi \otimes \nabla\phi$ where $\frac{\partial\phi}{\partial t} = \tau_g\phi$ then geometric flow (1.2) called harmonic-Ricci flow.

Now we present our main results about the equation (1.1) as follows.

**Theorem 1.1.** *Suppose $(M, g(t))$ is the family of complete Riemannian manifolds evolving by (1.2). Let $M$ be complete under the initial metric g(0). Given $x_0 \in M$, and $M_1, R > 0$, let $u$ be a positive solution to the nonlinear equation* (1.1) *with $u \geq M_1$ in the cube $Q_{2R,T} = \{(x,t)|d(x,x_0,t) \leq 2R, 0 \leq t \leq T\}$. Suppose that there exist constants $K_1, K_2, K_3, K_4 \geq 0$ such that*

$$Ric \geq -K_1 g, \quad -K_2 g \leq s \leq K_3 g, \quad |\nabla s| \leq K_4$$

*on $Q_{2R,T}$. Moreover, assume that there exist positive constants $\theta_a, \theta_b, \gamma_a, \gamma_b$ such that $\triangle a \leq \theta_a$, $|\nabla a| \leq \gamma_a$, $\triangle b \leq \theta_b$ and $|\nabla b| \leq \gamma_b$ in $Q_{2R,T}$. Then for any constant* $0 < \beta < 1$ *and $(x,t) \in Q_{2R,T}$ if $\beta < p, q < 1$ we have*

$$\beta\frac{|\nabla u|^2}{u^2} + au^{p-1} + bu^{q-1} - \frac{u_t}{u} \leq H_1 + H_2 + \frac{n}{\beta}\frac{1}{t},$$

*where,*

$$H_1 = \frac{n}{\beta}\Big(\frac{(n-1)(1+\sqrt{K_1}R)c_1^2 + c_2 + 2c_1^2}{R^2} + \sqrt{c_3}K_2 + |a|(1-p)M_1^{(p-1)}$$
$$+ |b|(1-q)M_1^{(q-1)} + \frac{nc_1^2}{2R^2(\beta-\beta^2)}\Big),$$

$$H_2 = \Big[\frac{n^2}{4\beta^2(1-\beta)^2}\big(2(1-\beta)K_3 + 2\beta K_1 + \frac{3}{2}K_4\big)^2$$
$$+ \frac{n}{\beta}\{M_1^{(p-1)}\theta_a + M_1^{(q-1)}\theta_b + n\big(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4\big)\}$$
$$- \frac{n}{\beta}\{\frac{[(p-\beta)M_1^{(p-1)}\gamma_a + (q-\beta)M_1^{(q-1)}\gamma_b]^2}{|a|(p-\beta)(p-1)M_1^{(p-1)} + |b|(q-\beta)(q-1)M_1^{(q-1)}}\}\Big]^{\frac{1}{2}}.$$

When $R$ approaches infinity, we get the global Li-Yau type gradient estimates (see [6]) for equation (1.1) as follows.

**Corollary 1.2.** *Let $(M, g(0))$ be a complete noncompact Riemannian manifold without boundary, and suppose that $g(t)$ evolves by $\frac{\partial g_{ij}}{\partial t} = 2s_{ij}$ for $t \in [0,T]$ and satisfies*

$$Ric \geq -K_1 g, \quad -K_2 g \leq s \leq K_3 g, \quad |\nabla s| \leq K_4.$$

*Also, assume that $\triangle a \leq \theta_a$, $\triangle b \leq \theta_b$, $|\nabla a| \leq \gamma_a$ and $|\nabla b| \leq \gamma_b$ in $M \times [0,T)$ for some constants $\theta_a, \theta_b, \gamma_a$ and $\gamma_b$. Let $u$ be a positive solution of (1.1) with $u \geq M_1$. Then for any constant $0 < \beta < 1$, if $\beta < p, q < 1$, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + au^{p-1} + bu^{q-1} - \frac{u_t}{u} \leq \overline{H_1} + H_2 + \frac{n}{\beta}\frac{1}{t},$$

*where*

$$\overline{H_1} = \frac{n}{\beta}\big(\sqrt{c_3}K_2 + |a|(1-p)M_1^{(p-1)} + |b|(1-q)M_1^{(q-1)}\big).$$

As an application, we get the following Harnack inequality.

**Corollary 1.3.** *Let $(M, g(0))$ be a complete noncompact Riemannian manifold without boundary, and suppose that $g(t)$ evolves by $\frac{\partial g_{ij}}{\partial t} = 2s_{ij}$ for $t \in [0,T]$ and satisfies*

$$Ric \geq -K_1 g, \quad -K_2 g \leq s \leq K_3 g, \quad |\nabla s| \leq K_4.$$

*Also, assume that* $\triangle a \leq \theta_a$*,* $\triangle b \leq \theta_b$*,* $|\nabla a| \leq \gamma_a$ *and* $|\nabla b| \leq \gamma_b$ *in* $M \times [0,T)$ *for some constants* $\theta_a, \theta_b, \gamma_a$ *and* $\gamma_b$*. Let* $u(x,t)$ *be a positive solution of (1.1) in* $M \times [0,T)$ *with* $u \geq M_1$ *where,* $a$ *and* $b$ *are positive constants. Then for any constant* $0 < \beta < 1$*, if* $\beta < p, q < 1$*, for any points* $(x_1,t_1)$ *and* $(x_2,t_2)$ *on* $M \times [0,T)$ *with* $0 < t_1 < t_2$*, we have the following Harnack inequality,*

$$u(x_1,t_1) \leq u(x_2,t_2)(\frac{t_2}{t_1})^{\frac{n}{\beta}} e^{\Psi(x_1,x_2,t_1,t_2)+(\overline{H_1}+H_2)(t_2-t_1)},$$

*where* $\Psi(x_1,x_2,t_1,t_2) = \inf_\gamma \int_{t_1}^{t_2} \frac{1}{4\beta}|\gamma'|^2 dt$*, and* $\overline{H_1} = \frac{n}{\beta}\big(\sqrt{c_3}K_2 + a(1-p)M_1^{(p-1)} + b(1-q)M_1^{(q-1)}\big)$*, and*

$$\begin{aligned} H_2 = \Big[&\frac{n^2}{4\beta^2(1-\beta)^2}\big(2(1-\beta)K_3 + 2\beta K_1 + \frac{3}{2}K_4\big)^2 \\ &+ \frac{n}{\beta}\{M_1^{(p-1)}\theta_a + M_1^{(q-1)}\theta_b + n\big(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4\big)\} \\ &- \frac{n}{\beta}\{\frac{[(p-\beta)M_1^{(p-1)}\gamma_a + (q-\beta)M_1^{(q-1)}\gamma_b]^2}{a(p-\beta)(p-1)M_1^{(p-1)} + b(q-\beta)(q-1)M_1^{(q-1)}}\}\Big]^{\frac{1}{2}}. \end{aligned}$$

## 2. Methods and Proofs

Let $u$ be a positive solution to (1.1). Let $w = \ln u$, then a simple computation shows that $w$ satisfies the following equation

$$w_t = \triangle w + |\nabla w|^2 + ae^{(p-1)w} + be^{(q-1)w}. \tag{2.1}$$

We need the following lemmas of [3, 9] to prove our main theorem.

**Lemma 2.1.** *If the metric evolves by (1.2) then for any smooth function* $w$*, we have*

$$\frac{\partial}{\partial t}|\nabla w|^2 = -2s(\nabla w, \nabla w) + 2\nabla w \nabla w_t$$

*and*

$$\frac{\partial}{\partial t}\triangle w = \triangle w_t - 2s\nabla^2 w - 2\nabla w\big(\text{div}s - \frac{1}{2}\nabla(tr_g s)\big),$$

*where,* div$s$ *denotes the divergence of* $s$*.*

**Lemma 2.2.** *Assume that* $(M, g(t))$ *satisfies the hypotheses of Proposition 1.1.*

*Then for any constant* $0<\beta<1$ *and* $(x,t)\in Q_{R,T}$*, if* $\beta<p,q<1$*, we have*

$$
\begin{aligned}
(\triangle-\frac{\partial}{\partial t})F \geq & -2\nabla w\nabla F+t\Big\{\frac{\beta}{n}(w_t-|\nabla w|^2-ae^{(p-1)w}-be^{(q-1)w})^2\\
&+[a(p-\beta)(p-1)e^{(p-1)w}+b(q-\beta)(q-1)e^{(q-1)w}+2(\beta-1)K_3\\
&-2\beta K_1-\frac{3}{2}K_4]|\nabla w|^2\\
&+2(p-\beta)e^{(p-1)w}\nabla w\nabla a+2(q-\beta)e^{(q-1)w}\nabla w\nabla b\\
&+e^{(p-1)w}\triangle a+e^{(q-1)w}\triangle b-n\big(\frac{1}{\beta}(K_2+K_3)^2+\frac{3}{2}K_4\big)\Big\}\\
&-a(p-1)e^{(p-1)w}F-b(q-1)e^{(q-1)w}F-\frac{F}{t},
\end{aligned}
$$

*where*

$$F=t\big(\beta|\nabla w|^2+ae^{(p-1)w}+be^{(q-1)w}-w_t\big).$$

*Proof.* Define

$$F=t\big(\beta|\nabla w|^2+ae^{(p-1)w}+be^{(q-1)w}-w_t\big).$$

By the Bochner formula, we can write

$$\triangle|\nabla w|^2\geq 2|\nabla^2 w|^2+2\nabla w\nabla(\triangle w)-2K_1|\nabla w|^2.$$

Note that

$$
\begin{aligned}
\triangle w_t =&(\triangle w)_t+2s\nabla^2 w+2\nabla w\big(\mathrm{div}s-\frac{1}{2}\nabla(tr_g s)\big)\\
=&w_{tt}-(|\nabla w|^2)_t-a_te^{(p-1)w}-ae^{(p-1)w}-b_te^{(q-1)w}-be^{(q-1)w}\\
&+2s\nabla^2 w+2\nabla w\big(\mathrm{div}s-\frac{1}{2}\nabla(tr_g s)\big)\\
=&2s(\nabla w,\nabla w)-2\nabla w\nabla w_t-a_te^{(p-1)w}-ae^{(p-1)w}\\
&-b_te^{(q-1)w}-be^{(q-1)w}+w_{tt}+2s\nabla^2 w+2\nabla w\big(\mathrm{div}s-\frac{1}{2}\nabla(tr_g s)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\triangle w &= -|\nabla w|^2-ae^{(p-1)w}-be^{(q-1)w}+w_t\\
&=(\frac{1}{\beta}-1)\big(ae^{(p-1)w}+be^{(q-1)w}-w_t\big)-\frac{F}{t\beta}\\
&=(\beta-1)|\nabla w|^2-\frac{F}{t}.
\end{aligned}
$$

We can write,

$$\triangle F = t\Big(\beta\triangle|\nabla w|^2 + \triangle(ae^{(p-1)w}) + \triangle(be^{(q-1)w}) - \triangle w_t\Big).$$

According to the above computations, we obtain

$$\begin{aligned}
\beta\triangle|\nabla w|^2 \geq & 2\beta|\nabla^2 w|^2 + 2\beta\nabla w\nabla(\triangle w) - 2\beta K_1|\nabla w|^2 \\
= & 2\beta|\nabla^2 w|^2 + 2\beta\nabla w\nabla\Big([(\frac{1}{\beta} - 1)\big(ae^{(p-1)w} + be^{(q-1)w} - w_t\big) - \frac{F}{t\beta}]\Big) \\
& -2\beta K_1|\nabla w|^2 \\
= & 2\beta|\nabla^2 w|^2 - \frac{2}{t}\nabla w\nabla F + 2(1-\beta)e^{(p-1)w}\nabla w\nabla a \\
& + 2(1-\beta)e^{(q-1)w}\nabla w\nabla b \\
& + 2a(1-\beta)(p-1)e^{(p-1)w}|\nabla w|^2 + 2b(1-\beta)(q-1)e^{(q-1)w}|\nabla w|^2 \\
& + 2(1-\beta)\nabla w\nabla w_t - 2K_1\beta|\nabla w|^2,
\end{aligned}$$

and, we know

$$\begin{aligned}
\triangle(ae^{(p-1)w}) &= e^{(p-1)w}\triangle a + 2(p-1)e^{(p-1)w}\nabla w\nabla a + a(p-1)^2e^{(p-1)w}|\nabla w|^2 \\
&\quad +a(p-1)e^{(p-1)w}\triangle w \\
&= e^{(p-1)w}\triangle a + 2(p-1)e^{(p-1)w}\nabla w\nabla a + a(p-1)^2e^{(p-1)w}|\nabla w|^2 \\
&\quad +a(p-1)e^{(p-1)w}[(\beta-1)|\nabla w|^2 - \frac{F}{t}].
\end{aligned}$$

So we have

$$\begin{aligned}
\triangle F \geq\ & t\Big\{2\beta|\nabla^2 w|^2 - \frac{2}{t}\nabla w\nabla F + 2(1-\beta)e^{(p-1)w}\nabla w\nabla a + 2(1-\beta)e^{(q-1)w}\nabla w\nabla b \\
& +2a(1-\beta)(p-1)e^{(p-1)w}|\nabla w|^2 + 2b(1-\beta)(q-1)e^{(q-1)w}|\nabla w|^2 \\
& +2(1-\beta)\nabla w\nabla w_t - 2k_1\beta|\nabla w|^2 + e^{(p-1)w}\triangle a + 2(p-1)e^{(p-1)w}\nabla w\nabla a \\
& +a(p-1)^2e^{(p-1)w}|\nabla w|^2 + a(p-1)e^{(p-1)w}[(\beta-1)|\nabla w|^2 - \frac{F}{t}] \\
& +e^{(q-1)w}\triangle b + 2(q-1)e^{(q-1)w}\nabla w\nabla b + b(q-1)^2e^{(q-1)w}|\nabla w|^2 \\
& +b(q-1)e^{(q-1)w}[(\beta-1)|\nabla w|^2 - \frac{F}{t}] - \Big[w_{tt} - (|\nabla w|^2)_t - a_te^{(p-1)w} \\
& -a(p-1)e^{(p-1)w}w_t - b_te^{(q-1)w} - b(q-1)e^{(q-1)w}w_t + 2s\nabla^2 w \\
& +2\nabla w(\mathrm{div}s - \frac{1}{2}\nabla(tr_g s))\Big]\Big\},
\end{aligned}$$

and

$$\begin{aligned} F_t &= \frac{F}{t} + t\Big\{2\beta(|\nabla w|^2)_t + a_t e^{(p-1)w} + a(p-1)e^{(p-1)w}w_t + b_t e^{(q-1)w} \\ &+ b(q-1)e^{(q-1)w}w_t - w_{tt}\Big\} \\ &= \frac{F}{t} + t\Big\{2\beta\nabla w\nabla w_t - 2\beta s(\nabla w,\nabla w) \\ &+ a_t e^{(p-1)w} + a(p-1)e^{(p-1)w}w_t + b_t e^{(q-1)w} \\ &\quad +b(q-1)e^{(q-1)w}w_t - w_{tt}\Big\}. \end{aligned}$$

This equation implies that

$$\begin{aligned} (\triangle - \frac{\partial}{\partial t})F \geq\; & -2\nabla w\nabla F + t\Big\{2\beta|\nabla^2 w|^2 + 2(\beta-1)s(\nabla w,\nabla w) \\ & +a(p-\beta)(p-1)e^{(p-1)w}|\nabla w|^2 + b(q-\beta)(q-1)e^{(q-1)w}|\nabla w|^2 \\ & +2(p-\beta)e^{(p-1)w}\nabla w\nabla a + 2(q-\beta)e^{(q-1)w}\nabla w\nabla b \\ & +e^{(p-1)w}\triangle a + e^{(q-1)w}\triangle b \\ & -2K_1\beta|\nabla w|^2 - 2s\nabla^2 w - 2\nabla w(\mathrm{div}s - \frac{1}{2}\nabla(tr_g s))\Big\} \\ & -a(p-1)e^{(p-1)w}F \\ & -b(q-1)e^{(q-1)w}F - \frac{F}{t}. \end{aligned}$$

By our assumptions, we have

$$-(K_2+K_3)g \leq s \leq (K_2+K_3)g$$

which implies that

$$|s|^2 \leq (K_2+K_3)^2|g|^2 = n(K_2+K_3)^2.$$

Using Young's inequality and applying those bounds yields

$$|s\nabla^2 w| \leq \frac{\beta}{2}|\nabla^2 w|^2 + \frac{1}{2\beta}|s|^2 \leq \frac{\beta}{2}|\nabla^2 w|^2 + \frac{n}{2\beta}(K_2+K_3)^2.$$

On the other hand,

$$|\mathrm{div}s - \frac{1}{2}\nabla(tr_g s)| = |g^{ij}\nabla_i s_{jl} - \frac{1}{2}g^{ij}\nabla_l s_{ij}| \leq \frac{3}{2}|g||\nabla s| \leq \frac{3}{2}\sqrt{n}K_4.$$

Finally, with the help of the following inequality,

$$|\nabla^2 w|^2 \geq \frac{1}{n}(tr\nabla^2 w)^2 = \frac{1}{n}(\triangle w)^2 = \frac{1}{n}(-|\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w} + w_t)^2.$$

We obtain

$$\begin{aligned}(\triangle - \frac{\partial}{\partial t})F \geq & -2\nabla w\nabla F + t\Big\{\frac{\beta}{n}(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 \\ & + a(p-\beta)(p-1)e^{(p-1)w}|\nabla w|^2 + b(q-\beta)(q-1)e^{(q-1)w}|\nabla w|^2 \\ & + 2(p-\beta)e^{(p-1)w}\nabla w\nabla a + 2(q-\beta)e^{(q-1)w}\nabla w\nabla b \\ & + e^{(p-1)w}\triangle a + e^{(q-1)w}\triangle b + 2(\beta-1)K_3|\nabla w|^2 \\ & - 2\beta K_1|\nabla w|^2 - \frac{n}{\beta}(K_2+K_3)^2 \\ & - 3\sqrt{n}K_4|\nabla w|\Big\} - a(p-1)e^{(p-1)w}F - b(q-1)e^{(q-1)w}F - \frac{F}{t}.\end{aligned}$$

Applying AM-GM inequality, we can write

$$3\sqrt{n}K_4|\nabla w| \leq 3K_4(\frac{n}{2} + \frac{|\nabla w|^2}{2}),$$

we get

$$\begin{aligned}(\triangle - \frac{\partial}{\partial t})F & \\ \geq & -2\nabla w\nabla F + t\Big\{\frac{\beta}{n}(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 \\ & + [a(p-\beta)(p-1)e^{(p-1)w} + b(q-\beta)(q-1)e^{(q-1)w} + 2(\beta-1)K_3 \\ & - 2\beta K_1 - \frac{3}{2}K_4]|\nabla w|^2 + 2(p-\beta)e^{(p-1)w}\nabla w\nabla a + 2(q-\beta)e^{(q-1)w}\nabla w\nabla b \\ & + e^{(p-1)w}\triangle a + e^{(q-1)w}\triangle b - n\big(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4\big)\Big\} \\ & - a(p-1)e^{(p-1)w}F - b(q-1)e^{(q-1)w}F - \frac{F}{t}.\end{aligned}$$

This completes the proof. □

Let's take a cut-off function $\tilde{\varphi}$ defined on $[0,\infty)$ such that $0 \leq \tilde{\varphi}(r) \leq 1$, $\tilde{\varphi}(r) = 1$ for $r \in [0,1]$ and, $\tilde{\varphi}(r) = 0$ for $r \in [2,\infty)$. Furthermore $\tilde{\varphi}$ satisfies the following inequalities for some positive constants $c_1$ and $c_2$.

$$-\frac{\tilde{\varphi}'(r)}{\tilde{\varphi}^{\frac{1}{2}}(r)} \leq c_1, \qquad \tilde{\varphi}''(r) \geq -c_2.$$

Define $r(x,t) := d(x,x_0,t)$ and, set

$$\varphi(x,t) = \tilde{\varphi}(\frac{r(x,t)}{R}).$$

Using Corollary in page 53 of [2], we can assume $\varphi(x,t) \in C^2(M)$ with support in $Q_{2R,T}$. A direct calculation indicates that on $Q_{2R,T}$, we have

$$\frac{|\nabla\varphi|^2}{\varphi} \leq \frac{c_1^2}{R^2}. \tag{2.2}$$

According to the Laplace comparison theorem in [1], we can write

$$\triangle\varphi \geq -\frac{(n-1)(1+\sqrt{K_1}R)c_1^2+c^2}{R^2}. \tag{2.3}$$

For any $0 < T_1 < T$, suppose that $\varphi F$ attains it maximum value at the point $(x_0, t_0)$ in the cube $Q_{2R,T_1}$. We can assume that this maximum value is positive (otherwise the proof of our main theorem will be trivial). At the maximum point $(x_0, t_0)$, we have

$$\nabla(\varphi F) = 0, \quad \triangle(\varphi F) \leq 0, \quad (\varphi F)_t \geq 0,$$

which follows that

$$0 \geq (\triangle - \frac{\partial}{\partial t})(\varphi F) = (\triangle\varphi)F - \varphi_t F + \varphi(\triangle - \frac{\partial}{\partial t})F + 2\nabla\varphi\nabla F.$$

So, we can write

$$(\triangle\varphi)F - \varphi_t F + \varphi(\triangle - \frac{\partial}{\partial t})F - 2F\varphi^{-1}|\nabla\varphi|^2 \leq 0. \tag{2.4}$$

Also, we know (see [9], p. 494) there exists a positive constant $c_3$ such that

$$-\varphi_t F \geq -\sqrt{c_3}K_2 F.$$

The inequality (2.4) together with the inequalities (2.2) and (2.3) yield

$$\varphi(\triangle - \frac{\partial}{\partial t})F \leq HF, \tag{2.5}$$

where

$$H = \frac{(n-1)(1+\sqrt{K_1}R)c_1^2 + c_2 + 2c_1^2}{R^2} + \sqrt{c_3}K_2.$$

*Proof of Theorem 1.1.* At the maximum point $(x_0, t_0)$, by (2.5) and Lemma 2.2, we have

$$\begin{aligned}
0 &\geq \varphi(\triangle - \frac{\partial}{\partial t})F - HF \geq -HF + \varphi\{-2\nabla w\nabla F + \frac{\beta t_0}{n}(w_t - |\nabla w|^2 - ae^{(p-1)w} \\
&- be^{(q-1)w})^2 + t_0[a(p-\beta)(p-1)e^{(p-1)w} + b(q-\beta)(q-1)e^{(q-1)w} + 2(\beta-1)K_3 \\
&- 2\beta K_1 - \frac{3}{2}K_4]|\nabla w|^2 + 2t_0(p-\beta)e^{(p-1)w}\nabla w\nabla a + 2t_0(q-\beta)e^{(q-1)w}\nabla w\nabla b \\
&+ t_0 e^{(p-1)w}\triangle a + t_0 e^{(q-1)w}\triangle b - nt_0\big(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4\big) \\
&- a(p-1)e^{(p-1)w}F - b(q-1)e^{(q-1)w}F - \frac{F}{t_0}\}
\end{aligned}$$

$$
\begin{aligned}
\geq & - HF + 2F\nabla w\nabla\varphi + \frac{\beta t_0}{n}\varphi(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 \\
& + t_0\varphi[a(p-\beta)(p-1)e^{(p-1)w} + b(q-\beta)(q-1)e^{(q-1)w} + 2(\beta-1)K_3 \\
& - 2\beta K_1 - \frac{3}{2}K_4]|\nabla w|^2 + 2t_0\varphi(p-\beta)e^{(p-1)w}\nabla w\nabla a + 2t_0\varphi(q-\beta)e^{(q-1)w}\nabla w\nabla b \\
& + t_0\varphi e^{(p-1)w}\triangle a + t_0\varphi e^{(q-1)w}\triangle b - nt_0\varphi\big(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4\big) \\
& - a(p-1)e^{(p-1)w}\varphi F - b(q-1)e^{(q-1)w}\varphi F - \varphi t_0^{-1}F
\end{aligned}
$$

$$
\begin{aligned}
\geq & - HF + 2F\nabla w\nabla\varphi + \frac{\beta t_0}{n}\varphi(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 \\
& - t_0\varphi[|a|(p-\beta)(p-1)M_1^{(p-1)} + |b|(q-\beta)(q-1)M_1^{(q-1)} + 2(1-\beta)K_3 \\
& + 2\beta K_1 + \frac{3}{2}K_4]|\nabla w|^2 + 2t_0\varphi(\beta-p)M_1^{(p-1)}\gamma_a|\nabla w| + 2t_0\varphi(\beta-q)M_1^{(q-1)}\gamma_b|\nabla w| \\
& - t_0\varphi M_1^{(p-1)}\theta_a - t_0\varphi M_1^{(q-1)}\theta_b - nt_0\varphi\big(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4\big) \\
& + |a|(p-1)M_1^{(p-1)}\varphi F + |b|(q-1)M_1^{(q-1)}\varphi F - \varphi t_0^{-1}F \\
= & - HF + 2F\nabla w\nabla\varphi + \frac{\beta t_0}{n}\varphi(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 \\
& - t_0\varphi[|a|(p-\beta)(p-1)M_1^{(p-1)} + |b|(q-\beta)(q-1)M_1^{(q-1)}]|\nabla w|^2 \\
& - t_0\varphi[2(1-\beta)K_3 + 2\beta K_1 + \frac{3}{2}K_4]|\nabla w|^2 - t_0\varphi[2(p-\beta)M_1^{(p-1)}\gamma_a \\
& + 2(q-\beta)M_1^{(q-1)}\gamma_b]|\nabla w| \\
& - t_0\varphi[M_1^{(p-1)}\theta_a + M_1^{(q-1)}\theta_b + n\big(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4\big)] \\
& + |a|(p-1)M_1^{(p-1)}\varphi F + |b|(q-1)M_1^{(q-1)}\varphi F - \varphi t_0^{-1}F.
\end{aligned}
$$

For the sake of simplicity, set

$$
\begin{aligned}
\widetilde{C_1} &= 2(1-\beta)K_3 + 2\beta K_1 + \frac{3}{2}K_4 \\
\widetilde{C_2} &= M_1^{(p-1)}\theta_a + M_1^{(q-1)}\theta_b + n\big(\frac{1}{\beta}(K_2+K_3)^2 + \frac{3}{2}K_4\big)
\end{aligned}
$$

and

$$
\widetilde{C_3} = -\frac{[(p-\beta)M_1^{(p-1)}\gamma_a + (q-\beta)M_1^{(q-1)}\gamma_b]^2}{|a|(p-\beta)(p-1)M_1^{(p-1)} + |b|(q-\beta)(q-1)M_1^{(q-1)}}.
$$

Using the inequality $ax^2 + bx \leq -\dfrac{b^2}{4a}$ which holds for $a < 0$, we obtain

$$\begin{aligned} 0 \geq & - HF + 2F\nabla w \nabla \varphi + \frac{\beta t_0}{n}\varphi(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 \\ & - t_0\varphi[\widetilde{C_3} + \widetilde{C_2} + \widetilde{C_1}|\nabla w|^2] + |a|(p-1)M_1^{(p-1)}\varphi F \\ & + |b|(q-1)M_1^{(q-1)}\varphi F - \varphi t_0^{-1} F. \end{aligned}$$

Noting the fact that $0 < \varphi < 1$ and multiplying both sides of the above inequality by $t_0\varphi$, leads to

$$\begin{aligned} 0 \geq & - Ht_0\varphi F + 2t_0\varphi F\nabla w \nabla \varphi + \frac{\beta t_0^2}{n}\varphi^2(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 \\ & - \widetilde{C_1} t_0^2\varphi^2|\nabla w|^2 - (\widetilde{C_2} + \widetilde{C_3})t_0^2\varphi^2 \\ & + |a|(p-1)M_1^{(p-1)}t_0\varphi F + |b|(q-1)M_1^{(q-1)}t_0\varphi F - \varphi F \\ \geq & - Ht_0\varphi F - \frac{2c_1}{R}t_0\varphi F|\nabla w|\varphi^{\frac{3}{2}} + |a|(p-1)M_1^{(p-1)}t_0\varphi F \\ & + |b|(q-1)M_1^{(q-1)}t_0\varphi F - \varphi F \\ & + \frac{\beta t_0^2}{n}\varphi^2[(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 - \frac{n}{\beta}\widetilde{C_1}|\nabla w|^2] - (\widetilde{C_2} + \widetilde{C_3})t_0^2\varphi^2, \end{aligned}$$

where in the last inequality the following fact is applied

$$-2\varphi\nabla w \nabla F = 2F\nabla w\nabla\varphi \geq -2F|\nabla w||\nabla\varphi| \geq -\frac{2c_1}{R}\varphi^{\frac{1}{2}}F|\nabla w|.$$

Assume that

$$y = \varphi|\nabla w|^2, \qquad z = \varphi(ae^{(p-1)w} + be^{(q-1)w} - w_t).$$

So, we can write

$$\begin{aligned} 0 \geq & \varphi F(-Ht_0 + |a|(p-1)M_1^{(p-1)}t_0 + |b|(q-1)M_1^{(q-1)}t_0 - 1) - \frac{2c_1}{R}t_0 F|\nabla w|\varphi^{\frac{3}{2}} \\ & + \frac{\beta t_0^2}{n}\varphi^2[(w_t - |\nabla w|^2 - ae^{(p-1)w} - be^{(q-1)w})^2 - \frac{n}{\beta}\widetilde{C_1}|\nabla w|^2] - (\widetilde{C_2} + \widetilde{C_3})t_0^2\varphi^2 \\ \geq & \varphi F(-Ht_0 + |a|(p-1)M_1^{(p-1)}t_0 + |b|(q-1)M_1^{(q-1)}t_0 - 1) \\ & + \frac{\beta t_0^2}{n}\{(y-z)^2 - \frac{n}{\beta}\widetilde{C_1}y - 2nc_1R^{-1}y^{\frac{1}{2}}(y - \frac{1}{\beta}z)\} - (\widetilde{C_2} + \widetilde{C_3})t_0^2. \end{aligned}$$

For all $a, b > 0$ the inequality $ax^2 - bx \geq -\dfrac{b^2}{4a}$ holds for every real number $x$. Using this inequality, we obtain

$$
\begin{aligned}
&\frac{\beta t_0^2}{n}\{(y-z)^2-\frac{n}{\beta}\widetilde{C_1}y-2nc_1R^{-1}y^{\frac{1}{2}}(y-\frac{1}{\beta}z)\}\\
=&\frac{\beta t_0^2}{n}\{\beta^2(y-\frac{z}{\beta})^2+(1-\beta^2)y^2-\frac{n}{\beta}\widetilde{C_1}y+[2(\beta-\beta^2)y-2\frac{nc_1}{R}y^{\frac{1}{2}}](y-\frac{z}{\beta})\}\\
\geq&\frac{\beta t_0^2}{n}\{\beta^2(y-\frac{z}{\beta})^2-\frac{n^2\widetilde{C_1}^2}{4\beta^2(1-\beta)^2}-\frac{n^2c_1^2}{2R^2(\beta-\beta^2)}(y-\frac{z}{\beta})\}\\
=&\frac{\beta}{n}(\varphi F)^2-\frac{n\widetilde{C_1}^2t_0^2}{4\beta(1-\beta)^2}-\frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}(\varphi F).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{\beta}{n}(\varphi F)^2+\big[-Ht_0+|a|(p-1)M_1^{(p-1)}t_0\\
&+|b|(q-1)M_1^{(q-1)}t_0-1-\frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}\big](\phi F)\\
&-\frac{n\widetilde{C_1}^2t_0^2}{4\beta(1-\beta)^2}-(\widetilde{C_2}+\widetilde{C_3})t_0^2\leq 0.
\end{aligned}
$$

As we know, the inequality $Ax^2-2Bx\leq C$, yields $x\leq\frac{2B}{A}+\sqrt{\frac{C}{A}}$. So, we get

$$
\begin{aligned}
\varphi F\leq&\frac{n}{\beta}\Big(Ht_0+|a|(1-p)M_1^{(p-1)}t_0+|b|(1-q)M_1^{(q-1)}t_0+1+\frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}\Big)\\
&+\Big[\frac{n}{\beta}(\frac{n\widetilde{C_1}^2}{4\beta(1-\beta)^2}+\widetilde{C_2}+\widetilde{C_3})\Big]^{\frac{1}{2}}t_0.
\end{aligned}
$$

If $d(x,x_0,T_1)\leq 2R$, we know that $\varphi(x,T_1)=1$. Then

$$
\begin{aligned}
F(x,T_1)&=T_1(\beta|\nabla w|^2+ae^{(p-1)w}+be^{(q-1)w}-w_t)\\
&\leq\varphi F(x_0,t_0)\\
&\leq\frac{n}{\beta}\Big(Ht_0+|a|(1-p)M_1^{(p-1)}t_0+|b|(1-q)M_1^{(q-1)}t_0+1+\frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}\Big)\\
&+\Big[\frac{n}{\beta}(\frac{n\widetilde{C_1}^2}{4\beta(1-\beta)^2}+\widetilde{C_2}+\widetilde{C_3})\Big]^{\frac{1}{2}}t_0.
\end{aligned}
$$

Since $T_1$ was supposed to be arbitrary, we can get the assertion. □

*Proof of Corollary 1.3.* For any points $(x_1,t_1)$ and $(x_2,t_2)$ on $M\times[0,T)$ with $0<t_1<t_2$, we take a curve $\gamma(t)$ parametrized with $\gamma(t_1)=x_1$ and $\gamma(t_2)=x_2$. In

the ray of Corollary 1.2, one can get

$$\begin{aligned}
&\log u(x_2,t_2)-\log u(x_1,y_1)\\
=&\int_{t_1}^{t_2}\big((\log u)_t+\langle\nabla\log u,\gamma'\rangle\big)dt\\
\geq&\int_{t_1}^{t_2}\Big(\beta|\nabla\log u|^2+au^{p-1}+bu^{q-1}-\overline{H_1}-H_2-\frac{n}{\beta t}-|\nabla\log u||\gamma'|\Big)dt\\
\geq&-\int_{t_1}^{t_2}\big(\frac{1}{4\beta}|\gamma'|^2-au^{p-1}-bu^{q-1}+\overline{H_1}+H_2+\frac{n}{\beta t}\big)dt\\
\geq&-\Big(\log(\frac{t_2}{t_1})^{\frac{n}{\beta}}+(\overline{H_1}+H_2)(t_2-t_1)+\int_{t_1}^{t_2}\frac{1}{4\beta}|\gamma'|^2dt\Big)
\end{aligned}$$

which means

$$\log\frac{u(x_1,t_1)}{u(x_2,t_2)}\leq\log(\frac{t_2}{t_1})^{\frac{n}{\beta}}+(\overline{H_1}+H_2)(t_2-t_1)+\int_{t_1}^{t_2}\frac{1}{4\beta}|\gamma'|^2dt.$$

Hence,

$$u(x_1,t_1)\leq u(x_2,t_2)(\frac{t_2}{t_1})^{\frac{n}{\beta}}e^{\Psi(x_1,x_2,t_1,t_2)+(\overline{H_1}+H_2)(t_2-t_1)},$$

where $\Psi(x_1,x_2,t_1,t_2)=\inf_\gamma\int_{t_1}^{t_2}\frac{1}{4\beta}|\gamma'|^2dt$, and $\overline{H_1}=\frac{n}{\beta}\big(\sqrt{c_3}K_2+a(1-p)M_1^{(p-1)}+b(1-q)M_1^{(q-1)}\big)$, and

$$\begin{aligned}
H_2=\Big[&\frac{n^2}{4\beta^2(1-\beta)^2}\big(2(1-\beta)K_3+2\beta K_1+\frac{3}{2}K_4\big)^2\\
&+\frac{n}{\beta}\{M_1^{(p-1)}\theta_a+M_1^{(q-1)}\theta_b+n\big(\frac{1}{\beta}(K_2+K_3)^2+\frac{3}{2}K_4\big)\}\\
&-\frac{n}{\beta}\{\frac{[(p-\beta)M_1^{(p-1)}\gamma_a+(q-\beta)M_1^{(q-1)}\gamma_b]^2}{a(p-\beta)(p-1)M_1^{(p-1)}+b(q-\beta)(q-1)M_1^{(q-1)}}\}\Big]^{\frac{1}{2}}.
\end{aligned}$$

□

# References


[1] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampre equations*, Springer, New York, 1982.

[2] E. Calabi, *An extension of E. Hopf s maximum principle with an application to Riemannian geometry*, Duke Math. J.,**25(1)**(1958), 45-56.

[3] B. Chow and D. Knopf, *The Ricci flow; An introduction*, Mathematical survey and monographs, vol. 110, AMS, New York, 2004.

[4] R. S. Hamilton,*The Harnack estimate for the Ricci flow*, J. Differ. Geom., **37(1)**(1993), 225-243.

[5] S. Kuang and Q. S. Zhang,*A gradient estimate for all positive solutions of the conjugate heat equation under Ricci flow*, J. Funct. Anal., **255(4)**(2008), 1008-1023.

[6] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math., **156(3-4)**(1986), 153-201.

[7] S. P. Liu, *Gradient estimates for solutions of the heat equation under Ricci flow*, Pacific J. Math., **243(1)**(2009), 165-180.

[8] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv: math/0211159, 2002.

[9] J. Sun, *Gradient estimates for positive solutions of the heat equation under geometric flow*, Pacific J. Math., **253(2)**(2011), 489-510.

[10] J. Y. Wu, *Gradient estimates for a nonlinear parabolic equation and Liouville theorems*, Manuscripta mathematica, **159(3-4)**(2019),511-547.