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On Coefficients of a Certain Subclass of Starlike and Bi–starlike Functions

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Abstract. In this paper we investigate a subclass $\mathcal{M}(\alpha)$ of the class of starlike functions in the unit disk $|z| < 1$. $\mathcal{M}(\alpha)$, $\pi/2 \leq \alpha < \pi$, is the set of all analytic functions f in the unit disk $|z| < 1$ with the normalization $f(0) = f'(0) - 1 = 0$ that satisfy the condition

$$1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\alpha}{2\sin\alpha} \quad (z \in \Delta).$$

The class $\mathcal{M}(\alpha)$ was introduced by Kargar et al. [*Complex Anal. Oper. Theory* **11**: 1639–1649, 2017]. In this paper some basic geometric properties of the class $\mathcal{M}(\alpha)$ are investigated. Among others things, coefficients estimates and bound are given for the Fekete-Szegö functional associated with the k–th root transform $[f(z^k)]^{1/k}$. Also a certain subclass of bi–starlike functions is introduced and the bounds for the initial coefficients are obtained.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ of all univalent functions f in Δ is denoted

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by $\mathcal{S}$. We denote by $\mathcal{P}$ the well-known class of analytic functions p with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$, $z \in \Delta$. We also denote by $\mathcal{B}$ the class of analytic functions $w(z)$ in Δ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \Delta$. If f and g are two functions in $\mathcal{A}$, then we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a function $w \in \mathcal{B}$ such that $f(z) = g(w(z))$ for all $z \in \Delta$. As a special case, if the function g is univalent in Δ, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$

A function $f \in \mathcal{S}$ is *starlike* (with respect to 0) if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0, 1]$. The class of starlike functions is denoted by $\mathcal{S}^*$. We say that $f \in \mathcal{S}^*(\gamma)$ $(0 \le \gamma < 1)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in \Delta).$$

The equality $\mathcal{S}^*(0) = \mathcal{S}^*$ is well known. Recently Kargar *et al*. (see [4]) introduced a certain subclass of starlike functions as follows.

Definition 1.1. Let $\pi/2 \le \alpha < \pi$. Then the function $f \in \mathcal{A}$ belongs to the class $\mathcal{M}(\alpha)$ if f satisfies

$$1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\alpha}{2\sin\alpha} \qquad (z \in \Delta). \tag{1.2}$$

Consider the function ϕ as follows

$$\phi(\alpha) := 1 + \frac{\alpha - \pi}{2\sin\alpha} \quad (\pi/2 \le \alpha < \pi).$$

It is clear that $\phi(\pi/2) = 1 - \pi/4 \approx 0.2146$ and

$$\lim_{\alpha\to\pi^-} \phi(\alpha) = \frac{1}{2}.$$

Thus, the class $\mathcal{M}(\alpha)$ is a subclass of the class $f \in \mathcal{S}^*(\phi(\pi/2))$ of starlike functions of order $\phi(\pi/2) = 1 - \pi/4$.

By the subordination principle we have the following lemma.

Lemma 1.2. (see [4]) *Let $f(z) \in \mathcal{A}$ and $\pi/2 \le \alpha < \pi$. Then $f \in \mathcal{M}(\alpha)$ if and only if*

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \mathcal{B}_\alpha(z) \qquad (z \in \Delta), \tag{1.3}$$

where

$$\mathcal{B}_\alpha(z) := \frac{1}{2i\sin\alpha}\log\left(\frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}}\right) \qquad (z \in \Delta). \tag{1.4}$$

The function $\mathcal{B}_\alpha(z)$ is convex univalent in Δ and maps Δ onto

$$\Omega_\alpha := \left\{ w : \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}(w) < \frac{\alpha}{2\sin\alpha} \right\}, \tag{1.5}$$

in other words, the image of Δ is a vertical strip when $\pi/2 \le \alpha < \pi$. For other α, $\mathcal{B}_\alpha(z)$ is convex univalent in Δ and maps Δ onto the convex hull of three points (one of which may be that point at infinity) on the boundary of Ω_α. Therefore, in other cases, we obtain a trapezium, or a triangle, see [3]. Also, we have that

$$\mathcal{B}_\alpha(z) = \sum_{n=1}^{\infty} A_n z^n \qquad (z \in \Delta), \tag{1.6}$$

where

$$A_n = \frac{(-1)^{(n-1)} \left(e^{in\alpha} - e^{-in\alpha}\right)}{2in \sin\alpha} \qquad (n = 1, 2, \ldots). \tag{1.7}$$

The following lemma will be useful.

Lemma 1.3. (see [9]) *Let $q(z) = \sum_{n=1}^{\infty} Q_n z^n$ be analytic and univalent in Δ, and suppose that $q(z)$ maps Δ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} P_n z^n$ is analytic in Δ and satisfies the following subordination*

$$p(z) \prec q(z) \qquad (z \in \Delta),$$

then

$$|P_n| \le |Q_1| \qquad n \ge 1.$$

This paper is organized as follows. In Section 2 we study the class $\mathcal{M}(\alpha)$. We consider the coefficient estimates and Fekete-Szegö inequality. Also, in Section 3 we introduce a certain subclass $\mathcal{M}_\sigma(\alpha)$ of bi–univalent functions and we estimate the initial coefficients of functions belonging to $\mathcal{M}_\sigma(\alpha)$.

2. Coefficient Estimates

Theorem 2.1. ([10]) *Let $\pi/2 \le \alpha < \pi$. If a function $f \in \mathcal{A}$ of the form* (1.1) *belongs to the class $\mathcal{M}(\alpha)$, then*

$$|a_n| \le 1 \qquad (n = 2, 3, 4, \ldots). \tag{2.1}$$

Here, we consider the problem of finding sharp upper bounds for the Fekete-Szegö coefficient functional associated with the k–th root transform for functions in the class $\mathcal{M}(\alpha)$. For a univalent function $f(z)$ of the form (1.1), the k–th root transform is defined by

$$F(z) = [f(z^k)]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \qquad (z \in \Delta). \tag{2.2}$$

In order to prove next result, we need the following lemma due to Keogh and Merkes [5].

Lemma 2.2. (see [5]) *Let the function $g(z)$ given by*

$$g(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

be in the class $\mathcal{P}$. Then, for any complex number μ

$$|c_2 - \mu c_1^2| \le 2\max\{1, |2\mu - 1|\}.$$

The result is sharp.

Theorem 2.3. *Let $\pi/2 \le \alpha < \pi$. Suppose also that $f \in \mathcal{M}(\alpha)$ and let F be the k–th root transform of f defined by* (2.2)*. Then, for any complex number μ,*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le \frac{1}{2k}\max\left\{1, \left|\frac{2\mu - k - 1 + k\cos\alpha}{2k}\right|\right\}. \tag{2.3}$$

The result is sharp.

Proof. Since $f \in \mathcal{M}(\alpha)$, from Lemma 1.2 and by definition of subordination, there exists a function $w \in \mathcal{B}$ such that

$$zf'(z)/f(z) = 1 + \mathcal{B}_\alpha(w(z)). \tag{2.4}$$

We define

$$p(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots, \tag{2.5}$$

and note that $p \in \mathcal{P}$. Relationships (1.6) and (2.5) give us

$$1 + \mathcal{B}_\alpha(w(z)) = 1 + \frac{1}{2}A_1 p_1 z + \left(\frac{1}{4}A_2 p_1^2 + \frac{1}{2}A_1\left(p_2 - \frac{1}{2}p_1^2\right)\right) z^2 + \cdots, \tag{2.6}$$

where $A_1 = 1$ and $A_2 = -\cos\alpha$. If we equate the coefficients of z and z^2 on both sides of (2.4), then we get

$$a_2 = \frac{1}{2}p_1, \tag{2.7}$$

and

$$a_3 = \frac{1}{8}(1 - \cos\alpha)p_1^2 + \frac{1}{4}\left(p_2 - \frac{1}{2}p_1^2\right). \tag{2.8}$$

For each f given by (1.1) and with a simple calculation we have

$$F(z) = [f(z^{1/k})]^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2\right) z^{2k+1} + \cdots. \tag{2.9}$$

Moreover by (2.2) and (2.9), we obtain

$$b_{k+1} = \frac{1}{k}a_2 \quad \text{and} \quad b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{2.10}$$

By inserting (2.7) and (2.8) into (2.10), we get

$$b_{k+1} = \frac{p_1}{2k},$$

and

$$b_{2k+1} = \frac{1}{8k}\left(1 - \cos\alpha - \frac{k-1}{k}\right)p_1^2 + \frac{1}{4k}\left(p_2 - \frac{1}{2}p_1^2\right).$$

Therefore,

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{1}{4k}\left[p_2 - \frac{2\mu + k - 1 + k\cos\alpha}{2k}p_1^2\right]. \tag{2.11}$$

Applying Lemma 2.2 in (2.11) with

$$\mu' = \frac{2\mu + k - 1 + k\cos\alpha}{2k},$$

gives the inequality (2.3). For the sharpness it is sufficient to consider the k–th root transforms of the function

$$f(z) = z\exp\left(\int_0^z \frac{\mathcal{B}_\alpha(w(t))}{t}\mathrm{d}t\right). \tag{2.12}$$

It is clear that $f \in \mathcal{M}(\alpha)$. If we take in (2.12) $w(z) = z$, then from (2.5) we obtain $p_1 = p_2 = 2$ hence from (2.11) we get

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| = \frac{1}{2k}\left|\frac{2\mu - k - 1 + k\cos\alpha}{2k}\right|.$$

If we take in (2.12) $w(z) = z^2$, then from (2.5) we obtain $p_1 = 0$ while $p_2 = 2$ hence from (2.11) we get for this case

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| = \frac{1}{2k}.$$

It shows the sharpness of (2.3) and ends the proof. □

The problem of finding sharp upper bound for the coefficient functional $|a_3 - \mu a_2^2|$ for different subclasses of the class $\mathcal{A}$ is known as the Fekete-Szegö problem. Putting $k = 1$ in the Theorem 2.3 gives us:

Corollary 2.4. *Let $\alpha \in [\pi/2, \pi)$. Suppose also that $f \in \mathcal{M}(\alpha)$. Then, for any complex number μ,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{2}\max\left\{1, \left|\frac{2\mu - 2 + \cos\alpha}{2}\right|\right\}. \tag{2.13}$$

The result is sharp.

Putting $\alpha = \pi/2$, in the Corollary 2.4, we get:

Corollary 2.5. *Assume that the function f given by (1.1) satisfies in the following two-sided inequality:*

$$1 - \frac{\pi}{4} < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\pi}{4} \qquad z \in \Delta,$$

then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2}\max\{1, |\mu - 1|\} \qquad (\mu \in \mathbb{C}). \tag{2.14}$$

If we take $\alpha \to \pi^-$ in the Corollary 2.4, then we have:

Corollary 2.6. *Assume that the function f given by (1.1) satisfies in the following inequality:*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 1 - \frac{\pi}{4} \qquad z \in \Delta,$$

then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{2}\max\{1, |(2\mu - 3)/2|\} \qquad (\mu \in \mathbb{C}). \tag{2.15}$$

Corollary 2.7. *Let the function f, given by (1.1), be in the class $\mathcal{M}(\alpha)$. Also let the function $f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n$ be the inverse of f. Then*

$$|b_2| \leq 1, \tag{2.16}$$

and

$$|b_3| \leq \frac{1}{2}|6 - \cos\alpha| \qquad \pi/2 \leq \alpha < \pi. \tag{2.17}$$

We remark that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z$ $(z \in \Delta)$ and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0; \quad r_0 \geq 1/4),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2.18}$$

Proof Comparing (2.18) with $f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n$, gives us

$$b_2 = -a_2 \quad \text{and} \quad b_3 = 2a_2^2 - a_3.$$

Applying Theorem 2.1 we get

$$|b_2| = |a_2| \leq 1.$$

The second inequality (2.17) follows by taking $\mu = -2$ in the Corollary 2.4. $\square$

3. Bi–Univalent Functions

First, we recall that a function $f \in \mathcal{A}$ is said to be bi–univalent in Δ if f univalent in Δ and f^{-1} has an univalent extension from $|w| < r_0 < 1$ to Δ. We denote by σ the class of bi–univalent functions in the unit disk Δ.

In 1967 Lewin [6] introduced the class σ of bi-univalent functions. He obtained the bound for the second coefficient. Recently, several authors have subsequently studied similar problems in this direction (see [2, 7]). For example, Brannan and Taha [1] considered certain subclasses of bi–univalent functions, similar to the familiar subclasses of univalent functions including of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients.

In this section we introduce by $\mathcal{M}_\sigma(\alpha)$ a certain subclass of bi–starlike functions as follows. Also, we obtain the bound for the initial coefficients.

Definition 3.1. *A function $f \in \sigma$ is said to be in the class $\mathcal{M}_\sigma(\alpha)$, if the following inequalities hold:*

$$1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\alpha}{2\sin\alpha} \qquad (z \in \Delta). \tag{3.1}$$

and

$$1 + \frac{\alpha - \pi}{2\sin\alpha} < \mathrm{Re}\left\{\frac{wg'(w)}{g(w)}\right\} < 1 + \frac{\alpha}{2\sin\alpha} \qquad (w \in \Delta), \tag{3.2}$$

where $g(w) = f^{-1}(w)$ and $\pi/2 \le \alpha < \pi$.

For functions in the class $\mathcal{M}_\sigma(\alpha)$, the following result is obtained.

Theorem 3.2. *Let the function $f \in \mathcal{A}$ of the form* (1.1) *belongs to the class $\mathcal{M}_\sigma(\alpha)$. Then*

$$|a_2| \le \frac{1}{\sqrt{2 + \cos\alpha}} \qquad \pi/2 \le \alpha < \pi, \tag{3.3}$$

and

$$|a_3| \le 2 + \cos\alpha \qquad \pi/2 \le \alpha < \pi. \tag{3.4}$$

Proof. Let $f \in \mathcal{M}_\sigma(\alpha)$ and $g = f^{-1}$. Then using Lemma 1.2, there are analytic functions $u, v \in \mathcal{B}$, satisfying

$$zf'(z)/f(z) = 1 + \mathcal{B}_\alpha(u(z)) \quad \text{and} \quad wg'(w)/g(w) = 1 + \mathcal{B}_\alpha(v(z)), \tag{3.5}$$

where $\mathcal{B}_\alpha(.)$ defined by (1.4). Define the functions k and l by

$$k(z) = \frac{1+u(z)}{1-u(z)} = 1+k_1z+k_2z^2+\cdots \quad \text{and} \quad l(z) = \frac{1+v(z)}{1-v(z)} = 1+l_1z+l_2z^2+\cdots,$$

or, equivalently,

$$u(z) = \frac{k(z)-1}{k(z)+1} = \frac{1}{2}\left(k_1z + \left(k_2 - \frac{k_1^2}{2}\right)z^2 + \cdots\right), \tag{3.6}$$

and

$$v(z) = \frac{l(z)-1}{l(z)+1} = \frac{1}{2}\left(l_1z + \left(l_2 - \frac{l_1^2}{2}\right)z^2 + \cdots\right). \tag{3.7}$$

It is clear that the functions $k(z)$ and $l(z)$ belong to class $\mathcal{P}$ and we have $|k_i| \le 2$ and $|l_i| \le 2$ $(i = 1, 2, \ldots)$ (see [8]). However, clearly

$$\frac{zf'(z)}{f(z)} = 1 + \mathcal{B}_\alpha\left(\frac{k(z)-1}{k(z)+1}\right) \quad \text{and} \quad \frac{wg'(w)}{g(w)} = 1 + \mathcal{B}_\alpha\left(\frac{l(z)-1}{l(z)+1}\right). \tag{3.8}$$

From (1.6), (3.6) and (3.7), we have

$$1 + \mathcal{B}_\alpha\left(\frac{k(z)-1}{k(z)+1}\right) = 1 + \frac{1}{2}A_1k_1z + \left(\frac{1}{2}A_1\left(k_2 - \frac{k_1^2}{2}\right) + \frac{1}{4}A_2k_1^2\right)z^2 + \cdots, \tag{3.9}$$

and

$$1 + \mathcal{B}_\alpha\left(\frac{l(z)-1}{l(z)+1}\right) = 1 + \frac{1}{2}A_1l_1z + \left(\frac{1}{2}A_1\left(l_2 - \frac{l_1^2}{2}\right) + \frac{1}{4}A_2l_1^2\right)z^2 + \cdots, \tag{3.10}$$

where $A_1 = 1$ and $A_2 = -\cos\alpha$, are given by (1.7). By suitably comparing coefficients of (3.5), we get

$$a_2 = \frac{1}{2}A_1k_1, \tag{3.11}$$

$$2a_3 - a_2^2 = \frac{1}{2}A_1\left(k_2 - \frac{k_1^2}{2}\right) + \frac{1}{4}A_2k_1^2, \tag{3.12}$$

$$-a_2 = \frac{1}{2}A_1l_1, \tag{3.13}$$

and

$$3a_2^2 - 2a_3 = \frac{1}{2}A_1\left(l_2 - \frac{l_1^2}{2}\right) + \frac{1}{4}A_2l_1^2. \tag{3.14}$$

From (3.11) and (3.13), we get

$$k_1 = -l_1 \tag{3.15}$$

Also, from (3.12)-(3.15), we find that
(3.16)

$$a_2^2 = \frac{A_1^3(k_2+l_2)}{4(A_1^2+A_1-A_2)} = \frac{k_2+l_2}{4(2+\cos\alpha)} \qquad (\text{with } \ \mathrm{A}_1 = 1 \ \text{ and } \ \mathrm{A}_2 = -\cos\alpha).$$

Therefore, we have

$$|a_2^2| \le \frac{|k_2|+|l_2|}{4(2+\cos\alpha)} \le \frac{1}{2+\cos\alpha}.$$

This gives the bound on $|a_2|$ as asserted in (3.3). Now, further computations from (3.12) and (3.14)-(3.16) lead to

$$a_3 = \frac{1}{8}\left(A_1(3k_2+l_2) + 2k_1^2(A_2-A_1)\right) = \frac{1}{8}\left(3k_2+l_2+2k_1^2(-\cos\alpha-1)\right).$$

Since $|k_i| \le 2$ and $|l_i| \le 2$, we have

$$|a_3| \le 1 + |1+\cos\alpha|.$$

Therefore, the proof of Theorem 3.2 is completed. □

Corollary 3.3. *Let the function f be in the class $\mathcal{M}_\sigma(\pi/2)$. Then*

$$|a_2| \le \sqrt{2}/2 \approx 0.7071068\ldots,$$

and

$$|a_3| \le 2.$$

Also, if we take $\alpha \to \pi^-$, in Theorem 3.2 we get

$$|a_i| \le 1 \qquad (i = 2, 3).$$

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