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3-Dimensional Trans-Sasakian Manifolds with Gradient Generalized Quasi-Yamabe and Quasi-Yamabe Metrics

Mohammed Danish Siddiqi
Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia
e-mail : msiddiqi@jazanu.edu.sa

Sudhakar Kumar Chaubey
Section of Mathematics, Department of Information Technology, University of Technology and Applied Sciences, Shinas, P. O. box 77, Postal Code 324, Oman
e-mail : sudhakar.chaubey@shct.edu.om

Ghodratallah Fasihi Ramandi*
Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran
e-mail : fasihi@sci.ikiu.ac.ir

Abstract. This paper examines the behavior of a 3-dimensional trans-Sasakian manifold equipped with a gradient generalized quasi-Yamabe soliton. In particular, It is shown that α-Sasakian, β-Kenmotsu and cosymplectic manifolds satisfy the gradient generalized quasi-Yamabe soliton equation. Furthermore, in the particular case when the potential vector field ζ of the quasi-Yamabe soliton is of gradient type $\zeta = grad(\psi)$, we derive a Poisson's equation from the quasi-Yamabe soliton equation. Also, we study harmonic aspects of quasi-Yamabe solitons on 3-dimensional trans-Sasakian manifolds sharing a harmonic potential function ψ. Finally, we observe that 3-dimensional compact trans-Sasakian manifold admits the gradient generalized almost quasi-Yamabe soliton with Hodge-de Rham potential ψ. This research ends with few examples of quasi-Yamabe solitons on 3-dimensional trans-Sasakian manifolds.

1. Introduction

In the past twenty years, geometric flows have emerged as versatile tools for de-

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* Corresponding Author.
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scribing geometric structures in Riemannian geometry. A specific class of solutions on which the metric evolves by dilation and diffeomorphisms plays a vital part in the study of singularities of the flows as they appear as possible singularity models. They are often called soliton solutions.

The theory of Yamabe flow was popularized by Hamilton in his prime research work [12] as a tool for constructing metrics of constant scalar curvature on an n-dimensional Riemannian manifold (M^n, g), $n \geq 3$. The Yamabe flow is an evolution equation for metrics on Riemannian manifolds. It is given by

$$\frac{\partial}{\partial t} g(t) = -r g(t), \quad g(0) = g_0, \tag{1.1}$$

where r is the scalar curvature corresponding to Riemannian metric g and t is time. It is used to deform a metric by smoothing out its singularities.

A Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter a family of diffeomorphisms generated by a fixed vector field E on M^n with a real constant λ satisfying the following equation

$$\frac{1}{2}\mathcal{L}_E g = (r - \lambda) g. \tag{1.2}$$

Here $\mathcal{L}_E g$ is the Lie derivative of the metric g along the vector field E, called the soliton vector field of the Yamabe soliton [12]. If $\lambda < 0$, $\lambda > 0$, or $\lambda = 0$, then the (M^n, g) is called a *Yamabe shrinker*, *Yamabe expander*, or *Yamabe steady soliton*, respectively.

When the vector field E is the gradient of a smooth function $\psi : M^n \longrightarrow \Re$, the manifold will be called a *gradient Yamabe soliton*. The function ψ is called the *potential function* of the gradient Yamabe soliton. In this case equation (1.2) becomes

$$Hess\psi = (r - \lambda) g, \tag{1.3}$$

where $Hess\psi$ stands for the Hessian of the potential function ψ. The gradient Yamabe soliton equation (1.3) links geometric information about the curvature of the manifold to the scalar curvature tensor and the geometry of the level sets of the potential function by means of their second fundamental form. This makes gradient Yamabe solitons under some curvature conditions an interesting topic of study.

An Einstein manifold [2] with a constant potential function is called a trivial gradient Ricci soliton. Gradient Yamabe solitons [14] play an important role in Yamabe flow as they correspond to self-similar solutions, and often arise as singularity models [18].

Introduced by Chen and Desahmukh in [4], a Riemannian manifold (M^n, g) is called a *quasi-Yamabe soliton*if it admits a vector field E such that

$$\mathcal{L}_E g + 2(\lambda - r) g = 2\mu E^\sharp \otimes E^\sharp, \tag{1.4}$$

for some real constant λ and smooth function μ, where $E^\sharp$ is the dual 1-form of E. The vector field E is also called a soliton vector field for the quasi-Yamabe soliton. We denote the quasi-Yamabe soliton satisfying (1.4) by $(M^n, g, E, \lambda, \mu)$. If $E = \nabla\psi$, then (1.4) becomes

$$\nabla^2\psi = (r - \lambda)g + \mu d\psi \otimes d\psi, \tag{1.5}$$

which is the gradient generalized quasi-Yamabe soliton studied by Huang et al. [10] and Leandro et al. [13], where $\nabla^2\psi$ denotes *Hessian* of ψ. This class of closely related Yamabe solitons hase be extensively studied; for further details see ([1], [5], [8], [19], [20], [22], [23], [25]).

According to Pigola et al. [17], if we replace the constant λ in (1.4) and (1.5) with a smooth function $\lambda \in C^\infty(M)$, called soliton function, then we can say that (M^n, g) is an almost quasi-Yamabe and gradient generalized almost quasi-Yamabe soliton, respectively.

On one hand, in 1985, Oubina [15] introduced a new class of almost contact metric manifolds, known as trans-Sasakian manifolds. This class consists the Sasakian, the Kenmotsu and the cosymplectic structures. The properties of trans-Sasakian manifolds have been studied by several authors, like Blair [3] and Marrero [14]. The main goal of this paper is to characterize the three-dimensional trans-Sasakian manifolds equipped with gradient generalized quasi-Yamabe solitons, quasi-Yamabe metrics, and gradient generalized almost quasi-Yamabe metrics.

2. Preliminaries

Let M be a connected almost contact metric manifold equipped with almost contact metric structure $(\varphi, \zeta, \eta, g)$ consisting of a $(1,1)$ tensor field φ, a vector field ζ, a 1-form η and a positive definite metric g such that

$$\varphi^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1, \quad \eta \circ \varphi = 0, \quad \varphi\zeta = 0, \tag{1.1}$$

$$g(\varphi E, \phi F) = g(E, F) - \eta(E)\eta(F), \quad \eta(E) = g(E, \zeta) \tag{1.2}$$

for all $E, F \in \chi(M)$, where $\chi(M)$ denotes the collection of all smooth vector fields of M and $dim\, M = 2m + 1$.

In the Grey and Harvella [9] classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds. In their classification, the class $C_6 \oplus C_5$ (*see* [3], [6], [15], [16]) coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely. An almost contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian [21] if $(M \times \mathfrak{R}, J, G)$ belongs to the class W_4, where J is an almost complex structure on $M \times \mathfrak{R}$ defined by

$$J\left(E, f\frac{d}{dt}\right) = \left(\varphi E - f\zeta,\, \eta(E)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times \mathfrak{R}$. Here G is the product metric on $M \times \mathfrak{R}$ and $\mathfrak{R}$ denotes the set of real numbers. This may be expressed by the condition

$$(\nabla_E \varphi)F = \alpha(g(E,F)\zeta - \eta(F)E) + \beta(g(\varphi E,F)\xi - \eta(F)\varphi E) \tag{1.3}$$

where α and β are some scalars functions on M and ∇ denotes the Levi-Civita connection with respect to g. We note that the trans-Sasakian structures of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are the cosymplectic, α-Sasakian and β-Kenmotsu structures, respectively. In particular, if $\alpha = 1, \beta = 0$, $\alpha = 0, \beta = 1$ and $\alpha = 0, \beta = 0$, then the trans-Sasakian manifold reduces to Sasakian, Kenmotsu and cosymplectic manifolds, respectively. From (1.3), it follows that

$$\nabla_E \zeta = -\alpha\varphi E + \beta[E - \eta(E)\zeta], \tag{1.4}$$

equivalent to

$$(\nabla_E \eta)F = -\alpha g(\varphi E, F) + \beta[g(E,F) - \eta(E)\eta(F)], \forall\ E,\ F \in \chi(M). \tag{1.5}$$

In a 3-dimensional trans-Sasakian manifold M, we have the following relations [7]

$$\begin{aligned} R(E,F)\zeta &= (\alpha^2 - \beta^2)[\eta(F)E - \eta(E)F] + 2\alpha\beta[\eta(F)\varphi E - \eta(E)\varphi F] \\ &\quad + [(E\alpha)\varphi E - (X\alpha)\varphi F + (F\beta)\varphi^2 E - (E\beta)\varphi^2 F], \end{aligned} \tag{1.6}$$

$$S(E,\zeta) = [(2(\alpha^2 - \beta^2) - (\zeta\beta)]\eta(E) + ((\varphi E)\alpha) + (E\beta), \tag{1.7}$$

$$Q\zeta = (2(\alpha^2 - \beta^2) - (\zeta\beta))\zeta + \varphi(grad\alpha) - (grad\beta), \tag{1.8}$$

where R, S and Q denote the curvature tensor, Ricci tensor and Ricci operator of g, respectively. Also *grad* stands for gradient. Further, in a three-dimensional trans-Sasakian manifold we have

$$\varphi(grad\alpha) = grad\beta, \tag{1.9}$$

and

$$2\alpha\beta + (\zeta\alpha) = 0. \tag{1.10}$$

Using (1.9) and (1.10), for constants α and β, we have

$$R(\zeta,E)F = (\alpha^2 - \beta^2)[g(E,F)\zeta - \eta(F)E], \tag{1.11}$$

$$R(E,F)\zeta = (\alpha^2 - \beta^2)[\eta(F)E - \eta(E)F], \tag{1.12}$$

$$S(E,\zeta) = [2(\alpha^2 - \beta^2)]\eta(E). \tag{1.13}$$

3. Gradient Generalized Quasi-Yamabe Soliton on Three-dimensional Trans-Sasakian Manifolds

For a smooth function ψ on M, the gradient and Hessian of ψ are, respectively, defined by

(1.1)
$$g(grad\psi, E) = E(\psi) \text{ and } (Hess\psi)(E, F) = g(\nabla_E grad\psi, F),\ \forall\ E, F \in \Gamma(TM).$$

For $E \in \Gamma(TM)$, we define $E^\sharp \in \Gamma(\bar{T}M)$ by

$$E^\sharp(F) = g(E, F). \tag{1.2}$$

The generalized quasi-Yamabe soliton equation [4] in a Riemannian manifold M is defined by

$$\frac{1}{2}\mathcal{L}_E g = \mu E^\sharp \odot E^\sharp + (r - \lambda)g. \tag{1.3}$$

Equation (1.3) is a generalization of Einstein manifold [10]. Note that if $E = grad\psi$, where $\psi \in C^\infty(M)$, the gradient generalized quasi-Yamabe soliton equation is given by [10]:

$$Hess\psi = \mu d\psi \odot d\psi + (r - \lambda)g. \tag{1.4}$$

Main Result:

Theorem 3.1. *Let M be a three-dimensional trans-Sasakian manifold satisfy the gradient generalized quasi-Yamabe soliton equation (1.4) with condition $\mu[\lambda+6(\alpha^2-\beta^2)] = 0$, then ψ is a constant function. Furthermore, if $\mu \neq 0$, then $\lambda = -6(\alpha^2-\beta^2)$ is negative, that is, a three-dimensional trans-Sasakian manifold admits a shrinking gradient generalized quasi-Yamabe soliton.*

From Theorem 3.1, we get the following remarks:

Remark. Let a three-dimensional trans-Sasakian manifold M satisfy the gradient generalized quasi-Yamabe soliton equation $Hess\psi = (r - \lambda)g$, then ψ is constant and M is η-Einstein.

Remark. In a three-dimensional trans-Sasakian manifold M, there is no non-constant smooth function ψ such that $Hess\psi = \lambda g$ for some constant λ.

To prove the Theorem 3.1, we have to demonstrate the following lemmas.

Lemma 3.2. *Let M be a three-dimensional trans-Sasakian manifold. Then we have*

(1.5)
$$(\mathcal{L}_\zeta(\mathcal{L}_E g))(F, \zeta) = (\alpha^2 - \beta^2)\{g(E, F) - \eta(E)\eta(F)\} + g(\nabla_\zeta\nabla_\zeta E, F) + Fg(\nabla_\zeta E, \xi),$$

where $E, F \in \Gamma(TM)$.

Proof. From the property of Lie-derivative we note that

$$(\mathcal{L}_\zeta(\mathcal{L}_E g))(E, \zeta) = \zeta((\mathcal{L}_E g)(F, \zeta)) - (\mathcal{L}_E g)(\mathcal{L}_\zeta F, \zeta) - (\mathcal{L}_E g)(F, \mathcal{L}_\zeta \zeta).$$

Since $\mathcal{L}_\zeta F = [\zeta, F]$ and $\mathcal{L}_\zeta \zeta = [\zeta, \zeta]$, therefore the above equation can be written as

$$(\mathcal{L}_\zeta(\mathcal{L}_E g))(F,\zeta) = \zeta g(\nabla_F E,\zeta) + \zeta g(\nabla_\zeta E, F) - g(\nabla_{[\zeta,F]}E,\zeta) - g(\nabla_\zeta E,[\zeta,F])$$

$$= g(\nabla_\zeta\nabla_F E,\zeta) + g(\nabla_F E,\nabla_\zeta\zeta) + g(\nabla_\zeta\nabla_\zeta E,F)$$

$$+g(\nabla_\zeta E,\nabla_\zeta F) - g(\nabla_\zeta E,\nabla_\zeta F) - g(\nabla_{[\zeta,F]}E,\zeta) + g(\nabla_\zeta E,\nabla_F\zeta).$$

From (1.4) we get $\nabla_\zeta\zeta = 0$, so the last equation gives

$$(\mathcal{L}_\zeta(\mathcal{L}_E g))(F,\zeta) = g(\nabla_\zeta\nabla_F E,\zeta) + g(\nabla_\zeta\nabla_\zeta E,F) - g(\nabla_{[\zeta,F]}E,\zeta)$$

$$+Fg(\nabla_\zeta E,\zeta) - g(\nabla_F\nabla_\zeta E,\zeta),$$

which gives

$$(\mathcal{L}_\zeta(\mathcal{L}_E g))(F,\zeta) = g(R(\zeta,F)E,\zeta) + g(\nabla_\zeta\nabla_\zeta E,F) + Yg(\nabla_\zeta E,\zeta). \tag{1.6}$$

From (1.12), we lead

$$g(R(\zeta,F)E,\zeta) = g(R(F,\zeta)\zeta,E) = (\alpha^2-\beta^2)\{g(E,F) - \eta(E)\eta(F)\}.$$

The Lemma 3.2 follows from the last two equations. Particularly, if Y is orthogonal to ζ then equation (1.5) assumes the form

$$(\mathcal{L}_\zeta(\mathcal{L}_E g))(E,\zeta) = (\alpha^2-\beta^2)g(E,F) + g(\nabla_\zeta\nabla_\zeta E,F) + Fg(\nabla_\zeta E,\zeta)$$

for all $E \in \chi(M)$ and F orthogonal to ζ. □

Lemma 3.3. *Let M be a Riemannian manifold, and let $\psi \in C^\infty(M)$. Then we have*

$$(\mathcal{L}_\zeta(d\psi\odot d\psi))(F,\zeta) = F(\zeta(\psi))\zeta(\psi) + F(\psi)\zeta(\zeta(\psi)). \tag{1.7}$$

Proof. We calculate:

$$(\mathcal{L}_\zeta(d\psi\odot d\psi))(F,\zeta) = \zeta(F(\psi)\zeta(\psi)) - [\zeta,F](\psi)\zeta(\psi) - F(\psi)[\zeta,\zeta](\psi)$$

$$= \zeta(F(\psi))\zeta(\psi) + F(\psi)\zeta(\zeta(\psi)) - [\zeta,F](\psi)\zeta(\psi).$$

Since $[\zeta,F](\psi) = \zeta(F(\psi)) - F(\zeta(\psi))$, therefore the above equation becomes

$$(\mathcal{L}_\zeta(d\psi\odot d\psi))(F,\zeta) = [\zeta,F](\psi)\zeta(\psi) + F(\zeta(\psi))\zeta(\psi) + F(\psi)\zeta(\zeta(\psi)) - [\zeta,F](\psi)\zeta(\psi)$$

$$= F(\zeta(\psi))\zeta(\psi) + F(\psi)\zeta(\zeta(\psi)).$$

Hence the statement of Lemma 3.3 is proved. □

Lemma 3.4. *Let a three-dimensional trans-Sasakian manifold M satisfy the gradient generalized quasi-Yamabe soliton equation* (1.4). *Then we have*

$$\nabla_\zeta \, grad\psi = -[\lambda - 6(\alpha^2 - \beta^2)]\zeta + \mu\, \zeta(\psi) grad\psi. \tag{1.8}$$

Proof. Let $F \in \Gamma(TM)$, then form the definition of Ricci tensor S, scalar curvature r and the curvature condition (1.12), we have

$$S(E, F) = \sum_{i=1}^{3} g(R(\zeta, e_i)e_i, F) = \sum_{i=1}^{3} g(R(e_i, F)\zeta, e_i) = 2(\alpha^2 - \beta^2),$$

$$r = 6(\alpha^2 - \beta^2),$$

where $\{e_1, e_2, e_3\}$ is an orthonormal frame on M. From the above equations, we infer

$$\lambda g(\zeta, F) + rg(\zeta, F) = [\lambda + 6(\alpha^2 - \beta^2)]g(\zeta, F). \tag{1.9}$$

From (1.4) and (1.9), we obtain

$$\begin{aligned}(Hess\psi)(\zeta, F) &= \mu\, \zeta(\psi)F(\psi) + [6(\alpha^2 - \beta^2) - \lambda]g(\zeta, F)\\ &= \mu\, \zeta(\psi)g(grad\psi, F) + [6(\alpha^2 - \beta^2) - \lambda]g(\zeta, F).\end{aligned} \tag{1.10}$$

The Lemma 3.3 follows from equation (1.10) and the definition of *Hessian* (see (1.1)). □

Now, we are going to prove our main Theorem 3.1 by using Lemma 3.2, Lemma 3.3 and Lemma 3.4.

Proof of Theorem 3.1. Let us suppose that the three-dimensional trans-Sasakian manifold satisfying the gradient generalized quasi-Yamabe soliton equation (1.4) and $\lambda, \mu \in \mathfrak{R}$. Let $Y \in \Gamma(TM)$, then Lemma together with $E = grad\ \psi$ leads to

$$\begin{aligned}2(\mathfrak{L}_\zeta(Hess\psi))(F, \zeta) &= (\alpha^2 - \beta^2)\{F(\psi) - \zeta(\psi)\eta(F)\}\\ &\quad + g(\nabla_\zeta \nabla_\zeta\ grad\psi, F) + Fg(\nabla_\zeta\ grad\psi, \zeta).\end{aligned} \tag{1.11}$$

From Lemma 3.4 and equations (1.1), (1.2), (1.4), (1.11), we get

$$\begin{aligned}2(\mathfrak{L}_\zeta(Hess\psi))(F, \zeta) = {} & F(\psi)[(\alpha^2 - \beta^2) + \mu(\zeta(\zeta(\psi))) + (\mu(\zeta(\psi)))^2]\\ & + \{\mu\zeta(\psi)[6(\alpha^2 - \beta^2) - \lambda] - \zeta(\psi)(\alpha^2 - \beta^2)\}\eta(F)\\ & + F[6(\alpha^2 - \beta^2) - \lambda + \mu(\zeta(\psi))^2]\end{aligned}$$

for all $F \in \Gamma(TM)$. Taking F orthogonal to ζ and therefore the above equation becomes

$$\begin{aligned}2(\mathfrak{L}_\zeta(Hess\psi))(F, \zeta) &= F[6(\alpha^2 - \beta^2) - \lambda + \mu(\zeta(\psi))^2]\\ &\quad + F(\psi)[(\alpha^2 - \beta^2) + \mu(\zeta(\zeta(\psi))) + (\mu(\zeta(\psi)))^2].\end{aligned} \tag{1.12}$$

Next, the Lie derivative of the gradient generalized quasi-Yamabe soliton equation (1.4) along the vector field ζ yields

$$2(\mathcal{L}_\zeta(Hess\psi))(F,\zeta) = \mu(\mathcal{L}_\zeta(d\psi \odot d\psi))(F,\zeta). \tag{1.13}$$

The last two equations together with Lemma infer

$$F(\psi) - \mu\zeta(\zeta(\psi))F(\psi) + \mu^2\zeta(\psi)^2F(\psi) - 2\mu\zeta(\psi)F(\zeta(\psi)) \tag{1.14}$$
$$= -2\mu F(\zeta(\psi))\zeta(\psi) - 2\mu F(\psi)\zeta(\zeta(\psi)),$$

which is equivalent to

$$F(\psi)[1 + \mu\zeta(\zeta(\psi)) + \mu^2\zeta(\psi)^2] = 0. \tag{1.15}$$

According to Lemma 4.3, we have

$$\begin{aligned} \mu\zeta(\zeta(\psi)) &= \mu\zeta g(\zeta, grad\ \psi) \\ &= ag(\zeta, \nabla_\zeta\ grad\psi) \\ &= \mu[\lambda + 6(\alpha^2 - \beta^2)] - \mu^2\zeta(\psi)^2, \end{aligned} \tag{1.16}$$

by equations (1.15) and (1.16), we obtain

$$F(\psi)[\lambda + 6(\alpha^2 - \beta^2)] = 0, \tag{1.17}$$

since $[\lambda + 6(\alpha^2 - \beta^2)] \neq 0$, we find that $F(\psi) = 0$, i.e., $grad\psi$ is parallel to ζ. Hence $grad\ \psi = 0$ as $D = ker\eta$ is not integrable any where, which means ψ is a constant function. □

Now, for particular values of α and β we turn up the following cases:
Case: For $\alpha = 0$, $(\beta = 1)$ and $(\alpha = \beta = 0)$ we can state the following results:

Corollary 3.5. *Let M be a 3-dimensional β-Kenmotsu (or Kenmotsu) manifold satisfies the gradient generalized quasi-Yamabe soliton(1.4) condition $\mu[\lambda - 6\beta^2)] \neq 0$, then ψ is a constant function. Furthermore, if $\mu \neq 0$, implies $\lambda = 6\beta^2)$, then M is expanding.*
Case: For $\beta = 0$, or $(\alpha = 1)$ we can state:

Corollary 3.6. *Let M be a 3-dimensional α-Sasakian (or Sasakian) manifold satisfies the gradient generalized quasi-Yamabe soliton(1.4) condition $\mu[\lambda + 6\alpha^2)] \neq 0$, then ψ is a constant function. Furthermore, if $\mu \neq 0$, implies $\lambda = -6\alpha^2)$, then M is shrinking.*
Case: For $\alpha = \beta = 0$, we can state:

Corollary 3.7. *Let M be a 3-dimensional cosymplectic manifold satisfies the gradient generalized quasi-Yamabe soliton (1.4) condition $\mu[\lambda] \neq 0$, then ψ is a constant function. Furthermore, if $\mu \neq 0$, implies $\lambda = 0$, then M is steady.*

4. Quasi-Yamabe Soliton on 3-dimensional Trans-Sasakian Manifolds

Again, assume the equation

$$\mathcal{L}_{\zeta} g + (\lambda - R)g + \mu E^{\sharp} \otimes E^{\sharp} = 0 \tag{4.1}$$

where g is a Riemannian metric and R is the scalar curvature, ζ is vector field, $E^{\sharp}$ is a 1-form and λ and μ are real constant. The data (g, ζ, λ, μ) satisfies the equation (4.1) is called the *quasi-Yamabe soliton.* In particular, if $\mu = 0$, (g, ζ, λ) is a Yamabe soliton.

Using the definition of Lie derivative and (4.1), we obtain

$$(R - \lambda)g(F, G) = -\mu E^{\sharp}(F)E^{\sharp}(G) - \frac{1}{2}[g(\nabla_F \zeta, G) + g(F, \nabla_G \zeta)], \tag{4.2}$$

for any $F, G \in \chi(M)$.
Contracting (4.2) we get

$$3\lambda - \mu = 3R - div(\zeta). \tag{4.3}$$

Let $(M, g, \varphi, \eta, \zeta)$ be a 3-dimensional trans-Sasakian manifold and (g, ζ, λ, μ) be a quasi-Yamabe soliton on M. Writing (4.2) for $F = G = \zeta$, we obtain

$$\lambda - \mu = 6(\alpha^2 - \beta^2). \tag{4.4}$$

Therefore

$$\begin{cases} \lambda = -6(\alpha^2 - \beta^2) + \frac{div(\zeta)}{2} \\ \mu = -12(\alpha^2 - \beta^2) + \frac{div(\zeta)}{2} \end{cases} \tag{4.5}$$

Using (4.5) we can state the following results.

Theorem 4.1. *Let $(M, \eta, \varphi, \zeta, g)$ be a 3-dimensional trans-Sasakian manifold and $E^{\sharp}$ be the g-dual 1-form of the gradient vector field $\zeta = grad(\psi)$. If (4.1) define a quasi-Yamabe soliton with non vanishing μ in M, then the Poisson equation satisfied by ψ becomes*

$$\Delta(\psi) = 2[\mu + 12(\alpha^2 - \beta^2)]. \tag{4.6}$$

Once again, considering the equation (4.5) we can also obtain

$$\Delta(\psi) = 2[\lambda + 6(\alpha^2 - \beta^2)]. \tag{4.7}$$

Remark.([24]) A C^{∞} function $f : M \longrightarrow \mathbb{R}$ is said to be harmonic if $\Delta f = 0$, where Δ is the Laplacian operator in M.

Now, from equation (4.7) and using above remark, we obtain the following results:

Theorem 4.2. *Let $(M, \eta, \varphi, \zeta, g)$ be a 3-dimensional trans-Sasakian manifold and $E^\sharp$ be the g-dual 1-form of the gradient potential vector field $\zeta = grad(\psi)$. If the potential function ψ is harmonic, then quasi-Yamabe soliton is shrinking for the value of $\lambda = -3(\alpha^2 - \beta^2)$.*

Corollary 4.3. *Let $(M, \eta, \varphi, \zeta, g)$ be a 3-dimensional α-Sasakian (or Sasakian) manifold and $E^\sharp$ be the g-dual 1-form of the gradient potential vector field $\zeta = grad(\psi)$. If the potential function ψ is harmonic, then quasi-Yamabe soliton is shrinking for the value of $\lambda = -3\alpha^2$.*

Corollary 4.4. *Let $(M, \eta, \varphi, \zeta, g)$ be a 3-dimensional β-Kenmotsu (or Kenmotsu) manifold and $E^\sharp$ be the g-dual 1-form of the gradient potential vector field $\zeta = grad(\psi)$. If the potential function ψ is harmonic, then quasi-Yamabe soliton is expanding for the value of $\lambda = 3\beta^2$.*

Corollary 4.5. *Let $(M, \eta, \varphi, \zeta, g)$ be a 3-dimensional cosymplectic manifold and $E^\sharp$ be the g-dual 1-form of the gradient potential vector field $\zeta = grad(\psi)$. If the potential function ψ is harmonic, then quasi-Yamabe soliton is steady for the value of $\lambda = 0$.*

5. Example of a Trans-Sasakian Manifold of Type $(\alpha, 0)$ 3-metric as Quasi Yamabe Soliton

Example 5.1. Let $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) is the standard coordinates of $\mathbb{R}^3$.

The vector fields are

$$e_1 = \frac{\partial}{\partial z} - y\frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = 2\frac{\partial}{\partial x}$$

Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \qquad g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$$

that is, the form of the metric becomes Let η be the 1-form defiend by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Also, let φ be the $(1, 1)$ tensor field defined by

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0.$$

Thus, using the linearity of φ and g, we have

$$\eta(e_3) = 0, \quad , \eta(e_1) = 0, \quad \eta(e_2) = 0,$$

$$[e_1, e_2] = \frac{1}{2}e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = 0,$$

$$\varphi^2 Z = -Z + \eta(Z)e_3$$

,

$$g(\varphi Z, \varphi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (φ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$

$$-g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is known as Koszul's formula.

Using Koszul's formula we have

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = -\frac{1}{4}e_3, \quad \nabla_{e_1} e_3 = \frac{1}{4}e_3,$$

$$\nabla_{e_2} e_1 = \frac{1}{4}e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = -\frac{1}{4}e_1,$$

$$\nabla_{e_3} e_1 = \frac{1}{4}e_2, \quad \nabla_{e_3} e_2 = -\frac{1}{4}e_1, \quad \nabla_{e_3} e_0 = 0. \tag{5.1}$$

From (5.1) we find that the structure (φ, ξ, η, g) satisfies the formula (4.5) for $\alpha = \frac{1}{4}$ and $\xi = e_3$. Hence the manifold is a 3-dimensional trans-Sasakian manifold of type $(\alpha, 0)$ with the constant structure function $\alpha = \frac{1}{4}$ and $\beta = 0$.

Then the Riemannian and Ricci curvature tensor fields are given by:

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = \frac{1}{16}e_2, \quad R(e_1, e_3)e_3 = \frac{1}{16}e_1,$$

$$R(e_1, e_2)e_2 = -\frac{3}{16}e_1, \quad R(e_2, e_3)e_2 = -\frac{1}{16}e_3, \quad R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = \frac{3}{16}e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e = -\frac{1}{16}e_3.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -\frac{1}{8}$$

similarly we have

$$S(e_1, e_1) = S(e_2, e_2) = -\frac{1}{8}, \quad \text{and} S(e_3, e_3) = \frac{1}{8}.$$

Now, we have constant scalar curvature as follows,

$$R = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -\frac{1}{8}.$$

By the definition of quasi-Yamabe soliton and using (1.4), we obtain

$$2\beta[g(e_i, e_i) + \eta(e_i)\eta(e_i)] + 2(\lambda - R)g(e_i, e_i) + 2\mu X^\sharp(e_i)X^\sharp(e_i) = 0$$

for all $i \in \{1, 2, 3\}$, and we have

$$2(1 + \delta_{i3}) + 2(\lambda - R) + 2\mu\delta_{i3} = 0$$

for all $i \in \{1, 2, 3\}$.
Therefore $\lambda = -\frac{1}{2}$ and $\mu = \frac{3}{8}$ the data (g, ξ, λ, μ) admitting the shrinking quasi-Yamabe soliton on 3-dimensional trans-Sasakian manifolds with $\lambda < 0$.

6. Gradient Almost Quasi-Yamabe Soliton in a Compact Trans-Sasakian Manifold

In [7] De and Sarkar proved that if a 3-dimensional trans-Sasakian manifold is of constant curvature is compact and connected. .

On the other hand, The classical theorem of de-Rham-Hodge asserts that the cohomology of an oriented closed Riemannian manifold can be represented by harmonic forms. The similar one still holds for an oriented compact Riemannian manifold with boundary by imposing certain boundary conditions, such as absolute and relative ones.

We consider M as a compact orientable trans-Sasakian manifold and $X \in \chi(M)$. Then Hodge-de Rham decomposition theorem [11] implies that E can be expressed as

$$E = \nabla h + F, \tag{5.1}$$

where $h \in C^\infty(M)$ and $div(F) = 0$. The function h is called the Hodge-de Rham potential [11].

Theorem 6.1. *If (g, E, λ, μ) is a compact gradient almost quasi-Yamabe soliton on trans-Sasakian manifold M. If M is also a gradient almost quasi-Yamabe soliton with potential function ψ, then up to a constant, f equals to the Hodge-de Rham potential.*

Proof. Since (g, E, λ, μ) is a compact almost quasi-Yamabe soliton, now taking the trace of (1.4), we find

$$div(E) = (R - \lambda)n + trce(\mu E^\sharp \otimes E^\sharp), \tag{5.2}$$

Hodge-de Rham decomposition implies that $div(E) = \Delta h$, hence the above equation, we get

$$R = \lambda - \frac{\Delta h}{n} + \frac{1}{3}trce(\mu E^\sharp \otimes E^\sharp). \tag{5.3}$$

Again since M is generalized gradient almost quasi-Yamabe soliton with Perelman potential f, hence taking trace of (1.5), we have

$$R = \lambda - \frac{\Delta\psi}{3} + \frac{1}{3}\mu\,|E|^2\,. \tag{5.4}$$

Now, equating the equations (5.3) and (5.4), we find $\frac{1}{3}\Delta(\psi - h) = 0$. Hence $\psi - h$ is a harmonic function in compact trans-Sasakian manifold. Hence $f = h + c$, for some constant c. □

7. Example of a Trans-Sasakian Manifold of Type $(0, \beta)$ 3-metric as Quasi Yamabe Soliton

Example. Let $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates of $\mathbb{R}^3$. The vector fields are

$$e_1 = z\frac{\partial}{\partial x}, \qquad e_2 = z\frac{\partial}{\partial y} \qquad e_3 = z\frac{\partial}{\partial z}$$

Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \qquad g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$$

that is, the form of the metric becomes

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Also, let φ be the $(1, 1)$ tensor field defined by

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0.$$

Thus, using the linearity of φ and g, we have

$$\eta(e_3) = 0, \quad ,\eta(e_1) = 0, \quad \eta(e_2) = 0,$$

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_1, e_3] = -e_1,$$

$$\varphi^2 Z = -Z + \eta(Z)e_3$$

,

$$g(\varphi Z, \varphi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (φ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$

$$-g(Y,[X,Z])+g(Z,[X,Y]),$$

which is known as Koszul's formula.
Using Koszul's formula we have

$$\nabla_{e_1}e_1=e_3,\quad \nabla_{e_1}e_2=0,\quad \nabla_{e_1}e_3=-e_1,$$

$$\nabla_{e_2}e_1=0,\quad \nabla_{e_2}e_2=e_3,\quad \nabla_{e_2}e_3=-e_2,$$

$$\nabla_{e_3}e_1=0,\quad \nabla_{e_3}e_2=0,\quad \nabla_{e_3}e_3=0. \tag{5.1}$$

From (5.1) we find that the manifold satisfies (1.4) for $\alpha=0$ and $\beta=-1$ and $\xi=e_3$. Hence the manifold is a 3-dimensional trans-Sasakian manifold of type $(0,\beta)$ with the constant structure function $\alpha=0$ and $\beta=-1$ [7] .

Then the Riemannian and Ricci curvature tensor fields are given by:

$$R(e_1,e_2)e_3=0,\quad R(e_2,e_3)e_3=-e_3,\quad R(e_1,e_3)e_3=-e_1,$$

$$R(e_1,e_2)e_2=-e_1,\quad R(e_2,e_3)e_2=e_3,\quad R(e_1,e_3)e_2=0,$$

$$R(e_1,e_2)e_1=e_2,\quad R(e_2,e_3)e_1=0,\quad R(e_1,e_3)e=e_3.$$

From the above expressions of the curvature tensor we obtain

$$S(e_1,e_1)=g(R(e_1,e_2)e_2,e_1)+g(R(e_1,e_3)e_3,e_1)=-2$$

similarly, we have

$$S(e_1,e_1)=S(e_2,e_2)=S(e_3,e_3)=-2.$$

Now, the scalar curvature

$$R=S(e_1,e_1)+S(e_2,e_2)+S(e_3,e_3)=-6.$$

Because of scalar curvature $r=6$, from Theorem (), we can conclude that M is an Einstein manifold.

By the definition of quasi-Yamabe soliton and using (1.4), we obtain

$$2\beta[g(e_i,e_i)+\eta(e_i)\eta(e_i)]+2(\lambda-R)g(e_i,e_i)+2\mu X^\sharp(e_i)X^\sharp(e_i)=0$$

for all $i\in\{1,2,3\}$, and we have

$$-2(1+\delta_{i3})+2(\lambda-R)+2\mu\delta_{i3}=0$$

for all $i\in\{1,2,3\}$.
Therefore $\lambda=-1$ and $\mu=\frac{3}{8}$ the data (g,ξ,λ,μ) admitting the shrinking quasi-Yamabe soliton on 3-dimensional trans-Sasakian manifolds with $\lambda<0$.

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