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Oh's 8-Universality Criterion is Unique

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Abstract. We partially characterize criteria for the n-universality of positive-definite integer-matrix quadratic forms. We then obtain the uniqueness of Oh's 8-universality criterion [11] as a corollary.

1. Introduction

A degree-two homogeneous polynomial in n independent variables is called a *quadratic form* (or just *form*) *of rank* n. For a rank-n quadratic form $Q(x_1,\dots,x_n) = \sum_{i,j} a_{ij}x_ix_j$ (where $a_{ij} = a_{ji}$), the matrix given by $L = (a_{ij})$ is the *Gram Matrix* of a $\mathbb{Z}$-lattice L equipped with a symmetric bilinear form $B(\cdot,\cdot)$ such that $B(L,L) \subseteq \mathbb{Z}$. Then, $Q(\mathbf{x}) = \mathbf{x}^T L\mathbf{x} = B(L\mathbf{x},\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$.

A rank-n quadratic form Q is said to *represent* an integer k if there exists an $\mathbf{x} \in \mathbb{Z}^n$ such that $Q(\mathbf{x}) = k$. More generally, a $\mathbb{Z}$-lattice L *represents* another $\mathbb{Z}$-lattice ℓ if there exists a $\mathbb{Z}$-linear, bilinear form-preserving injection $\ell \to L$. A quadratic form is called *universal* if it represents all positive integers. Analogously, a lattice is called n*-universal* if it represents all rank-n positive-definite integer-matrix $\mathbb{Z}$-lattices. Connecting the two notions of universality, we observe that a rank-n quadratic form Q is universal if and only if it is 1-universal, as for an integer k,

$$k = Q(x_1,\dots,x_n) \iff Q(x_1x,\dots,x_nx) = kx^2.$$

In 1993, Conway and Schneeberger announced their celebrated *Fifteen Theorem*, giving a criterion characterizing the universal positive-definite integer-matrix

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quadratic forms. Specifically, they showed that any positive-definite integer-matrix form that represents the set of nine critical numbers

$$\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$$

is universal (see [1, 3]). Kim, Kim, and Oh [6] presented an analogous criterion for 2-universality, showing that a positive-definite integer-matrix lattice is 2-universal if and only if it represents the set of forms

$$\mathcal{S}_2 = \{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}2 & 0\\ 0 & 3\end{smallmatrix}\right), \left(\begin{smallmatrix}3 & 0\\ 0 & 3\end{smallmatrix}\right), \left(\begin{smallmatrix}2 & 1\\ 1 & 2\end{smallmatrix}\right), \left(\begin{smallmatrix}2 & 1\\ 1 & 3\end{smallmatrix}\right), \left(\begin{smallmatrix}2 & 1\\ 1 & 4\end{smallmatrix}\right)\}.$$

Oh [11] gave a similar criterion for 8-universality, which we state in Theorem 4.1 of Section 4.

A set $\mathcal{S}$ of rank-n lattices having the property that a lattice L is n-universal if and only if L represents every lattice in $\mathcal{S}$ is called an *n-criterion set*. Thus, for example, the set $\mathcal{S}_2$ obtained by Kim, Kim, and Oh [6] is a 2-criterion set and the set of integers found by Conway [3] naturally gives the 1-criterion set

$$\mathcal{S}_1 = \left\{x^2, 2x^2, 3x^2, 5x^2, 6x^2, 7x^2, 10x^2, 14x^2, 15x^2\right\}.$$

The set $\mathcal{S}_1$ is known to be the unique minimal 1-criterion set (see [7]), in the sense that if $\mathcal{S}'_1$ is a 1-criterion set, then $\mathcal{S}_1 \subseteq \mathcal{S}'_1$. The author [9] obtained an analogous uniqueness result for the 2-criterion set $\mathcal{S}_2$.

Kim, Kim, and Oh [7] have proven that n-criterion sets exist for all positive integers n. However, the problems of finding and determining the uniqueness of these sets have proven to be difficult (see the discussion in [7]). Here, we advance both problems: We obtain two simple (partial) characterization results for arbitrary n-criterion sets, from which we obtain the uniqueness of Oh's 8-universality criterion as a corollary.

Since we first circulated this paper, there has been renewed attention in characterizing criterion sets: Elkies, Kane, and the author [5] identified several families of lattices for which there exist multiple universality criteria of different sizes, including one based on the $\mathbb{Z}^n$ and E_8 lattices that builds on our work here. More recently, Lee [10] and Kim, Lee, and Oh [8] showed that the minimal n-criterion sets are *not* unique for $n \geq 9$, and introduced an elegant theory of recoverable lattices that substantially generalizes [5]. (See also recent work of Chan and Oh [2] characterizing classes of exceptional sets for rank-n quadratic forms, which in some sense can be thought of as building blocks for criterion sets.)

2. Notation and Terminology

We use the lattice-theoretic language of quadratic form theory. A complete introduction to this approach may be found in [12]. In addition, we use the lattice notation of [4], under which I_n is the rank-n lattice of the form $\langle 1, \ldots, 1\rangle$ and E_8 is the unique even unimodular lattice of rank 8.

For a $\mathbb{Z}$-lattice (or hereafter, just *lattice*) L with basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, we write $L \cong \mathbb{Z}\mathbf{x}_1 + \cdots + \mathbb{Z}\mathbf{x}_n$. If L is of the form $L = L_1 \oplus L_2$ for sublattices L_1 and L_2 of L with $B(L_1, L_2) = 0$, then we write $L \cong L_1 \perp L_2$ and say that L_1 and L_2 are *orthogonal*.

For a sublattice ℓ of $L_1 \perp L_2$ that can be expressed in the form

$$\ell \cong \mathbb{Z}(\mathbf{x}_{1,1} + \mathbf{x}_{2,1}) + \cdots + \mathbb{Z}(\mathbf{x}_{1,n} + \mathbf{x}_{2,n})$$

with $\mathbf{x}_{i,j} \in L_i$, we denote $\ell(L_i) := \mathbb{Z}\mathbf{x}_{i,1} + \cdots + \mathbb{Z}\mathbf{x}_{i,n}$. We naturally extend this notation to lattices ℓ represented by $L_1 \perp L_2$. We then say that a lattice is *additively indecomposable* if either $\ell(L_1) \cong 0$ or $\ell(L_2) \cong 0$ whenever $L_1 \perp L_2$ represents ℓ. Otherwise, we say that ℓ is *additively decomposable*.

3. Partial Characterization of n-Criterion Sets

In this section, we prove two results that partially characterize the contents of arbitrary n-criterion sets.

Proposition 3.1. *Any n-criterion set must include the lattice I_n.*

Proof. If $\mathcal{T}$ is a finite, nonempty set of rank-n lattices not containing I_n, then every lattice $T \in \mathcal{T}$ may be written in the form $T \cong I_k \perp T'$, where $0 \le k < n$, the sublattice T' is of rank $n-k$, and the first minimum of T' is larger than 1. Indeed, any I_k-sublattice of T is unimodular and therefore splits T; the condition on T' follows from Minkowski reduction.

We may therefore write $\mathcal{T}$ in the form

$$\mathcal{T} = \bigcup_{k=0}^{n-1} \{I_k \perp T_{k,i}\}_{i=1}^{i_k},$$

where $0 < |\mathcal{T}| = \sum_{k=0}^{n-1} i_k$ and each $T_{k,i}$ is a rank-$(n-k)$ lattice with first minimum greater than 1. Then, the lattice

$$I_{n-1} \perp \left(\left(\perp_{i=1}^{i_0} T_{0,i} \right) \perp \cdots \perp \left(\perp_{i=1}^{i_{n-1}} T_{n-1,i} \right) \right)$$

represents all of $\mathcal{T}$ but does not represent I_n. It follows that $\mathcal{T}$ is not an n-criterion set; hence, any n-criterion set must contain I_n. □

Proposition 3.2. *Let $\mathcal{E}$ be the set of additively indecomposable unimodular lattices of rank n. If $\mathcal{E} \neq \emptyset$, then any n-criterion set must include at least one lattice $E \in \mathcal{E}$.*

Proof. Suppose that $\mathcal{E} \neq \emptyset$. If $\mathcal{T} = \{T_i\}_{i=1}^{k}$ is a finite, nonempty set of rank-n lattices with $\mathcal{T} \cap \mathcal{E} = \emptyset$, then every lattice $T_i \in \mathcal{T}$ is either additively decomposable or not unimodular (or both). Now, we consider the lattice

$$T_1 \perp \cdots \perp T_k,$$

which of course represents all of $\mathcal{T}$ by construction.

If $T_1 \perp \cdots \perp T_k$ were to represent some $E \in \mathcal{E}$, then under any such representation we would have $E(T_i) \cong 0$ for all but one i (with $1 \leq i \leq k$) because E is additively indecomposable. Then, for some i (again, with $1 \leq i \leq k$), the lattice T_i would represent E. In that case, as E is unimodular, the associated sublattice of T_i would split T_i as $T_i \cong E \perp T'$—and since both E and T_i are of rank n, we would have $T' \cong 0$; hence, $T_i \cong E$. But this is impossible because T_i is either additively decomposable or not unimodular, whereas $E \in \mathcal{E}$ is both additively indecomposable and unimodular.

Thus, we have found a lattice that represents all of $\mathcal{T}$ but cannot represent any $E \in \mathcal{E}$. As $\mathcal{E} \neq \emptyset$ by hypothesis, we see that $\mathcal{T}$ must not be an n-criterion set; the result follows. □

Remark 3.3. It is clear that direct analogues of Propositions 3.1 and 3.2 hold in the more general setting of S-universal lattices discussed in [7]. In particular, suppose that S is an infinite set of lattices. Then, if $n = \max\{k : I_k \in \mathsf{S}\} > 0$, any finite set $\mathcal{S}_{\mathsf{S}} \subset \mathsf{S}$ with the property that a lattice L represents every $\ell \in \mathsf{S}$ if and only if L represents every $\ell \in \mathcal{S}_{\mathsf{S}}$ must contain I_n. Similarly, such a set $\mathcal{S}_{\mathsf{S}}$ must contain an additively indecomposable unimodular lattice if S does.

4. Uniqueness of The 8-Criterion Set

Oh [11] obtained the following 8-criterion set.

Theorem 4.1.([11, remark on Theorem 3.1]) *The set $\mathcal{S}_8 = \{I_8, E_8\}$ is an 8-criterion set.*

The set $\mathcal{S}_8$ is clearly a *minimal* 8-criterion set, as for each $\ell \in \mathcal{S}_8$ there is a lattice that represents $\mathcal{S}_8 \setminus \ell$ but does not represent ℓ. (The single lattice in $\mathcal{S}_8 \setminus \ell$ suffices.) Meanwhile, our characterization results imply the following corollary, which strengthens Theorem 4.1.

Corollary 4.2. *Every 8-criterion set must contain $\mathcal{S}_8$ as a subset.*

Proof. As E_8 is the unique additively indecomposable unimodular lattice of rank 8, the result follows directly from Propositions 3.1 and 3.2. □

Corollary 4.2, when combined with Theorem 4.1, shows that $\mathcal{S}_8$ is the unique minimal 8-criterion set.

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