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Problem Posing in the Instruction of Proof: Bridging Everyday Lesson and Proof[1]

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Proof serves a critical role in mathematical practices as well as in fostering student's mathematical understanding. However, the research literature accumulates results that there are not many opportunities available for students to engage with proving-related activities and that students' understanding about proof is not promising. This unpromising state of instruction of proof calls for a novel approach to address the aforementioned issues. This study investigated an instruction of proof to explore a pedagogy to teach how to prove. The teacher utilized the way of problem posing to make proving a routine part of everyday lesson and changed the classroom culture to support student proving. The study identified the teacher's support for student proving, the key pedagogical changes that embraced proving as part of everyday lesson, and what changes the teacher made to cultivate the classroom culture to be better suited for establishing a supportive community for student proving. The results indicate that problem posing has a potential to embrace proof into everyday lesson.

Keywords: proof, reasoning, problem posing, high school, pedagogy.
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I. INTRODUCTION

Introduction Proof has been considered as central in the context of school mathematics with distinctive roles that it plays in mathematics (Bieda, 2010; Bieda, Ji, Drwencke, & Picard, 2014; Hanna, 2000; Hanna & Jahnke, 1996; Knuth, 2002a, 2002b, 2002c; National Council of Teachers of Mathematics [NCTM], 2000) and the importance of it is well reflected across curricula and recommendations (ACARA, 2015; CCSSO, 2010;

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DoE, 2014; MoE, 2011, 2012; NCTM, 2000). For example, in *Principles and Standards for School Mathematics* (NCTM, 2000), "Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena" (p. 56). NCTM (2000) also suggested that examples are used as a base line that students can refer back to or leveraging ideas from when formulating and evaluating mathematical conjectures (Lockwood et al., 2013). However, proof yields difficulty for students to understand and is hard for teachers to teach (Knuth, 2002a; Senk, 1985; Stylianides, Stylianides, & Weber, 2016). The "hard-to-teach" issue may be attributable partly due to teacher's not very sophisticated understanding about proof (Coe & Ruthven, 1994; Knuth et al., 2020; Martin & Harel, 1989; Stylianides & Stylianides, 2009) and the lack of opportunities available in textbook (Bieda et al., 2014; Thompson, Senk, & Johnson, 2012). As a result, the vast majority of students who took a full year of geometry did not reach at the level of mastery in proof writing and tend to surpass their counterpart's performance (Senk, 1985).

The guiding assumption of this study is that more opportunities for students to engage with proving in a mathematically meaningful way enhance students' understanding of proof. Understanding about proof includes that it is *a tool* to readily establish the truth of a mathematical claim (de Villiers, 1990; Harel & Sowder, 1998; Knuth, 2002b; Lakatos, 1976), that it offers *insight* into mathematically similar problems (Alcock & Inglis, 2008; Ellis et al., 2019; Epstein & Levy, 1995; Lynch & Lockwood, 2019), and, ultimately, that it is *a fundamental aspect* of recording and doing mathematics (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Schoenfeld, 1994). To offer such opportunities for students to engage with proof-related activities, this study aims to argue that proof is not a mere content to be learned and can be taught across many contexts, culminating in that everyday lesson affords an initiative for discussing need for proof and proving with teacher's pedagogical changes. To align with this regard the focus of this study, the research questions guided this study were as follows:

a) In what way(s) does a teacher use everyday lesson as an initiative for proving?;

b) What pedagogical change(s) did the teacher make to promote student proving and establish a supportive community for student proving?; and

c) What support does the teacher provide to facilitate student proving?

To address the questions above, this study will first explore how a teacher uses everyday lesson as an initiative by way of problem posing to begin to discuss why and when proof becomes required in mathematics. By problem posing, I refer to one's act of formulating problems by altering some of the given attributes of a problem (Brown & Walter, 2014). This behavior of problem posing can be either "generation of new problems or re-formulation of given problems" (Silver, 1994, p. 19). I will then examine what classroom norm(s) the teacher established to form a supportive community for student proving.

Finally, I will focus on the changes in classroom culture that the teacher made throughout the study to make viable student proving. As Schoenfeld (1979) exemplified that the instruction of problem-solving strategy became effective when accompanied with the corresponding change in the classroom culture to be better suited for the particular instruction and students, the key aspects of the teacher's pedagogical changes that contributed to cultivating a classroom culture that promoted students' participation and engagement in proving were identified and are to be reported. At the end of this paper, I will discuss the findings, limitations of the study and suggestions for the future research.

II. LITERATURE REVIEW

In this section, I will review the research of close relevance to this study and define the terms worth clarification for the subsequent discussions. I would like to share the quote that I found worth considering to highlight the issues underlying instruction of proof in school mathematics that are pertinent to this study:

> *Be a free thinker and don't accept everything you hear as truth. Be critical and evaluate what you believe in*. (Aristotle, n.d.)

The well-articulated quote generated some questions: What is the state of affairs regarding the instruction of proof and proving? How do mathematics teachers perceive proof in school mathematics? How sophisticated are teachers' and students' understandings about proof? This section was shaped by the aforementioned questions and organized in two parts: *Proof in School Mathematics* and *Example Use in Proving-related activities*. In the first subsection, *Proof in School Mathematics*, I consider proof in the context of school mathematics including the notion of proof, the teachers' perceptions of proof, and students' understanding of proof. In the following subsection, *Example Use in Proving-related activities*, example use in proving-related activities will be discussed and what is void or needed will also be considered.

1. PROOF IN SCHOOL MATHEMATICS

For the clarity in the subsequent discussions, some terms are worth clarifying: *Proof*, *proving*, *reasoning* and *proving-related activities*. As I define *proving* based on proof, I shall begin first by defining the terms but proof and proceed to provide the definition of proof later. I take to *proving* the process involved in or one's attempt of developing or writing a proof (Cai & Cirillo, 2014). I take *reasoning* as one's search for or providing reasons to justify steps or inferences involved in a proof (Pólya, 1954). By *proving-related activities*, I refer to “a wide range of activities involved in mathematical argumentation, including generalizing the mathematical relationship in a given pattern, producing a conjecture, generating a justification or proof, and evaluating a given justification or proof”(Bieda et al., 2014, p. 72). Proving and proving-related activities might seem to be interchangeable, however, I shall qualify proving as developing and writing a proof as a culmination of proving-related activities. In summary, *proof* is the end result of proving while proving is an act of developing and writing a proof. *Reasoning* is a kind of the proving-related activities that occur during one undertakes proving in order to justify part of a proof. Finally, the definition of *proof* that I will use in this study is borrowed from the part of the work by A. J. Stylianides (2007):

> *Proof* is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291, italics in original)

In recognition that there is need to define what it means to be a proof across school mathematics, Stylianides (2007) defined proof in the way that it encompasses cognitive, social, and cultural aspects of what constitutes a mathematical proof in the immediate community to which one belongs at a given time. The definition clearly places emphasis on the role of the classroom community to which both student and teacher have memberships. As any well-controlled research can hardly be independent of the context (e.g., classroom, school, society) where the study is situated (Yackel & Cobb, 1996), the definition would be appropriate for the analyses that I sought in this study so as to take the context of the class where the study was situated into account. However, what is implicit or absent from the definition is that it lacks clarity in what function a mathematical proof serves at a given time. This absence may engender issues in communicating need of proof: Why one is pressed for a proof by another. These issues arise due to the multiplicity of the meaning of proof (Hoyles, 1997) or the mismatch between the perceptions of teachers and students as to why proof is necessary when asked to provide a proof by someone else (de Villiers, 1990).

To effectively communicate the need and function of a proof at a given time, I shall borrow from the work by Hanna (2000) who laid out the functions of proof and proving are as follows: *verification*, *explanation*, *discovery*, and *communication*. The one that has been paid much attention to in school mathematics is *explanation* concerns how convincing one's explanation is to someone else as prototypical of the proofs in geometry textbook (Stylianides, Stylianides, & Weber, 2016); *verification* that concerns the truth of a statement; *discovery* concerns one's invention or discovery of new results; and *communication* concerns delivery (or the transmission) of mathematical knowledge (Hanna, 2000).

The instruction of proof should not be limited both *horizontally* (i.e. certain contents at a grade level) and *vertically* (i.e. across grade levels). Quite often, as proof is associated with geometry from teacher's outlook (Knuth, 2002a), this tends to limit *when* (i.e. in what grade) and *where* (i.e. in what subject area) proof is taught and *who* can learn proof (Knuth, 2002c). As Maher & Martino's (1996) longitudinal study documented that even a student at an early grade successfully developed a proof over an extended period of time, the conception of proof should be developed consistently (Stylianides, 2007) and persistently (NCTM, 2000).

The issue underlying the limited learning experience about proof in regards to both content and method is neither insignificant nor negligible. Proofs in geometry do not involve other ways of proving but direct proofs of form so called "two-column" and proof by mathematical induction may reduce proof to a kind of algorithm (Rowland, 2002). As reported by Knuth (2002a), the majority of teachers in his study considered proof as a topic that was not intended for *all* students and saw it as a mere content to teach with specific topics such as geometry. The problem of limiting the instruction of proof to the context of geometry is that the limited context of the instruction tends to contribute to the limited learning experiences of proof and accordingly the limited understanding about proof (Balacheff, 1988; Coe & Ruthven, 1994; Harel & Sowder, 1998; Healy & Hoyles, 2000; Moore, 1994). The issue was well depicted in the work of Harel & Sowder (1998). They studied undergraduate mathematics major students' proof schemes by analyzing the data that consist of the classroom discussion, individual interviews, and the written assessments. In their analysis, they identified *authoritarian proof scheme* (see Harel & Sowder, 1998, pp. 247-250 for the detail). To explain what they meant by authoritarian proof scheme, they described five episodes that highlighted how students' learning experiences about proof had impact on their perceptions of proof. Particularly, the source from which students gain conviction about the validity of a proof (in terms with form, language, or representation) stems from the mathematical authority such as textbook and teacher. Students who possessed this proof scheme tend to simply justify or evaluate proofs based on the work of the mathematical authority they rely on. This brings to the fore the teacher's role in teaching proof and raises a question: As a mathematical authority, how sophisticated are teachers' understanding about proof?

Some teachers tend to fail to draw distinction between what is *valid* mode of argumentation (Knuth et al., 2020; Stylianides & Stylianides, 2009; Stylianides et al., 2016). That is, teachers tend to fail to reject example-based proofs (i.e. empirical justification) that justify mathematical claims with resort to a few examples and are deemed as invalid at mathematicians' eyes. Recently, in the study by Knuth et al. (2020), middle school teachers were given four sample approaches to a given conjecture that were example-based and asked to rank order the approaches in terms of their judgements on how valid each approach was to them and their students. Even though they tended to favor the approaches that involved *generic exampl*es (Balacheff, 1988), the vast majority of the teachers responded that they thought the approaches with the generic examples were valid proofs though the mode of reasoning involved in the approaches was purely empirical in nature (Knuth et al., 2020). Based on Balacheff (1988)'s description on generic example that "involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class"(p. 219), the definition for the term generic example I will use in this report is an example that demonstrates an operation or a construction which is readily applicable to all other examples in the same class albeit its particularity as being an example rather than a general case that is seen in proof. Generic examples might be developed to be general cases, however, such transitions to general cases from generic examples tend to engender cognitive struggles for both mathematicians and students. As the authors (Knuth et al., 2020) suggested, "It is certainly possible that the teachers deemed the generic example justifications presented in the survey as not appropriate for middle school students, and perhaps they might have deemed other generic example justifications as more appropriate" (Knuth et al., 2020, p. 123). However, it is hard to deter one from arguing that teachers need more extensive and intensive experience about proof (Martin & Harel, 1989).

There are not many opportunities available in textbooks for students to engage with proving- related tasks and lack of variety of proving-related activities (Bieda et al., 2014; Thompson et al., 2012). Bieda et al. (2014) examined 7 elementary mathematics textbooks and documented that textbooks contribute a small fraction (3.7%) of their volumes to proving-related tasks as written. The authors went further to identify the types of proving-related tasks and the frequency of each type. The most frequent (varies between 46% and 84%) type across the textbooks was "making and justifying claims" and there was no task of the type "evaluating claims" (p. 78). Thompson et al. (2012) examined 20 high school mathematics textbooks (namely, Algebra 1, 2, & Precalculus) and only 11.2% of the properties addressed (on average, 19.2%) in the textbook narrative were left to students to potentially engage with reasoning and proving. Furthermore, the proof methods introduced to students are far from being of variety. Majority of the proofs presented in school mathematics are those called two-column proofs and do not show the variety of ways of proof (e.g., proof by contrapositive and *reductio ad absurdum*) (Thompson et al., 2012).

2. EXAMPLE USE IN INSTRUCTION OF PROOF

Though example plays a critical role in proving-related activities (Alcock & Inglis, 2008; Ellis et al., 2019; Knuth et al., 2009; Lakatos, 1976; Lockwood, Ellis, Knuth, Dogan, & Williams, 2013; Ozgur, Ellis, Vinsonhaler, Dogan, & Knuth, 2019; Weber & Alcock, 2004), it is both an obstacle for students (Fischbein & Kedem, 1982; Harel & Sowder, 1998; Porteous, 1990) and a leverage for mathematicians (Alcock & Inglis, 2008; Ellis et al., 2013; Lockwood et al., 2013; Lynch & Lockwood, 2017). In *Proofs and Refutations*, Lakatos (1976) exemplified the use of examples when exploring, formulating, or qualifying a conjecture, developing a proof and making revisions when one was encountered with counter examples—either *global* or *local*. Particularly, more attention seemed to be paid to how to revise a proof or a conjecture at hand rather than how one could use examples to develop a proof. Lockwood et al. (2013) studied how a mathematician used examples when proving and disproving. They found out that the mathematician gained insight into proof by leveraging idea of an insightful example, referring back and forth to an example pertaining to a conjecture. In line with what Balacheff (1988) coined as *generic example*, a representative example of the domain of a conjecture may possess the potential to be developed to a proof after *syntactic* or *semantic proof production* (Weber & Alcock, 2004). This view is well reflected in the work of the criteria-affordances-purposes-strategies framework (Ellis et al., 2019). In the study by Knuth et al. (2009), students were asked to justify the select conjectures from Number theory and the majority of their justifications were empirical in nature.

Teachers' views about students' example use are not surprisingly different from the result. Knuth et al. (2020) conducted a survey on teachers' views about the role of examples in proving-related activities, teachers' judgements about students' use of examples, and teachers' judgements on the given justifications. The teachers' responses related to students' uses of examples in proving-related activities indicated that the students' intent toward using examples is mostly concerned with *checking* whether a conjecture is true (65%), *proving* that the conjecture is true (50%), and *explaining* to someone else that the conjecture is true (46%) (Knuth et al., p. 122). What is deemed problematic about the students' predominant use of empirical arguments as proof is attributed, in part, to that students may not have robust understanding about the limitation of empirical arguments as valid proofs that are readily accepted by professional mathematicians (Fischbein & Kedem, 1982; Porteous, 1990). In other words, students may not develop a robust understanding about the distinction between proof and evidence in mathematics (Chazan, 1993).

Some research literature (e.g., Coe & Ruthven, 2002; Harel & Sowder, 1998; Knuth, 2002c; Knuth et al., 2020; Martin & Harel, 1989) documented that some teachers show lack of understanding about the limitation of examples as proof. In other words, some teachers do not have a robust understanding about what examples can and can't do: Examples do not prove a theorem and a counterexample disproves a conjecture. Martin & Harel (1989) studied 101 preservice elementary teachers' judgements about inductive and deductive justifications of a statement (either familiar or unfamiliar to them). The results indicated that, in both familiar and unfamiliar contexts, the majority of the participants considered inductive arguments as acceptable and accordingly rated them high. What comes as a surprise is that 80% of the participants deemed at least one of the inductive arguments as acceptable as proof. Furthermore, only fewer than 10% of the participants found all the inductive arguments not acceptable as proof. This can be explained by the convincingness of arguments to teachers or by the conception of proof that teachers have. Knuth (2002c) interviewed 16 secondary mathematics teachers and reported that 6 teachers found arguments that involved specific examples convincing. The author went on to say that "The teachers also displayed varying abilities in distinguishing between arguments that constituted proofs and those that did not; they tended to be very proficient at recognizing proofs but had more difficulty recognizing nonproof" (Knuth, 2002c, p. 401).

III. RESEARCH METHODLOGY

1. THE CONTEXT OF THE STUDY

The participants of this study were 420 students (Grade 10) and a mathematics teacher who had 3 years of teaching by the beginning of the academic year the study was begun. The locus of the study was a public high school located in the middle of South Korea and the students in this study enrolled the school to pursue undergraduate degrees upon graduation. Per the governmental policy on the mathematics curriculum, all the students learned about proof in the context of Euclidian geometry: Similarity and congruence of triangles and quadrilaterals in Grade 8, Pythagorean theorem and the properties of circles in Grade 9 (MoE, 2011). Also, they were expected to learn how to prove in Algebra (e.g., formulae for factorization of polynomials or expansion of multiplication of polynomials) albeit the more focus on application of the formulae (MoE, 2011) rather than the corresponding proofs of the formulae. Each classroom was occupied by 35 male students and the teacher taught fifty-minute-long class periods each classroom five times a week.

It should be noted that the teacher in the study is the author of this paper. When the study was underway, the teacher conducted a research on the interplay between problem solving and problem posing with not much focus placed on proving. This study was part of the teacher's own research on his instructional practice. The teacher participated in the regional instructional practice showcase in 2015 and the final report of the practice and reflection was reported to the local school district for review resulting in being awarded in the second best in the academic year. The author analyzed the data collected during the study with latency of five years taking a different lens.

During the first quarter of this research, I focused on how "What if?" strategy (Brown & Walter, 2014) might help students to gain insight into solution methods as a heuristic (Schoenfeld, 1983) and possibly extend their understandings to a broader context than the context (or domain) of a problem that they already considered when solving the problem. Note that two subject pronouns are used to distinguish the perspectives of the first to speak as the teacher and the third person as the author. For instance, after a student solved a problem of which domain is the set of whole numbers 1-10, the teacher asked the student to consider a broader domain such as whole numbers 1-100 or more. During the rest of the year, my focus on the instruction was gradually shifted to proving-related activities and everyday lesson was accordingly attuned to the explicit instruction about what constitutes an acceptable proof and negotiation of necessary norms reiteratively until they became consistent. The norms concerned participation, presentation of one's idea and expectation for others when one is presenting, writing one's proof, and so forth.

The teacher's discussions with individual students were always followed by other discussions to help them to articulate their thinking. To that end, the teacher did not provide any holistic evaluation or judgement to a student's (working or purported to be wrought) proof in the first place but solicited the student's thinking or reasoning implicit in the proof that the student provided. This was due to my belief that was well in resonance with a line from Zack (1999): "Although the children's talk is embedded in what sounds like everyday conversation, at the same time it reveals a complex mathematical structure"(p. 129). I valued students' ideas that were often masked by the medium (e.g., form, language), encouraged students to gradually articulate their thinking with their own language, and prompted them to revise their own languages to the language that is acceptable by the community of mathematicians. As this kind of gradual articulation may not likely occur between students until the relevant norm was established (and shared) in the classroom community, the teacher ensured that his discussions with individual students were ensued by discussions with their peers.

2. DATA COLLECTION

The data of this study consists of student's written assignments, teacher's written prompts that appeared on the students' written work, two video-taped lessons of which duration were about 50 minutes, and the teacher's reports (one submitted past half the year and the other reported at the end of the year) for the purpose of asking for advice or feedbacks and disseminating the results of the instructional practice. The students' written assignments were designed based on "What if?" strategy (Brown & Walter, 2014).

Students' written assignments were collected on a weekly basis and the worksheet for the assignment was structured by the teacher. The teacher collected mostly one from each student weekly and it accumulated to 105 worksheets in total per week with exceptions including the weeks with or before field trips, mid-terms, and finals. The students' work was collected by the teacher in chronological order so that the feature of the data allowed the author to analyze the changes in the teacher's pedagogy and draw the results concerning such pedagogical changes over the course of the study.

3. DATA ANALYSIS

The data including students' written work and teacher's prompts were analyzed not only to identify representative cases to highlight themes of close relevance to the foci of this study but to capture the teacher's pedagogical shifts based on inductive coding (Creswell & Poth, 2016). The relevant themes were pertaining to the potential of problem posing as a strategic tool to embrace proving into everyday lesson in general and how the teacher used problem posing as an initiative to necessitate the use of proof and guided students to develop proofs in particular.

The author took a sample of 200 (about 20 percent) from the students' written work. Since the data was collected in chronological order, in order to draw the sample of 200, the author took ten intervals with twenty in each. Given the focus of the study to identify pedagogical changes over time, the author made the decision to conduct the interval sampling which allowed for a more thorough look at the teacher's pedagogical emphasis during a specific time period and a longitudinal comparison in the emphasis over time.

The teacher's pedagogical shifts in learning goals set for the problem posing activity were captured by emergence of new codes based on analysis on what the teacher's feedback was concerned with. For example, before the teacher turned emphasis on generating examples for formulating conjectures, his feedback was concerned with computation, expressions, and representations used in students' solutions. This shift was captured based on the change in what the teacher's feedback asked for. Due to the subtlety of emphasis placed on feedback, the author complemented the analysis with the teacher's mid-year and end-of-year reports and analyzed the teacher's written feedback with taken into consideration two aspects: (1) what the teacher asked for at a given time and (2) what the teacher had asked before the time.

The first round of coding resulted in emergent codes in what the teacher asked for by providing feedbacks: *convergent* and *divergent*. These codes only separated the teacher's feedback in: One that he attended only to the substance inside a problem at a student's hand and the other that he pushed a student to consider a broader domain beyond the problem. Under the code *convergent*, the sub-codes computation, language, expression, and evaluating student's own argument emerged in the second round of coding. Take language for an example. This code meant that the teacher's feedback was calling a student's attention to the use of language in the student's argument so as to have the argument to be understandable by other students. Under the code *divergent*, there emerged the sub-codes strategically generating examples, making generalizations, evaluating conjectures, and proving.

The representative cases were identified based on the pedagogical shifts that coincided with the changes in emphasis captured by the coding and consistent with the teacher's reports. For instance, during the first month of the academic year, the teacher's feedbacks were concerned with computations, clarity in language, and posing problems. Later, the teacher provided feedback that invited students to generate examples for formulating conjectures. This shift coincided with the change in emphasis between the teacher's reports.

Due to the limitation in the data available, the critical events were re-constructed with collection of worksheets that preceded the pedagogical shifts. Though somewhat fictitious in nature, the critical events are to be reported as vignettes that are "aligned with relevant research paradigms and methodologies, reflecting realistic and identifiable settings that resonate with participants for the purpose of provoking responses, including but not limited to beliefs, perceptions, emotions, effective responses, reflections, and decision making" (Skilling & Stylianides, 2020, pp. 542-543).

The changes in classroom culture were analyzed based on the teacher's reports and the video-taped lessons. Since the primary data source of the teacher reports was the teacher's implementation of the practice and his reflection during and after the study, the results were akin to those encountered in design research. The recordings, as supplemental source of the data, allowed the author to report the changes in classroom culture that coincided with the reports.

IV. RESULTS AND DISCUSIONS

1. TEACHER’S PEDAGOGICAL CHANGES THAT GRADUALLY EMBRACED PROVING INTO EVERYDAY LESSON

1) Problem Posing as a Heuristic

Over the course of the study, the teacher continued to persistently build on or gradually revise the learning goals set for the problem posing activity that was not initially intended for teaching proof. At the beginning of the study, the teacher considered problem posing (or “What if” strategy) only as *a heuristic* that might offer students with insight into solution methods of a problem at hand (Schoenfeld, 1983) and the learning goal that he set initially was the students’ mastery of the strategy with hope that students would use it on their own when they stalemate in solving problems. Based on the video-recordings, he placed emphasis on students’ getting to know the interplay between what given constraints of a problems were modified and what changes were accordingly made in the newly posed problem.

2) Problem Posing as a Tool to Strategically Generate Examples

The following pedagogical change the teacher made for the problem posing activity was *a tool to strategically generate examples*. Here, I use strategic in a stark contrast to general, random, or serendipitous. As noted by Knuth et al. (2020), the vast majority of teachers seemed to think that their students select examples in not strategic ways such as first come to mind, easy to work with, or traditional. However, the teacher asked students to employ the way of problem posing in order to systematically generate examples when evaluating mathematical conjectures. The teacher’s prompt and the student’s attempts emerged in the student’s written work. For example, when evaluating a given proposition (a sum of two irrational numbers is irrational), the teacher called a student’s attention to look for a potential counterexample (a sum of two irrational numbers that is rational). To that end, the teacher asked the student to generate examples by way of problem posing. The student first came up with irrational numbers such as square roots of 2 and 3 and his attempts did not seem to be successful to disprove the proposition. Then it came into his realization that he only considered positive irrational numbers and he soon tried combinations of both positive and negative irrational numbers. The kind of proof that the student developed in this instance was a proof that argues against or contradicts a proposition and it functioned as verification and explanation.

Some students seemed to generate examples to evaluate the truth of their working conjectures and gain insight into how proofs for the conjectures would look like. There were a few instances of relevance. What the instances shared was that students changed the particular constraints of the problems at their hands and focused on how the modifications they made engendered changes in solutions correspondingly. For instance, after solving the problem (find the remainder of $2^{20} + 2^{22} + 2^{24}$ divided by 31), a student posed the new problem (find the remainder of $3^{20} + 3^{22} + 3^{24}$ divided by 242). The student noticed the relationship between the factor of the dividend and the divisor: $2^5 = 31 + 1$. Then, he also noted that $2^{5k} = (31 + 1)^k$ given k is a positive integer and its remainder is equal to 1 regardless of the value of k. Along a similar line of reasoning, he posed the new problem with a mathematically similar relationship between the factor of the dividend and the divisor ($3^5 = 242 + 1$) and noticed that $3^{5k} = (242 + 1)^k$ given k is a positive integer and its remainder is equal to 1 regardless of the value of k. Thus, he thought that the answers for both problems would be $1 + 2^2 + 2^4 = 21$ and $1 + 3^2 + 3^4 = 91$. After solving both problems, he further presented a conjecture that, when divided by $n^5 - 1$, the remainder of $n^{5k} + n^{5k+2} + n^{5k+4}$ where n (≥ 2) and k are whole numbers is equal to $1 + n^2 + n^4$. The constraint on n (namely, greater than or equal to 2) was added after he substituted 1 in the place of n. This kind of proof served as an entry into discovery of a new knowledge, explanation and verification of the knowledge. Moreover, the endeavor to develop the proof might improve the student's understanding about the initial problem that he worked with as mathematicians do. This dynamic (or non-linear) nature of evaluating and qualifying conjectures resonates well with the quasi-empirical view on proof (Lakatos, 1976). In particular, according to Lakatos, conjectures are developed through proving them vice versa. This instance is also pertaining to the following pedagogical change.

3) *Posing as a Tool to Make Generalizations*

The teacher considered problem posing as a support for students to *make generalizations*. While he thought problem posing might support student's problem solving by reiterating mathematically similar problems in his mid-year report, he later considered the problem posing activity could aid students in formulating conjectures, revising conjectures, and making generalizations based on examples generated as a byproduct of the activity (p. 14 of the end-year report). There was an instance where the teacher reconsidered the learning goal set for the activity. The protagonist of the anecdote was Sungjae (pseudonym) who began with the problem (find the value of the polynomial $\omega + \omega^2 + \omega^3$ given that ω is a complex root of $x^2 + x + 1 = 0$) and posed a new problem (find the value of the polynomial $\omega + \omega^2 + \cdots + \omega^{10}$ given that ω is a complex root of $x^2 + x + 1 = 0$). In his first attempt to solve the initial problem, he literally calculated the root ω and found the value of the polynomial in question. When posing a problem based on the initial problem, he thought that the computational approach to the problem would work for a more generalized cases and posed a problem with a longe r polynomial with respect to ω, intentionally attempting to apply the same approach to a similar but more generalized problem. Later, he found that his computational approach did not work as he had hoped and felt a need to look for other approaches to the problem that might work. Though he took some time of deliberation, he noticed that $\omega^2 + \omega + 1 = 0$ is true, came to understand the meaning of the equation $\omega^2 + \omega + 1 = 0$ that three consecutive terms in ascending order resulted in 0, and solved the ensuing problem $\omega + \omega^2 + \cdots + \omega^{10} = \omega$ or ω^{10} by grouping three consecutive terms knowing that either case equates another. Then he said he generalized the observation even further and provided a conjecture: given $\omega + \omega^2 + \cdots + \omega^n$ where n is a whole number, the equation is equal to either -1, 0, or ω with the justification that a sum of any three consecutive terms in the polynomial of ω is equal to 0 and this leaves three possibilities: ω, $-1 = \omega + \omega^2$, or $0 = \omega(1 + \omega^1 + \omega^2)$. Finally, he went on to say that the value of the polynomial of ω had to do with the remainder of n with division by 3. Here, Sungjae employed the problem posing activity to undertake an experiment to generalize his solution method. He gradually increased the level of generality of the initial problem and extended his prior understanding about it. He came to understand the mathematical structure of the initial problem and made a conjecture that needed a proof. The formulation of the conjecture necessitated a proof at the time. The proof functioned as discovery of a mathematical knowledge, explanation on his reasoning, and verification of the knowledge.

The context described above is quite different from what we often encounter in geometry in that it contains theorems that are known to be true with no room for students to evaluate the truth of the theorems before one puts an effort to prove them. This instance seemed to lead the teacher to recalibrate the learning goal set for the problem posing activity: Problem posing as *a tool to strategically generate examples* to formulate a conjecture, *revise the conjecture* based on the examples, and, finally, *make generalizations*. Note that, at this point, the teacher still did not seem to consider proving as part of the problem posing activity. This somewhat cyclic nature of implementing the activity can also be explained by the terms *the preparation for the experiment*, *the teaching experiment*, and *the retrospective analysis* (Gravemeijer & van Eerde, 2009).

4) Problem Posing as a Tool to Embrace Proving into Everyday Lesson

Later, the teacher asked for proofs from the students. This change was coincidental with the end-of-year report and the emergence of the code that was not observed earlier in the data. The teacher's prompts that pressed for proofs were apparent in writing such as "how do you prove this (pointing to a conjecture)?", "is this conjecture true?". The teacher asked proofs for conjectures as part of students' proofs or proofs as whole. At this point, the teacher did not appear to recognize the need for negotiating the meaning of proof in terms of true, valid, and appropriate (Stylianides, 2007). However, in other students' work, the teacher put an effort to make explicit the definition of proof with regard to what true statements one can use at a given time, what valid mode of reasoning one may take, and what appropriate representations one may use. For example, the teacher left a comment on a student's work: "This property has not been covered per the national curriculum, so you might need to consider using other properties that we've covered or have been covered in earlier grades." He seemed to consider true statements that students could use in their proofs as those covered previously. This criteria for true statements was consistent across his comments that asked students to use theorems that they had learned so far.

The change in the learning goal set for the problem posing activity engendered other issues after proving became part of the problem posing activity. The issue seemed to result from the lack of understanding about what it meant to be a proof. For example, the teacher attended to and pointed out true, valid, and appropriate aspects of students' proofs. The underlying causes of the issue included, lack or absence of justifying each step (i.e. reasoning), incorrect or peculiar use of representations, and gap between what was (commonly) known to the classroom community and what was known to the author of the proof. The causes brought to the fore the issues related to the aspects described in the definition of proof, namely, *appropriate mode of representations* (Stylianides, 2007).

The teacher appeared to notice that the issue of difficulty in understanding one's proof was primarily due to the absence of discussion about what the classroom community shared in regard to proof and that there must be continual and persistent negotiation of what it meant to be a valid proof in the classroom community. This was evident in the videorecording of the lesson that he began with discussions around what appropriate mode of representations students might use. Otherwise, some students used incorrect representations, the teacher seemed to have a firm reservation on limiting representations that students could use. He made explicit throughout his written feedback that any representations are deemed appropriate as long as they convey the intended meaning to the reader.

2. CULTIVATING THE CLASSROOM CULTURE TO SUPPORT STUDENT PROVING THAT MAKES STUDENT PROVING MORE VALUED AND VIABLE

The changes that the teacher made to cultivate a classroom culture where student proving was valued and viable were two-fold: One was promoting student's attempt (rather than the outcome of it) and the other was establishing a supportive community of students. When the learning goal of the problem posing activity on proving was in practice, the changes seemed to give rise to issues in regard to the classroom culture that discouraged or jeopardized student proving (or attempts to prove). For example, students were inclined to give evaluation or judgement of other student's work when they were expected to give constructive feedbacks. Some students refused to share their work with the excuse that their work was not yet complete.

Due to the changes in expectation, a class period was not sufficient for most students to finish the assignment. The teacher ran out of class time and that might hinder students learning. Then, the teacher decided to unshackle himself from the constraint on time by *not limiting the time for mathematical investigation to a single class time*. Rather, he expanded the learning beyond the classroom and the assigned class period. This was explicit in the latter video-recording while implicit in the former. During the class period, he and students only shared and discussed their work. Most questions were addressed between the teacher and students. This is not to say that there was no discussion between students outside the classroom, but, rather, to say that there were discussions among students which were unknown to the teacher (and the author). Finally, to communicate student thinking and solutions by writing, there arose a need for communication in classroom.

The teacher appeared to situate students as mathematical authorities who should respect others and be respected by others. With mutual respect, they must give constructive feedbacks to enable improvement in other's work rather than evaluation (or judgement). The teacher persistently instilled to students that proof is not an easy task for anyone and that proof is undergone continual revisions until it becomes acceptable by others, showing his erring in formulating false (or incomplete) conjectures and struggling in developing proofs. Furthermore, he valued students' attempts and potential of their work more than accuracy or completeness of the attempts and the constructs that students presented at a given time. This was well captured in both the video-recordings and the end-of-year report.

The teacher's pedagogical principles in supporting student proving were apparent in his writing. Notwithstanding implicitness in his mid-year-report, his principles in this intervention were explicit in the end-of-year report: (a) reinforce the strength of student's argument, (b) focus on processes rather than results, (c) do not evaluate student's argument that only leaves a mark, and (d) be a facilitator who helps student to elaborate/articulate his/her thinking (p. 4). His principles were well reflected in the following. The teacher extended a student's reasoning first by prompting reflection on the proof the student provided and accordingly by attending to where more reasoning is necessary. In an instance where, to denote the number of subsets of a given set, a student used his own representation that was understandable by some of other students, however, the teacher pointed to that and asked the student to re-represent it. In another instance, a student solved the problem that find a maximum of a function defined on a subset of integers from 1 to 15 with the constraints that $f(2n) = f(n)$ and $f(2n+1) = f(n) + 1/2$ and that $f(1) = 1/2$. After testing some whole numbers from 1 to 15, the student successfully developed a proof for the problem. The teacher posed a question in writing: Can you explain what the function is if the domain of the function is the set of all non-negative integers? Since the teacher might notice that the student gained insight into the underlying mathematical structure of the given function, the teacher posed a question to *extend* the student's *understanding* of the problem in a more generalized way by having the student to consider a broader domain to which the domain of the problem would be expanded. On the one hand, the newly posed problem seemed to be a daunting task for the student, on the other hand, it was appropriately and cognitively demanding for the student. The teacher's intention seemed to be in resonance with Stylianides (2009):

> examples are important because they can provide students with a powerful and easily reached means of conviction and explanation and can allow students to prove mathematical claims even when they lack mathematical language to express their proofs in more sophisticated ways (p. 264).

V. DISCUSSIONS AND IMPLICATIONS

Learning experiences about proof may not guarantee a better or more sophisticated understanding about proof. According to the national curriculum, the students who participated in this study were expected to learn proofs in plane geometry at Grade 8, Pythagorean theorem and properties of circles at Grade 9. It can be said that, on the one hand, the students had some level of familiarity with proof and deductive reasoning when the study was underway. In the other hand, their experience of learning about proof was exclusively limited to geometry of which a hallmark is deductive reasoning (or two-column proofs) and direct proof. However, the fact that they learned about proof at earlier grades did not guarantee that they had either robust understanding about proof or competence to undertake proving on their own. Some students tended to fall short of understanding the limitations of examples.

Problem posing can offer a strategic and systematic way to investigate mathematical conjectures for students as well as teachers. In this study, students used problem posing (or "what if") to strategically generate examples when formulating, investigating, or revising mathematical conjectures. The teacher used it as a way of extending student's thinking by challenging student's understanding at a given time or by prompting to consider a broader domain than that of what a student considered earlier. Iannone et al. (2011) suggested,

> […] *simply* asking students to generate examples about a concept may not substantially improve their abilities to write proofs about that concept. […] This suggests that if example generation is to be a useful pedagogical strategy, more *nuance* is needed in its implementation (p. 11, italics added).

Such nuance can be explained by the difference in example use between experts and students (Lynch & Lockwood, 2019). They also argued that students rarely generated examples strategically when students attempted to better understand and develop a proof for a conjecture. This study contributes to the literature in that problem posing (or "what if") enables example generation in a strategic and systematic way.

The implications of this study should be understood as problem posing *for* proving as not only a way of orchestrating classroom discourse but also a tool with potential to bridge proving and everyday lesson. Problem posing enables teachers to ask questions that extend students' mathematical understanding at a given time to the next level, challenging students' cognitive plateau. These questions can lead students to conduct further investigation on what has already been investigated, and reflect on what they already understand. Classroom discourse may be extended by the questions, offer opportunities for students to reinforce their understanding, and, ultimately, be followed by developing proofs. In light of Bieda's (2010) suggestion that "greater emphasis is needed for middle school teacher preparation, professional development, and curricular support to make justifying and proving a routine part of middle school students' opportunities to learn" (p. 380), this study is an initiative to answer the call for making justifying and proving a routine part of students' learning.

The teacher's role in the instruction of proof is crucial and critical for student's learning of proof. As Begle (1973) noted, "most student learning is directed by *the text* rather than the teacher" (p. 209, italics added). Although there exists variation among teachers in enacting curricular materials (Remillard & Bryans, 2004; Thompson & Senk, 2014), it may not be reasonable to have an expectation for teachers that they could identify proving-related activities outside textbook and enact them in their instruction to teach how to prove when proof is not an explicit goal of a lesson. However, the expectation may be seen as an imperative that professional development and teacher preparation programs need to include explicit instruction about instructional practices particular to teaching how to prove and knowledge about proof (see Selden & Selden (2003), pp. 27-28 for the discussion about the explicit instruction about proof validation and its potential to improve learner's performance of proof validation) so as for teachers to be better equipped with teaching how to prove beyond what are available in textbook and understanding strategic knowledge of instructional practices and its relation to student's learning of proof (Stylianides & Ball, 2008).

The results of this study should be scrutinized, involving different student population or curricular settings. As described earlier, this study was conducted at a male-student-only high school in South Korea with 10^{th} grade students in 2015. Given that the students had some level of understanding about proof, the instruction might have been encountered with different issues if implemented it in a classroom with different student populations. The content covered throughout the study ranged from sets and inequalities (including Cauchy-Schwarz inequality and inequality of arithmetic and geometric means) to proof methods (including proof by contrapositive, *reductio ad absurdum*, and mathematical induction). This particularity of this study necessitates examination on what support students who have less (lack thereof) experiences about proof need, or how strategically problem posing helps students to generate examples when engaging with proving-related activities. Due to the fact that the focus of this study was to explore possible instructional practices that brings proof into classroom as a routine part of student learning, the question should require an examination:

How impactful the instruction is on student's performance on proof writing or students' understanding about proof. Finally, due to the fact that the instruction about proof relied heavily on the teacher's knowledge, this requires future research on what a teacher should know to teach proof, what a teacher should know to implement the instruction described in this study, and the relationship between the instructional practice and student learning about proof (Stylianides & Ball, 2008).

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