Bull. Korean Math. Soc. **58** (2021), No. 5, pp. 1079–1095
https://doi.org/10.4134/BKMS.b200359
pISSN: 1015-8634 / eISSN: 2234-3016

NEW CONGRUENCES WITH THE GENERALIZED CATALAN NUMBERS AND HARMONIC NUMBERS

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Abstract. In this paper, we give new congruences with the generalized Catalan numbers and harmonic numbers modulo p^2. One of our results is as follows: for prime number $p > 3$,

$$\sum_{k=(p+1)/2}^{p-1} k^2 B_{p,k} B_{p,k-(p-1)/2} H_k \equiv (-1)^{(p-1)/2} \left(-\frac{521}{36}p - \frac{1}{p} - \frac{41}{12} + pH_{3(p-1)/2}^2 - 10pq_p^2(2) + 4\left(\frac{10}{3}p + 1\right) q_p(2) \right) \pmod{p^2},$$

where $q_p(2)$ is Fermat quotient.

1. Introduction

The Catalan numbers have important applications in combinatorics, number theory and the analysis algorithms in [7, 8, 12, 14].

In [14], Shapiro gave the generalized Catalan numbers $B_{n,k}$ defined by

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k},\ 0 \le k \le n.$$

The numbers $B_{n,k}$ are the entries of the Catalan triangles and satisfy the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1},\quad k \ge 2$$

with the initial conditions $B_{n,0} = 0 = B_{n,n+m}$, $m \ge 1$ [5, 9, 14]. Note that for $k = 1$, $B_{n,1}$ are the well known Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n},\quad n \ge 1.$$

Harmonic numbers are those rational numbers given by

$$H_0 = 0,\quad H_n = \sum_{k=1}^{n} \frac{1}{k},\quad n \in \mathbb{Z}^+,$$

Received April 19, 2020; Revised February 15, 2021; Accepted June 4, 2021.
2010 *Mathematics Subject Classification.* Primary 11B50, 11A07, 11B65.
Key words and phrases. Congruences, harmonic numbers and binomial coefficients.

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and for $m \in \mathbb{Z}^+$, harmonic numbers of order m are those rational numbers given by

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \quad n \in \mathbb{Z}^+.$$

For $x \in \mathbb{N}$, Bernoulli polynomials $B_n(x)$ and Euler polynomials are defined by the generating functions

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} \text{ and } \frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!}.$$

The rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called Bernoulli numbers and Euler numbers, respectively. These polynomials and numbers appear in many combinatorial and number theory contexts.

For a prime p and an integer a with $p \nmid a$, we write the Fermat quotient $q_p(a) = \left(a^{p-1} - 1\right)/p$.

In [19], Wolstenholme discovered that for any prime number $p > 3$,

$$H_{p-1} \equiv 0 \pmod{p^2}. \tag{1}$$

Some interesting congruences involving harmonic numbers are given as follows [17, 18]: for $0 \leq k \leq p-1$,

$$\binom{2p-1}{k} \equiv (-1)^k (1 - 2pH_k) \pmod{p^2}, \tag{2}$$

$$H_{p-1-k} \equiv H_k \pmod{p}, \tag{3}$$

$$H_{p+k} \equiv H_k + \frac{1}{p} \pmod{p}, \tag{4}$$

$$H_{p+k,2} \equiv H_{k,2} + \frac{1}{p^2} \pmod{p}. \tag{5}$$

In [16], Sun obtained that for prime number $p \geq 3$ and $m \in \{1, 2, 3, \ldots, p-4\}$,

$$H_{p-1,m} \equiv \frac{m(m+1)}{2}\frac{B_{p-2-m}}{p-2-m}p^2 \pmod{p^3}, \tag{6}$$

and for prime number $p > 5$ and $m \in \{2, 4, \ldots, p-5\}$

$$H_{(p-1)/2,m} \equiv \frac{m\left(2^{m+1} - 1\right)}{2(m+1)} pB_{p-1-m} \pmod{p^2}. \tag{7}$$

In [11], Lehmer derived that for prime number $p > 3$,

$$H_{[p/4],2} \equiv (-1)^{(p-1)/2} 4E_{p-3} \pmod{p}, \tag{8}$$

$$\sum_{k=1}^{[p/4]} \frac{1}{p-4k} \equiv \frac{3}{4}q_p(2) - \frac{3}{8}pq_p^2(2) \pmod{p^2}, \tag{9}$$

and for odd prime number p,

$$H_{(p-1)/2} \equiv pq_p^2(2) - 2q_p(2) \pmod{p^2}. \tag{10}$$

Some interesting properties of harmonic numbers are given as follows [2, 4, 6, 15]:

$$\sum_{k=1}^{n} k^{\underline{m}} H_k = n^{\underline{m}} \frac{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right), \tag{11}$$

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2} \left(H_n^2 + H_{n,2} \right), \tag{12}$$

where $x^{\underline{m}}$ stands for the falling factorial defined by

$$x^{\underline{0}} = 1, \quad x^{\underline{m}} = x\,(x-1)\cdots(x-m+1)\,.$$

In [15], Spieß obtained that for any $n, m \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=1}^{n} \binom{k}{m} H_k^2 = {} & \binom{n+1}{m+1} H_n^2 - \left(2 \binom{n}{m+1} + (-1)^m \right) \frac{H_n}{m+1} \\ & + \frac{2}{(m+1)^2} \binom{n}{m+1} - \frac{1}{(m+1)!} \sum_{k=2}^{m+1} s(m+1,k) H_{n,2-k}, \end{aligned} \tag{13}$$

where $s(n,k)$ are Stirling numbers of the first kind.

In [1], every pair of families $(a_k)_{k=1}^n$ and $(b_k)_{k=1}^n$ of complex numbers verifies the identity

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} \left[(a_k - a_{k+1}) \left(\sum_{j=1}^{k} b_j \right) \right] + a_n \left(\sum_{j=1}^{n} b_j \right). \tag{14}$$

This identity is called as Abel's partial summation formula.

Recently, some authors studied the binomial sums $\sum_{k=1}^{n} k^m \binom{2n}{n-k}^2$ [3, 7, 8, 12].

In [13], Ömür and Koparal obtained various congruences involving the numbers $B_{p,k-d}$. For example, for $1 \le d \le p-1$,

$$\sum_{k=1}^{p-1} B_{p,k} B_{p,k-d} \equiv 4\,(-1)^d \left(-2d + p \left(8d\,(H_d - 1) + \frac{2}{d} - 3 \right) \right) \pmod{p^2},$$

and in [10], they proved some congruences involving the generalized Catalan numbers and harmonic numbers modulo p^2. For example, for $0 \le m < p-1$ and $1 \le d \le p-1$,

$$\begin{aligned} \sum_{k=d}^{p-1} k^{\underline{m}} H_{k-d} \equiv {} & (p+d-1)^m \frac{p+d}{p(m+1)} \\ & - (d-1)^m \frac{d}{m+1} \left(\frac{1}{p} + H_d - H_{m+1} \right) \pmod{p}. \end{aligned} \tag{15}$$

Now, in this paper, we give new congruences with the generalized Catalan numbers and harmonic numbers modulo p^2. One of our results is as follows: for prime number $p > 3$,

$$\begin{aligned}
&\sum_{k=(p+1)/2}^{p-1} k^2 B_{p,k} B_{p,k-(p-1)/2} H_k \\
&\equiv (-1)^{(p-1)/2} \left(-\frac{521}{36}p - \frac{1}{p} - \frac{41}{12} + pH_{3(p-1)/2}^2 - 10pq_p^2(2) \right. \\
&\quad \left. +4\left(\frac{10}{3}p + 1\right) q_p(2) \right) \pmod{p^2},
\end{aligned}$$

where $q_p(2)$ is Fermat quotient.

2. Some results on congruences

In this section, firstly, we will continue our work with the following lemmas for the next use.

Lemma 2.1. *Let p be prime number. We have*

$$H_{\lfloor p/4 \rfloor} \equiv -p\,(-1)^{(p-1)/2} E_{p-3} - 3q_p\,(2) + \frac{3}{2} p q_p^2\,(2) \pmod{p^2}. \tag{16}$$

Proof. Consider that

$$\begin{aligned}
\sum_{k=1}^{[p/4]} \frac{1}{p-4k} &= \sum_{k=1}^{[p/4]} \frac{p+4k}{p^2-16k^2} \equiv -\sum_{k=1}^{[p/4]} \frac{p+4k}{16k^2} \\
&= -\frac{p}{16} H_{[p/4],2} - \frac{1}{4} H_{[p/4]} \pmod{p^2}.
\end{aligned}$$

From (8) and (9), we have the conclusion. □

Lemma 2.2. *Let p be odd prime number. We have*

$$p^2 H_{3(p-1)/2}^2 \equiv 4p\,(1 - q_p(2)) + 1 \pmod{p^2}, \tag{17}$$

and for $p \equiv 1 \pmod 4$,

$$p^2 H_{5(p-1)/4}^2 \equiv 2p\,(4 - 3q_p(2)) + 1 \pmod{p^2}. \tag{18}$$

Proof. From (4), we have

$$\begin{aligned}
\left(pH_{3(p-1)/2}\right)^2 &\equiv \left(pH_{(p-3)/2} + 1\right)^2 \\
&= p^2 H_{(p-3)/2}^2 + 2pH_{(p-3)/2} + 1 \\
&\equiv 2p\left(H_{(p-1)/2} - \frac{2}{p-1}\right) + 1 \\
&\equiv 2pH_{(p-1)/2} + 4p + 1 \pmod{p^2}.
\end{aligned}$$

By (10), the desired result is obtained. Similarly, with the help of (9), the other congruence is given. □

Lemma 2.3. *Let p be prime number. For integers m, k and d such that $k \neq p, d$ and $p \nmid k-d$,*

$$\frac{k^m\,(k-d)}{(p-k)\,(p-k+d)} \equiv k^{m-2}\left(k+2p-d\frac{p}{d-k}\right) \pmod{p^2}. \tag{19}$$

Proof. By $\frac{1}{p^2+k} \equiv \frac{1}{k} \pmod{p^2}$, we have

$$\frac{k^m}{p-k} = \frac{k^{m-2}k^2\,(p+k)}{p^2-k^2} \equiv -k^{m-2}\,(p+k) \pmod{p^2},$$

and

$$\frac{k-d}{p-(k-d)} = \frac{(k-d)\,(p+k-d)}{p^2-(k-d)^2} \equiv -\frac{p+k-d}{k-d} \pmod{p^2}.$$

From multiply of these congruences, we get

$$\frac{k^m\,(k-d)}{(p-k)\,(p-k+d)} \equiv k^{m-2}\left(k+2p-d\frac{p}{d-k}\right) \pmod{p^2},$$

as claimed. □

Lemma 2.4. *Let p be odd prime number. For $1 \leq d < p-1$,*

$$\sum_{k=d+1}^{p-1} H_k \equiv d-(d+1)H_d-p+1 \pmod{p^3} \quad \textit{for } p \neq 3,$$

$$\sum_{k=d+1}^{p-1} kH_k \equiv -\binom{d+1}{2}H_d+\frac{1}{2}\binom{d}{2}-\frac{1}{2} \pmod{p},$$

and

$$\sum_{k=d+1}^{p-1} k^2H_k \equiv -\frac{2d+1}{3}\binom{d+1}{2}H_d-\frac{17}{36}p+\frac{1}{6}+\frac{4d+1}{18}\binom{d}{2} \pmod{p^2}.$$

Proof. From (1) and the equality (11), the desired congruences are given. □

Lemma 2.5. *Let p be odd prime number. For $1 \leq d < p-1$,*

$$\sum_{k=d+1}^{p-1} H_k^2 \equiv -(d+1)\left(H_d^2+2\right)+(2d+1)\,H_d \pmod{p},$$

$$\sum_{k=d+1}^{p-1} kH_k^2 \equiv \frac{1}{2}\left(-2\binom{d+1}{2}H_d^2+2\binom{d}{2}H_d-H_d-\binom{d}{2}+d+2\right) \pmod{p},$$

and for $p > 3$,

$$\sum_{k=d+1}^{p-1} k^2H_k^2 \equiv \frac{2}{3}\left(-\binom{d+1}{2}\frac{2d+1}{2}H_d^2+\frac{1}{12}\left(4d^3-3d^2-d+3\right)H_d\right.$$
$$\left.-\frac{1}{72}d\left(8d^2-15d+25\right)-\frac{2}{3}\right) \pmod{p}.$$

Proof. Taking $m = 0, 1, 2$ in (13) and the congruence (1), we have the proofs of congruences. □

Lemma 2.6 ([10])**.** *Let $p > 3$ be prime number. For $1 < d < p-1$,*

$$\begin{aligned}(20)\qquad &\sum_{k=d+1}^{p-1} \frac{1}{k} H_{k-d} \\ &\equiv H_{p+d-1}^2 - H_{p+d-1,2} - 2H_{d-1}^2 + 2H_{d-1,2} - \frac{2}{p} H_{d-1} \pmod{p}.\end{aligned}$$

Lemma 2.7. *Let $p > 3$ be prime number. For $1 < d < p-1$,*

$$\begin{aligned}&\sum_{k=d+1}^{p-1} \frac{H_k}{k-d} \\ &\equiv -H_{p+d-1}^2 + 2H_{d-1}^2 - H_{d-1,2} + \frac{2}{p} H_{d-1} + \frac{2}{d} H_d + \frac{1}{p^2} \pmod{p}, \\ &d \sum_{k=d+1}^{p-1} \frac{H_k}{k\,(k-d)} \\ &\equiv \frac{1}{p^2} + \frac{5}{2} H_{d-1}^2 - \frac{1}{2} H_{d-1,2} - H_{p+d-1}^2 + \frac{2}{p} H_{d-1} + \frac{3}{d} H_d \pmod{p}.\end{aligned}$$

Proof. Taking $a_k = H_{k+d}$ and $b_k = \frac{1}{k}$ in (14), by Abel sum, (1) and (3), we get

$$\begin{aligned}\sum_{k=d+1}^{p-1} \frac{H_k}{k-d} &= \sum_{k=1}^{p-1-d} \frac{H_{k+d}}{k} = -\sum_{k=1}^{p-2-d} \frac{H_k}{k+d+1} - H_{p-1} H_{p-1-d} \\ &\equiv -\sum_{k=2}^{p-1-d} \frac{H_{k-1}}{k+d} = -\sum_{k=1}^{p-1-d} \frac{H_k}{k+d} + \sum_{k=1}^{p-1-d} \frac{1}{(k+d)\,k} \\ &= -\sum_{k=d+1}^{p-1} \frac{H_{k-d}}{k} + \frac{1}{d} \sum_{k=1}^{p-1-d} \frac{1}{k} - \frac{1}{d} \sum_{k=1}^{p-1-d} \frac{1}{k+d} \\ &\equiv -\sum_{k=d+1}^{p-1} \frac{H_{k-d}}{k} + \frac{2}{d} H_d \pmod{p}.\end{aligned}$$

By (5), (12) and (20), we have

$$\sum_{k=d+1}^{p-1} \frac{H_k}{k-d} \equiv -H_{p+d-1}^2 + 2H_{d-1}^2 - H_{d-1,2} + \frac{2}{p} H_{d-1} + \frac{2}{d} H_d + \frac{1}{p^2} \pmod{p},$$

as claimed. Similarly, the other congruence is obtained. □

Lemma 2.8. *Let $p>3$ be prime number. For $1<d<p-1$,*

$$\sum_{k=d+1}^{p-1} H_k H_{k-d} \equiv d\left(H_{p+d-1}^2-2H_{d-1}^2+H_{d-1,2}+2\left(1-\frac{1}{p}\right)H_{d-1}-\frac{1}{p^2}-2\right) \pmod p, \tag{21}$$

$$\sum_{k=d+1}^{p-1} kH_k H_{k-d} \equiv \binom{d}{2}\left(H_{p+d-1}^2-2H_{d-1}^2+H_{d-1,2}+2\left(1-\frac{1}{p}\right)H_{d-1}-\frac{1}{p^2}-2\right) \pmod p, \tag{22}$$

and

$$\begin{aligned}&\sum_{k=d}^{p-1} k^2 H_k H_{k-d}\\ &\equiv \frac{1}{3}\left(\frac{1}{6}d\left(-15d+13d^2+8\right)H_d\right.\\ &\quad+\frac{1}{2}d\left(2d-1\right)\left(d-1\right)\left(H_{p+d-1}^2-2H_{d-1}^2+H_{d-1,2}-\frac{2}{p}H_{d-1}-\frac{1}{p^2}\right)\\ &\quad\left.-\frac{1}{36}\left(d+1\right)\left(-104d+71d^2+48\right)\right) \pmod p.\end{aligned}$$

Proof. Consider that

$$\begin{aligned}&\sum_{k=d+1}^{p-1} k^3 H_k H_{k-d}+p^3 H_p H_{p-d}\\ &=\sum_{k=d+1}^{p} k^3 H_k H_{k-d}=\sum_{k=d}^{p-1}(k+1)^3 H_{k+1}H_{k+1-d}\\ &=\sum_{k=d}^{p-1}(k+1)^3\left(H_k+\frac{1}{k+1}\right)\left(H_{k-d}+\frac{1}{k-d+1}\right)\\ &=\sum_{k=d}^{p-1}(k+1)^3\left(H_k H_{k-d}+\frac{1}{k-d+1}H_k+\frac{1}{k+1}H_{k-d}+\frac{1}{(k-d+1)(k+1)}\right).\end{aligned}$$

Hence, if necessary arrangements are made, we have

$$\begin{aligned}&-3\sum_{k=d}^{p-1} k^2 H_k H_{k-d}\\ &=-p^3 H_p H_{p-d}+3\sum_{k=d}^{p-1} kH_k H_{k-d}+\sum_{k=d}^{p-1} H_k H_{k-d}+\sum_{k=d}^{p-1} k^2 H_k\end{aligned}$$

$$+(d+2)\sum_{k=d}^{p-1}kH_k+\left(d^2+d+1\right)\sum_{k=d}^{p-1}H_k+d^3\sum_{k=d}^{p-1}\frac{H_k}{k-d+1}$$
$$+\sum_{k=d}^{p-1}k^2H_{k-d}+2\sum_{k=d}^{p-1}kH_{k-d}+\sum_{k=d}^{p-1}H_{k-d}+\sum_{k=d}^{p-1}k+(d+1)(p-d)$$
$$+d^2\sum_{k=d}^{p-1}\frac{1}{k-d+1}.$$

By (15), (21), (22), Lemmas 2.4 and 2.7, the desired result is obtained. Similarly, we have the proofs of the other congruences. Thus, the proof is complete. □

Now we give some congruences involving generalized Catalan numbers and harmonic numbers in the following theorems.

Theorem 2.9. *Let $p>3$ be prime number. For $1<d<p-1$,*

$$\begin{aligned}&\sum_{k=d+1}^{p-1}B_{p,k}B_{p,k-d}H_k\\ &\equiv 4(-1)^d\left(d+7p+8dp-3+\frac{1+2d}{p}+\frac{3p}{2d^2}+\frac{4p-2}{d}\right.\\ &\quad+p\left(6d+\frac{7}{2}\right)H_d^2+\left(3d-10p-8dp+1-\frac{2p}{d}\right)H_d\\ &\quad\left.-p(2d+1)H_{p+d-1}^2-p\left(2d+\frac{3}{2}\right)H_{d-1,2}\right)\pmod{p^2}.\end{aligned}$$

Proof. Observe that

$$\begin{aligned}&\sum_{k=d+1}^{p-1}B_{p,k}B_{p,k-d}H_k\\ &=\sum_{k=d+1}^{p-1}\frac{k}{p}\binom{2p}{p-k}\frac{k-d}{p}\binom{2p}{p-k+d}H_k\\ &=4\sum_{k=d+1}^{p-1}\frac{k}{p-k}\binom{2p-1}{p-k-1}\frac{k-d}{p-k+d}\binom{2p-1}{p-k+d-1}H_k.\end{aligned}$$

From (2) and (19), we get

$$\begin{aligned}&\sum_{k=d+1}^{p-1}B_{p,k}B_{p,k-d}H_k\\ &\equiv 4(-1)^d\sum_{k=d+1}^{p-1}\left(1+2\frac{p}{k}-d\frac{p}{(d-k)k}\right)(1-2pH_{p-k-1})(1-2pH_{p-k+d-1})H_k\end{aligned}$$

$$= 4\,(-1)^d \sum_{k=d+1}^{p-1} \left(1 + 2\frac{p}{k} - d\frac{p}{(d-k)\,k}\right)(1 - 2pH_k - 2pH_{k-d})\,H_k$$

$$= 4\,(-1)^d \sum_{k=d+1}^{p-1} \left(H_k + 2p\frac{H_k}{k} - 2pH_k^2 - 2pH_kH_{k-d} - d\frac{p}{(d-k)\,k}H_k\right) \pmod{p^2},$$

and hence

$$\begin{aligned}
&\sum_{k=d+1}^{p-1} B_{p,k}B_{p,k-d}H_k \\
&\equiv 4\,(-1)^d \left(\sum_{k=d+1}^{p-1} H_k + 2p\sum_{k=d+1}^{p-1}\frac{H_k}{k} - 2p\sum_{k=d+1}^{p-1} H_k^2\right. \\
&\quad \left. -2p\sum_{k=d+1}^{p-1} H_kH_{k-d} - dp\sum_{k=d+1}^{p-1}\frac{1}{(d-k)\,k}H_k\right) \pmod{p^2}.
\end{aligned}$$

Using Lemmas 2.4, 2.5 and (6), (12), we have

$$\begin{aligned}
&\sum_{k=d+1}^{p-1} B_{p,k}B_{p,k-d}H_k \\
&\equiv 4\,(-1)^d \left(d + 1 + 4pd + 3p + pH_{p-1}^2 + pH_{p-1,2}\right. \\
&\quad + p\,(2d+1)\,H_d^2 - pH_{d,2} - (4pd + 2p + d + 1)\,H_d \\
&\quad \left. -2p\sum_{k=d+1}^{p-1} H_kH_{k-d} + p\sum_{k=d+1}^{p-1}\frac{d}{k\,(k-d)}H_k\right) \pmod{p^2}.
\end{aligned}$$

By (21) and Lemma 2.7, we write

$$\begin{aligned}
&\sum_{k=d+1}^{p-1} B_{p,k}B_{p,k-d}H_k \\
&\equiv 4\,(-1)^d \left(d + 1 + 4pd + 3p + pH_{p-1}^2 + pH_{p-1,2}\right. \\
&\quad + p\,(2d+1)\,H_d^2 - pH_{d,2} - (4pd + 2p + d + 1)\,H_d \\
&\quad - 2pd\left(H_{p+d-1}^2 - 2H_{d-1}^2 + H_{d-1,2} + 2\left(1 - \frac{1}{p}\right)H_{d-1} - \frac{1}{p^2} - 2\right) \\
&\quad \left. + p\left(\frac{1}{p^2} + \frac{5}{2}H_{d-1}^2 - \frac{1}{2}H_{d-1,2} - H_{p+d-1}^2 + \frac{2}{p}H_{d-1} + \frac{3}{d}H_d\right)\right) \pmod{p^2}.
\end{aligned}$$

From (1) and (6), this concludes the proof of Theorem 2.9. □

For example, for $d = 2$, we write

$$\sum_{k=3}^{p-1} B_{p,k}B_{p,k-2}H_k \equiv 77p - 6 \pmod{p^2}.$$

Corollary 2.10. *Let $p > 3$ be prime number. We have*

$$\sum_{k=(p+1)/2}^{p-1} B_{p,k}B_{p,k-(p-1)/2}H_k \equiv 2\left(-1\right)^{(p-1)/2}\left(7p+1+3pq_p^2(2)+2\left(5p+1\right)q_p(2)\right) \pmod{p^2}.$$

Proof. Taking $d=(p-1)/2$ in Theorem 2.9, we write

$$\begin{aligned}&\sum_{k=(p+1)/2}^{p-1} B_{p,k}B_{p,k-(p-1)/2}H_k \\ &\equiv 4\left(-1\right)^{(p-1)/2}\left(\frac{7}{2}p+\frac{10}{p-1}+\frac{6}{\left(p-1\right)^2}+\frac{11}{2}+\frac{p}{2}H_{(p-1)/2}^2\right. \\ &\left.-\frac{1}{2}\left(3p-1\right)\frac{3p+1}{p-1}H_{(p-1)/2}-p^2H_{3(p-1)/2}^2-\frac{1}{2}pH_{(p-3)/2,2}\right) \pmod{p^2}.\end{aligned}$$

With the help of (10) and (17) ,we get

$$\begin{aligned}&\sum_{k=(p+1)/2}^{p-1} B_{p,k}B_{p,k-(p-1)/2}H_k \\ &\equiv 2\left(-1\right)^{(p-1)/2}\left(3p+1+3pq_p^2(2)+2(5p+1)q_p(2)-pH_{(p-3)/2,2}\right) \pmod{p^2}.\end{aligned}$$

Hence, by (7), we have the proof. □

Corollary 2.11. *Let $p > 3$ be prime number. For $p \equiv 1 \pmod 4$,*

$$\begin{aligned}&\sum_{k=(p+3)/4}^{p-1} B_{p,k}B_{p,k-(p-1)/4}H_k \\ &\equiv \left(-1\right)^{(p-1)/4}\left(\frac{2}{p}+19+133p-3\left(p-1\right)q_p\left(2\right)+\frac{147}{2}pq_p^2\left(2\right)-2pH_{5(p-1)/4}^2\right. \\ &\left.-\left(-1\right)^{(p-1)/2}17pE_{p-3}\right) \pmod{p^2},\end{aligned}$$

and for $p \equiv 3 \pmod 4$,

$$\begin{aligned}&\sum_{k=(p+1)/4}^{p-1} B_{p,k}B_{p,k-(p-3)/4}H_k \\ &\equiv \left(-1\right)^{(p-3)/4}\left(-\frac{2}{p}-\frac{13}{3}-\frac{67}{9}p+2pH_{(5p-7)/4}^2\right. \\ &\left.+5p\left(-1\right)^{(p-1)/2}E_{p-3}-\frac{87}{2}pq_p^2\left(2\right)+\left(15+19p\right)q_p\left(2\right)\right) \pmod{p^2}.\end{aligned}$$

Proof. Taking $d=(p-1)/4$ in Theorem 2.9, we write

$$\begin{aligned}&\sum_{k=(p+3)/4}^{p-1} B_{p,k}B_{p,k-(p-1)/4}H_k\\ &\equiv 4\,(-1)^{(p-1)/4}\left(\frac{1}{2p}+\frac{21}{4}+\frac{85}{4}p+2pH_{(p-1)/4}^2+\frac{1}{4}\,(1+3p)\,H_{(p-1)/4}\right.\\ &\quad\left.-\frac{1}{2}p\,(p+1)\,H_{5(p-1)/4}^2-pH_{(p-5)/4,2}\right)\pmod{p^2}.\end{aligned}$$

By (8), (16) and (18), hence the first congruence in Corollary 2.11 also holds. Similarly for $d=(p-3)/4$, the proof is obtained. Then we have the proof. □

Theorem 2.12. *Let $p>3$ be prime number. For $1<d<p-1$,*

$$\begin{aligned}&\sum_{k=d+1}^{p-1} kB_{p,k}B_{p,k-d}H_k\\ &\equiv\ (-1)^d\left(\frac{4d^2}{p}-4d^2pH_{p+d-1}^2+3p-9d+10d^2p+2dp+d^2-2-4d^2pH_{d-1,2}\right.\\ &\quad\left.-2\left(d-2p+6d^2p+6dp-3d^2\right)H_d+4dp\,(3d+1)\,H_d^2\right)\pmod{p^2}.\end{aligned}$$

Proof. Consider

$$\begin{aligned}&\sum_{k=d+1}^{p-1} kB_{p,k}B_{p,k-d}H_k\\ &=\sum_{k=d+1}^{p-1}\frac{k^2}{p}\binom{2p}{p-k}\frac{k-d}{p}\binom{2p}{p-k+d}H_k\\ &=4\sum_{k=d+1}^{p-1}\frac{k^2}{p-k}\binom{2p-1}{p-k-1}\frac{k-d}{p-k+d}\binom{2p-1}{p-k+d-1}H_k.\end{aligned}$$

From (2) and (19), we get

$$\begin{aligned}&\sum_{k=d+1}^{p-1} kB_{p,k}B_{p,k-d}H_k\\ &\equiv 4\,(-1)^d\sum_{k=d+1}^{p-1}\left(k+2p-d\frac{p}{d-k}\right)(1-2pH_{p-k-1})\,(1-2pH_{p-k+d-1})\,H_k\\ &\equiv 4\,(-1)^d\sum_{k=d+1}^{p-1}\left(k+2p-d\frac{p}{d-k}\right)\left(H_k-2pH_k^2-2pH_kH_{k-d}\right)\\ &=4\,(-1)^d\sum_{k=d+1}^{p-1}\left(kH_k+2pH_k-2pkH_kH_{k-d}-2pkH_k^2-d\frac{p}{d-k}H_k\right)\pmod{p^2},\end{aligned}$$

and hence

$$\begin{aligned}
&\sum_{k=d+1}^{p-1} kB_{p,k}B_{p,k-d}H_k \\
&\equiv 4(-1)^d \left(\sum_{k=1}^{p-1} kH_k - \sum_{k=1}^{d} kH_k + 2p \sum_{k=d+1}^{p-1} H_k - 2p \sum_{k=d+1}^{p-1} kH_k^2 \right. \\
&\quad \left. -2p \sum_{k=d+1}^{p-1} kH_kH_{k-d} - dp \sum_{k=d+1}^{p-1} \frac{1}{d-k} H_k \right) \pmod{p^2}.
\end{aligned}$$

Using Lemmas 2.4, 2.5 and (11), we have

$$\begin{aligned}
&\sum_{k=d+1}^{p-1} kB_{p,k}B_{p,k-d}H_k \\
&\equiv 4(-1)^d \left(\binom{p}{2} \left(H_p - \frac{1}{2} \right) - \binom{d+1}{2} \left(H_{d+1} - \frac{1}{2} \right) \right. \\
&\quad + 2p(d - (d+1)H_d - p + 1) \\
&\quad -p \left(-2\binom{d+1}{2} H_d^2 + 2\binom{d}{2} H_d - H_d - \binom{d}{2} + d + 2 \right) \\
&\quad \left. -2p \sum_{k=d+1}^{p-1} kH_kH_{k-d} + dp \sum_{k=d+1}^{p-1} \frac{1}{k-d} H_k \right) \pmod{p^2}.
\end{aligned}$$

By (22) and Lemma 2.7, we write

$$\begin{aligned}
&\sum_{k=d+1}^{p-1} kB_{p,k}B_{p,k-d}H_k \\
&\equiv 4(-1)^d \left(\binom{p}{2} \left(H_p - \frac{1}{2} \right) - \binom{d+1}{2} \left(H_{d+1} - \frac{1}{2} \right) \right. \\
&\quad + 2p(d - (d+1)H_d - p + 1) \\
&\quad -p \left(-2\binom{d+1}{2} H_d^2 + 2\binom{d}{2} H_d - H_d - \binom{d}{2} + d + 2 \right) \\
&\quad -2p\binom{d}{2} \left(H_{p+d-1}^2 - 2H_{d-1}^2 + H_{d-1,2} + 2\left(1 - \frac{1}{p}\right) H_{d-1} - \frac{1}{p^2} - 2 \right) \\
&\quad \left. +dp \left(-H_{p+d-1}^2 + 2H_{d-1}^2 - H_{d-1,2} + \frac{2}{p} H_{d-1} + \frac{2}{d} H_d + \frac{1}{p^2} \right) \right) \pmod{p^2}.
\end{aligned}$$

From (1), the desired result is obtained. □

Corollary 2.13. *Let $p>3$ be prime number. We have*

$$\begin{aligned}&\sum_{k=(p+1)/2}^{p-1} kB_{p,k}B_{p,k-(p-1)/2}H_k\\ &\equiv (-1)^{(p-1)/2}\left(\frac{23}{2}p+\frac{1}{p}+\frac{11}{4}-pH_{3(p-1)/2}^2-(5+14p)\,q_p(2)\right.\\ &\quad\left.+\frac{13}{2}pq_p^2(2)\right)\pmod{p^2}.\end{aligned}$$

Proof. Taking $d=(p-1)/2$ in Theorem 2.12, we write

$$\begin{aligned}&\sum_{k=(p+1)/2}^{p-1} kB_{p,k}B_{p,k-(p-1)/2}H_k\\ &\equiv (-1)^{(p-1)/2}\left(\frac{1}{p}+\frac{3}{4}+\frac{p}{2}-p\,(p-1)^2\,H_{3(p-1)/2}^2+\left(\frac{5}{2}+3p\right)H_{(p-1)/2}\right.\\ &\quad\left.+pH_{(p-1)/2}^2-p\,(p-1)^2\,H_{(p-3)/2,2}\right)\pmod{p^2},\end{aligned}$$

and using (7) and (10),

$$\begin{aligned}&\sum_{k=(p+1)/2}^{p-1} kB_{p,k}B_{p,k-(p-1)/2}H_k\\ &\equiv 4\,(-1)^{(p-1)/2}\left(\frac{1}{4p}+\frac{3}{16}+\frac{9}{8}p-\frac{1}{4}\left(\frac{1}{p}-2+p\right)p^2H_{3(p-1)/2}^2\right.\\ &\quad\left.+\left(\frac{5}{8}+\frac{3}{4}p\right)\left(pq_p^2(2)-2q_p(2)\right)+pq_p^2(2)\right)\pmod{p^2}.\end{aligned}$$

By (17), we have the proof. □

Corollary 2.14. *Let $p>3$ be prime number. For $p\equiv 1 \pmod 4$,*

$$\begin{aligned}&\sum_{k=(p+3)/4}^{p-1} kB_{p,k}B_{p,k-(p-1)/4}H_k\\ &\equiv (-1)^{(p-1)/4}\left(\frac{1}{4p}+\frac{5}{16}+\frac{35}{4}p-\frac{p}{4}H_{5(p-1)/4}^2-(-1)^{(p-1)/2}\,\frac{15}{8}pE_{p-3}\right.\\ &\quad\left.-\frac{15}{16}pq_p^2(2)-\left(\frac{21}{8}+18p\right)q_p(2)\right)\pmod{p^2},\end{aligned}$$

and for $p\equiv 3 \pmod 4$,

$$\sum_{k=(p+1)/4}^{p-1} kB_{p,k}B_{p,k-(p-3)/4}H_k$$

$$\equiv (-1)^{(p-3)/4}\left(\frac{9}{4p}+\frac{85}{16}+\frac{25}{2}p-\frac{9}{4}pH^2_{(5p-7)/4}+\frac{657}{16}pq_p^2(2)\right.$$
$$\left.-\left(\frac{39}{2}p+\frac{117}{8}\right)q_p(2)-(-1)^{\frac{p-1}{2}}\frac{111}{8}pE_{p-3}\right) \pmod{p^2}.$$

Proof. Taking $d=(p-1)/4$ in Theorem 2.12, we write

$$\sum_{k=(p+3)/4}^{p-1} kB_{p,k}B_{p,k-(p-1)/4}H_k$$
$$\equiv 4(-1)^{(p-1)/4}\left(\frac{1}{16p}-\frac{3}{64}+\frac{5}{4}p-\frac{p}{16}H^2_{5(p-1)/4}+\left(\frac{1}{8}-\frac{1}{16}p\right)p^2H^2_{5(p-1)/4}\right.$$
$$\left.+\left(\frac{7}{32}+\frac{5}{4}p\right)H_{(p-1)/4}-\frac{1}{16}pH^2_{(p-1)/4}-\frac{1}{16}pH_{(p-1)/4,2}\right) \pmod{p^2}.$$

By (16) and (18), we get

$$\sum_{k=(p+3)/4}^{p-1} kB_{p,k}B_{p,k-(p-1)/4}H_k$$
$$\equiv (-1)^{(p-1)/4}\left(\frac{1}{4p}+\frac{5}{16}+\frac{35}{4}p-\frac{p}{4}H^2_{5(p-1)/4}-\frac{15}{32}pH_{(p-1)/4,2}-\frac{15}{16}pq_p^2(2)\right.$$
$$\left.-\left(\frac{21}{8}+18p\right)q_p(2)\right) \pmod{p^2}.$$

From (8), the desired result is given. Similarly for $d=(p-3)/4$, the proof is obtained. Thus, the proof is obtained. □

Similarly, the proof of Theorem 2.15 is similar to the proofs of Theorems 2.9 and 2.12.

Theorem 2.15. *Let $p>3$ be prime number. For $1<d<p-1$,*

$$\sum_{k=d+1}^{p-1} k^2B_{p,k}B_{p,k-d}H_k$$
$$\equiv 4(-1)^d\left(\frac{1}{6}+\frac{1}{36}d(4d+1)(d-1)\right.$$
$$-\frac{1}{6}d(d+1)(2d+1)H_d(1-2pH_d)-\frac{1}{9}p\left(7d+17d^3+3\right)H_d$$
$$-\frac{1}{3}dp\left(2d^2+1\right)\left(H^2_{p+d-1}-2H^2_{d-1}+H_{d-1,2}-\frac{2}{p}H_{d-1}-\frac{1}{p^2}\right)$$
$$\left.+\frac{p}{108}\left(-8d+66d^2+158d^3+33\right)\right) \pmod{p^2}.$$

We can give the results of Theorem 2.15 without proof as follows:

Corollary 2.16. *Let $p>3$ be prime number. We have*

$$\begin{aligned}&\sum_{k=(p+1)/2}^{p-1}k^2B_{p,k}B_{p,k-(p-1)/2}H_k\\ &\equiv(-1)^{(p-1)/2}\left(-\frac{521}{36}p-\frac{1}{p}-\frac{41}{12}+pH_{3(p-1)/2}^2-10pq_p^2(2)\right.\\ &\quad\left.+4\left(\frac{10}{3}p+1\right)q_p(2)\right)\pmod{p^2}.\end{aligned}$$

Corollary 2.17. *Let $p>3$ be prime number. For $p\equiv 1\pmod 4$,*

$$\begin{aligned}&\sum_{k=(p+3)/4}^{p-1}k^2B_{p,k}B_{p,k-(p-1)/4}H_k\\ &\equiv(-1)^{(p-1)/4}\left(-\frac{5647}{288}p-\frac{3}{8p}-\frac{7}{3}+\frac{3}{8}pH_{5(p-1)/4}^2-\frac{285}{32}pq_p^2(2)\right.\\ &\quad\left.+\left(\frac{77}{4}p+\frac{33}{16}\right)q_p(2)+(-1)^{(p-1)/2}\frac{35}{16}pE_{p-3}\right)\pmod{p^2},\end{aligned}$$

and for $p\equiv 3\pmod 4$,

$$\begin{aligned}&\sum_{k=(p+1)/4}^{p-1}k^2B_{p,k}B_{p,k-(p-3)/4}H_k\\ &\equiv(-1)^{(p-3)/4}\left(-\frac{17}{8p}-\frac{127}{24}-\frac{3625}{288}p+\frac{17p}{8}H_{(5p-7)/4}^2-\frac{1395}{32}pq_p^2(2)\right.\\ &\quad\left.+3\left(\frac{57}{8}p+\frac{69}{16}\right)q_p(2)+(-1)^{(p-1)/2}\frac{205}{16}pE_{p-3}\right)\pmod{p^2}.\end{aligned}$$

Acknowledgement. The author would like to thank the referee for carefully reading the paper and valuable comments to improve its presentation.

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