

UNIFORMLY CONVERGENT NUMERICAL SCHEME FOR SINGULARLY PERTURBED PARABOLIC DELAY DIFFERENTIAL EQUATIONS[†]

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ABSTRACT. In this paper, numerical treatment of singularly perturbed parabolic delay differential equations is considered. The considered problem have small delay on the spatial variable of the reaction term. To treat the delay term, Taylor series approximation is applied. The resulting singularly perturbed parabolic PDEs is solved using Crank Nicolson method in temporal direction with non-standard finite difference method in spatial direction. A detail stability and convergence analysis of the scheme is given. We proved the uniform convergence of the scheme with order of convergence $O(N^{-1} + (\Delta t)^2)$, where N is the number of mesh points in spatial discretization and Δt is mesh length in temporal discretization. Two test examples are used to validate the theoretical results of the scheme.

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1. Introduction

In mathematical biology, there are a lot of models in which small values of the coefficient of diffusive term play an important role to capture the behavior of the physical phenomena. The small values of the coefficient of diffusive term in the differential equations exists in many real-life applications. For example in hemoglobin molecules and oxygen in blood have diffusion coefficient of the order 10^{-7} and $10^{-5}cm^2/s$, respectively [10]. The diffusion coefficient of the order $10^{-9} - 10^{-11}cm^2/s$ are discussed in models in the book “Mathematical Biology I” [11]. In [20, 21], various mathematical models exist for the determination of the behavior of a neuron to random synaptic inputs. The feasibility of recording single-neuron movement induces the development of accurate mathematical

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models of neuronal variability. The modelling of spiking movement of neuron to any level of exactness, one has to consider special features of each kind of neuron and its input processes. The stochastic movement of a neuron is first modelled by Stein [17] in 1965. After two years, the author generalize the model to handle distribution of past synaptic potential amplitudes [18].

Musila and Lansky [12] generalized the Stein's model and developed a model in terms of singularly perturbed parabolic delay differential equations to treat the time evolution trajectories of the membrane potential

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + \left(\mu_D - \frac{x}{\tau}\right) \frac{\partial u}{\partial x} + \lambda_s u(x+a_s, t) + \omega_s u(x+i_s, t) - (\lambda_s + \omega_s) u(x, t), \quad (1)$$

where μ_D and σ are diffusion moments characterizing the influence of dendrites synapses on the cell excitation. The excitation input contributes to the membrane potential with an amplitude a_s and intensity λ_s and the inhibition input contributes with an amplitude i_s and intensity ω_s . The first derivative term is because of the exponential decay between two consecutive jumps caused by the input processes. The membrane potential decays exponentially to the resting level with time constant τ .

The model in (1) is a form of singularly perturbed parabolic delay differential equation, one can hardly derive its exact solution. Thus, to find the solution to this model problem, one has to develop efficient numerical scheme. Numerical methods developed for solving regular problems turn out inapplicable for singular perturbation problem as the solution profile in this case depends on the value of the singular perturbation parameter. It is well-known that the standard numerical method in FDM, FEM and collocation methods on uniform meshes fail to converge uniformly with respect to the singular perturbation parameter [6],[24]. The efficiency of a numerical method is determined by its accuracy, simplicity in computing the discrete solution and its sensitivity to the parameters of the given problem. It is desirable to develop numerical methods which converges uniformly. This motivates the researcher for developing simple, easy to use, parameter uniformly convergent numerical methods for solving singularly perturbed parabolic delay differential equations.

In recent years, scholars have devoted to the study of numerical solution of singularly perturbed parabolic delay differential equations having delay on the spatial variable. In series of papers [1, 2, 3], Bansal and Sharma have developed numerical methods which converges uniformly for the problem with general shift arguments (where the delays are large). Kumar and Kadalbajoo [8] used implicit Euler in temporal direction and B-spline collocation method on a piece-wise uniform mesh in the spatial direction. Ramesh and Kadalbajoo [13] used implicit Euler in temporal direction and upwind and midpoint upwind finite difference method on a piecewise uniform mesh in the spatial direction. Rao and Chakravarthy [15] considered implicit Euler in temporal direction and exponentially fitted operator FDM is applied in the spatial direction. Woldaregay and Duressa [22] applied numerical schemes based on method of line with

non-standard finite difference method for the spatial discretization with Runge-Kutta method on the temporal direction.

Notations. Throughout this paper N and M are denoted for the number of mesh intervals in space and time direction respectively. The symbol C is denoted for positive constant independent of c_ε , N and Δt . The norm $\|\cdot\|$ denotes the maximum norm. We used the phrase ‘parameter uniform’ and ‘ ε -uniform’ as the same meaning.

2. Statement of the problem

Singularly perturbed parabolic delay differential equations of convection-diffusion-reaction type having delay on the spatial variable of the reaction term is given as

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + a(x) \frac{\partial u}{\partial x} + \alpha(x)u(x - \delta, t) + \beta(x)u(x, t) = f(x, t), & (x, t) \in D, \\ u(x, 0) = u_0(x), x \in D_0 = \{(x, 0) : x \in \bar{\Omega}\}, \\ u(x, t) = \phi(x, t), (x, t) \in D_L = \{(x, t) : -\delta \leq x \leq 0, t \in \Lambda\}, \\ u(1, t) = \psi(1, t), (x, t) \in D_R = \{(x, t) : x = 1, t \in \Lambda\}, \end{cases} \tag{2}$$

where $D = \Omega \times \Lambda = (0, 1) \times (0, T]$ with smooth boundary $\partial D = \bar{D} \setminus D = D_L \cup D_0 \cup D_R$ for some fixed positive number T , ε is a singular perturbation parameter satisfying $0 < \varepsilon \leq 1$ and δ is delay parameter assumed to be sufficiently small as order of $o(\varepsilon)$.

The coefficient functions $a(x), \alpha(x), \beta(x)$, source function $f(x, t)$ and the initial and boundary condition functions $u_0(x), \phi(x, t), \psi(1, t)$ are assumed to be sufficiently smooth, bounded and independent of ε . The coefficient functions α and β are assumed to satisfy

$$\alpha(x) + \beta(x) \geq \zeta > 0, \quad x \in \bar{\Omega}, \tag{3}$$

where ζ is constant. The condition in (3) ensure that the solution of (2) exhibit boundary layer in the neighbourhood of D_L or D_R depending on the sign of the convective term. When the delay parameter is zero (i.e., $\delta = 0$) the equation reduces to a singularly perturbed parabolic PDEs, with small ε exhibits layers depending upon the value of the coefficient function $a(x)$. When $a(x) < 0$ a regular boundary layer exist in the neighbourhood of D_L and in case $a(x) > 0$ corresponds to existence of a the layer near D_R , in addition to that if $a(x)$ change sign then inner layer exist on the solution of the problem [22]. The layer is maintained for $\delta \neq 0$ but sufficiently small.

2.1. Estimate for the delay term. For $\delta < \varepsilon$, using Taylor’s series approximation for the term containing delay is appropriate [19]. So, we approximate $u(x - \delta, t)$ as

$$u(x - \delta, t) \approx u(x, t) - \delta u_x(x, t) + (\delta^2/2)u_{xx}(x, t) + O(\delta^3). \tag{4}$$

Now substituting (4) into (2), we obtain

$$\begin{cases} \frac{\partial u}{\partial t} - c_\varepsilon(x) \frac{\partial^2 u}{\partial x^2} + p(x) \frac{\partial u}{\partial x} + q(x)u(x, t) = f(x, t), & (x, t) \in D, \\ u(x, 0) = u_0(x), & x \in \bar{\Omega}, \\ u(0, t) = \phi(0, t), & t \in \bar{\Lambda}, \\ u(1, t) = \psi(1, t), & t \in \bar{\Lambda}, \end{cases} \quad (5)$$

where $c_\varepsilon(x) = \varepsilon^2 - (\delta^2/2)\alpha(x)$, $p(x) = a(x) - \delta\alpha(x)$ and $q(x) = \alpha(x) + \beta(x)$. For small values of delay parameter, (2) and (5) are asymptotically equivalent, because the difference between the two equations is $O(\delta^3)$. Now, we assume $0 < c_\varepsilon(x) \leq \varepsilon^2 - \delta^2 M_1 = c_\varepsilon$ where $2M_1$ is the lower bounds for $\alpha(x)$. We also assume $p(x) = a(x) - \delta\alpha(x) \geq p^* > 0$, which implies that existence of boundary layer near the right side of the rectangular domain D . The other case, when $p(x) = a(x) - \delta\alpha(x) \leq p^* < 0$, leads to the existence of the boundary layer of thickness $O(c_\varepsilon)$ near the left side of the rectangular domain D and we can treat it in similar manner.

2.2. Some standard results of the analytical solution. The existence and uniqueness of the solution of (5) can be established by assuming that the data are Holder continuous and imposing appropriate compatibility at the corner points $(0,0)$, $(1,0)$.

Lemma 2.1. [16] *Let $u_0(x) \in C^2[0,1]$, $\phi \in C^1[0,T]$ and $\psi \in C^1[0,T]$ by imposing the compatibility conditions $u_0(0) = \phi(0,0)$, $u_0(1) = \psi(1,0)$ and*

$$\begin{aligned} \frac{\partial \phi(0,0)}{\partial t} - c_\varepsilon \frac{\partial^2 u_0(0)}{\partial x^2} + p(0) \frac{\partial u_0(0)}{\partial x} + q(0)u_0(0) &= f(0,0), \\ \frac{\partial \psi(1,0)}{\partial t} - c_\varepsilon \frac{\partial^2 u_0(1)}{\partial x^2} + p(1) \frac{\partial u_0(1)}{\partial x} + q(1)u_0(1) &= f(1,0), \end{aligned}$$

so that the data matches at the two corners $(0,0)$ and $(1,0)$. Let p, q and f be continuous on D . Then, (5) has unique solution $u \in C^2(D)$. In particular when the compatibility conditions are not satisfied, a unique classical solution will still exist but will not be differentiable on all of ∂D .

In the considered case, the boundary layer occurs near $x = 1$. By using compatibility conditions, we deduce that there exist a constant C independent of c_ε such that for all $(x, t) \in \bar{D}$, we have the following conditions that guarantee the existence of a constant C independent of c_ε such that

$$|u(x, t) - u_0(x)| \leq Ct, \text{ and } |u(x, t) - \phi(0, t)| \leq C(1 - x). \quad (6)$$

For the detail interested reader can refer [4], [16].

Remark 2.1. Note that there does not exist a constant C independent of c_ε such that $|u(x, t) - u(1, t)| = |u(x, t) - \psi(1, t)| \leq Cx$, because a boundary layer will occur near $x = 1$.

The problem obtained by setting $c_\varepsilon = 0$ in (5) is called reduced problem and given as

$$\begin{cases} \frac{\partial u^0}{\partial t} + p(x)\frac{\partial u^0}{\partial x} + q(x)u^0(x,t) = f(x,t), & (x,t) \in D, \\ u^0(x,0) = u_0(x), & x \in \bar{\Omega}, \\ u^0(0,t) = \phi_0(t), & t \in \bar{\Lambda}. \end{cases} \quad (7)$$

It is a first order hyperbolic PDE with initial data given along sides $t = 0$ and $x = 0$ of the domain \bar{D} . For small values of c_ε the solution $u(x,t)$ of the problem in (5) is very close to the solution $u^0(x,t)$ of (7).

In order to show on the bounds of the solutions $u(x,t)$ of (5), we assume, without loss of generality the initial condition to be zero [4]. Since $u_0(x)$ is sufficiently smooth and using the property of norm, we can prove the following lemma.

Lemma 2.2. *The bound on the solution $u(x,t)$ of (5) is given by*

$$|u(x,t)| \leq C, \quad (x,t) \in \bar{D}. \quad (8)$$

Proof. From the inequality $|u(x,t) - u(x,0)| = |u(x,t) - u_0(x)| \leq Ct$, we have $|u(x,t)| - |u_0(x)| \leq |u(x,t) - u_0(x)| \leq Ct$. Implies that $|u(x,t)| \leq Ct + |u_0(x)|$. Since $t \in [0, T]$ and $u_0(x)$ is bounded it implies $|u(x,t)| \leq C$. \square

Let L be denoted for the differential operator $Lu(x,t) = \frac{\partial u}{\partial t} - c_\varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + p(x)\frac{\partial u(x,t)}{\partial x} + q(x)u(x,t)$ for the problem in (5).

Lemma 2.3. (The maximum principle.) *Let z be a sufficiently smooth function defined on D which satisfies $z(x,t) \geq 0$, $(x,t) \in \partial D$. Then, $Lz(x,t) \geq 0$, $(x,t) \in D$ implies that $z(x,t) \geq 0$, $\forall (x,t) \in \bar{D}$.*

Proof. Let (x^*, t^*) be such that $z(x^*, t^*) = \min_{(x,t) \in \bar{D}} z(x,t)$, and suppose that $z(x^*, t^*) < 0$. It is clear that $z(x^*, t^*) \notin \partial D$. Since $z(x^*, t^*) = \min_{(x,t) \in \bar{D}} z(x,t)$ which implies $z_x(x^*, t^*) = 0$, $z_t(x^*, t^*) = 0$ and $z_{xx}(x^*, t^*) \geq 0$ and implies that $Lz(x^*, t^*) < 0$ which is contradiction to the assumption that made above $Lz(x^*, t^*) \geq 0$, $(x,t) \in D$. Therefore, $z(x,t) \geq 0$, $\forall (x,t) \in D$. \square

Lemma 2.4. (Uniform stability estimate.) *Let $u(x,t)$ be the solution of the continuous problem in (5). Then, it satisfies the bound*

$$|u(x,t)| \leq \frac{\|f\|}{\zeta} + \max|u(x,t)|_{\partial D}, \quad (9)$$

where $|u|_{\partial D}$ denoted for restriction of $|u|$ on ∂D .

Proof. Let us define a barrier functions ϑ^\pm as $\vartheta^\pm(x,t) = \frac{\|f\|}{\zeta} + \max|u|_{\partial D} \pm u(x,t)$. At the initial value we obtain, $\vartheta^\pm(x,0) = \frac{\|f\|}{\zeta} + \max|u(x,t)|_{\partial D} \pm u(x,0) \geq 0$.

On the boundary lines, we have

$$\begin{aligned}\vartheta^\pm(0, t) &= \frac{\|f\|}{\zeta} + \max|u(x, t)|_{\partial D} \pm u(0, t) \geq 0, \\ \vartheta^\pm(1, t) &= \frac{\|f\|}{\zeta} + \max|u(x, t)|_{\partial D} \pm u(1, t) \geq 0.\end{aligned}$$

For $(x, t) \in D$, we have

$$\begin{aligned}L\vartheta^\pm(x, t) &= \vartheta_t^\pm(x, t) - c_\varepsilon \vartheta_{xx}^\pm(x, t) + p(x)\vartheta_x^\pm(x, t) + q(x)\vartheta^\pm(x, t) \\ &= \frac{\partial}{\partial t} \left(\frac{\|f\|}{\zeta} + \max|u(x, t)|_{\partial D} \pm u(x, t) \right) - c_\varepsilon \frac{\partial^2}{\partial x^2} \left(\frac{\|f\|}{\zeta} + \max|u(x, t)|_{\partial D} \right. \\ &\quad \left. \pm u(x, t) \right) + p(x) \frac{\partial}{\partial x} \left(\frac{\|f\|}{\zeta} + \max|u(x, t)|_{\partial D} \pm u(x, t) \right) \\ &\quad + q(x) \left(\frac{\|f\|}{\zeta} + \max|u(x, t)|_{\partial D} \pm u(x, t) \right) \\ &= q(x) \left(\frac{\|f\|}{\zeta} + \max|u(x, t)|_{\partial D} \right) \pm Lu(x, t) \geq 0, \text{ since } q(x) \geq \zeta > 0.\end{aligned}$$

Which implies $L\vartheta^\pm(x, t) \geq 0$, $(x, t) \in D$. Hence, using the maximum principle stated in Lemma 2.3 we get, $\vartheta^\pm(x, t) \geq 0$, $\forall (x, t) \in \bar{D}$. Hence, the required bound is obtained. \square

Lemma 2.5. *The bound on the derivative of the solution $u(x, t)$ of the problem in (5) with respect to x is given by*

$$\left| \frac{\partial^i u(x, t)}{\partial x^i} \right| \leq C(1 + c_\varepsilon^{-i} e^{-p^*(1-x)/c_\varepsilon}), \quad (x, t) \in \bar{D}, \quad 0 \leq i \leq 4. \quad (10)$$

Proof. Refer in [5], [7]. \square

3. Numerical scheme formulation

3.1. Discretization in temporal direction. In this section, first we semi-discretize the temporal direction of the problem in (5) using the Crank Nicolson method. While doing this, we obtain a system of singularly perturbed boundary value problems. We sub-divided the temporal direction domain $[0, T]$ into M equal subintervals using grid points $t_0 = 0$, $t_j = j\Delta t$, $j = 0, 1, 2, \dots, M$, where $\Delta t = T/M$. Let $U_{j+1}(x)$ denotes the approximation of $u(x, t_{j+1})$ at the $(j+1)$ th time level grid point. Using the Crank Nicolson method on (5), we obtain

$$\begin{aligned}\frac{U_{j+1}(x) - U_j(x)}{\Delta t} - \frac{c_\varepsilon}{2} \left(\frac{d^2}{dx^2} U_{j+1}(x) + \frac{d^2}{dx^2} U_j(x) \right) + \frac{p(x)}{2} \left(\frac{d}{dx} U_{j+1}(x) + \frac{d}{dx} U_j(x) \right) \\ + \frac{q(x)}{2} (U_{j+1}(x) + U_j(x)) = \frac{f(x, t_{j+1}) + f(x, t_j)}{2}.\end{aligned} \quad (11)$$

Including the initial and the boundary conditions, we rewrite (11) in operator form as

$$\begin{cases} L^{\Delta t}U_{j+1}(x) = g(x, t_{j+1}), & x \in \bar{\Omega}, \quad j = 1, 2, \dots, M - 1, \\ U_0(x) = u_0(x), & x \in \Omega, \\ U_{j+1}(0) = \phi(t_{j+1}), \\ U_{j+1}(1) = \psi(t_{j+1}), \end{cases} \tag{12}$$

where $L^{\Delta t}U_{j+1}(x) = -\frac{c_\varepsilon}{2} \frac{d^2}{dx^2}U_{j+1}(x) + \frac{p(x)}{2} \frac{d}{dx}U_{j+1}(x) + d(x)U_{j+1}(x)$ and

$$g(x, t_{j+1}) = \frac{c_\varepsilon}{2} \frac{d^2}{dx^2}U_j(x) - \frac{p(x)}{2} \frac{d}{dx}U_j(x) - r(x)U_j(x) + \frac{f(x, t_{j+1}) + f(x, t_j)}{2}$$

for $d(x) = \frac{1}{\Delta t} + \frac{q(x)}{2}$ and $r(x) = \frac{q(x)}{2} - \frac{1}{\Delta t}$.

Lemma 3.1. (Semi-discrete maximum principle.) *Let z_{j+1} be a sufficiently smooth function on $\bar{\Omega}$. If $z_{j+1}(0) \geq 0$, $z_{j+1}(1) \geq 0$ and $L^{\Delta t}z_{j+1}(x) \geq 0$, $\forall x \in \Omega$, then $z_{j+1}(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Proof. Let x^* be such that $z_{j+1}(x^*) = \min_{x \in \bar{\Omega}} z_{j+1}(x)$ and suppose that $z(x^*) < 0$. From the above assumption it is clear that $x^* \notin \{0, 1\}$. Which implies that $x^* \in (0, 1)$. Since $z_{j+1}(x^*) = \min_{x \in \bar{\Omega}} z_{j+1}(x)$, using the property in calculus, we have $(z_{j+1})_{xx}(x^*) \geq 0$ and $(z_{j+1})_x(x^*) = 0$ then we obtain $L^{\Delta t}z_{j+1}(x^*) < 0$ which is contradiction to $L^{\Delta t}z_{j+1}(x^*) \geq 0$, $\forall x \in \Omega$. Therefore, we conclude $z_{j+1}(x) \geq 0$, $\forall x \in \bar{\Omega}$. \square

Next, let us analyse the error for the discretization made in temporal direction. Let us denote the local error at each time step as $e_{j+1}(x) := u(x, t_{j+1}) - U_{j+1}(x)$, $j = 0, 1, 2, \dots, M - 1$.

Lemma 3.2. *Suppose that*

$$\left| \frac{\partial^k}{\partial t^k} u(x, t) \right| \leq C, \quad (x, t) \in \bar{\Omega} \times \Lambda, \quad k = 0, 1, 2. \tag{13}$$

The local error in the temporal discretization is bounded as

$$\|e_{j+1}\| \leq C_1(\Delta t)^3. \tag{14}$$

Proof. Using Taylor's series to $u(x, t_j)$ and $u(x, t_{j+1})$, we obtain

$$\begin{aligned} u(x, t_j) &= u(x, t_{j+1/2}) - \frac{\Delta t}{2} u_t(x, t_{j+1/2}) + \frac{(\Delta t)^2}{8} u_{tt}(x, t_{j+1/2}) + O((\Delta t)^3), \\ u(x, t_{j+1}) &= u(x, t_{j+1/2}) + \frac{\Delta t}{2} u_t(x, t_{j+1/2}) + \frac{(\Delta t)^2}{8} u_{tt}(x, t_{j+1/2}) + O((\Delta t)^3). \end{aligned} \tag{15}$$

Substituting (15) into (5) gives

$$\begin{aligned} \frac{u(x, t_{j+1}) - u(x, t_j)}{\Delta t} &= u_t(x, t_{j+1/2}) + O((\Delta t)^2) \\ &= c_\varepsilon (u(x, t_{j+1/2}))_{xx} - p(x) (u(x, t_{j+1/2}))_x - q(x) u(x, t_{j+1/2}) \\ &\quad + f(x, t_{j+1/2}) + O((\Delta t)^2) \end{aligned}$$

where $u(x, t_{j+1/2}) = \frac{1}{2}[u(x, t_{j+1}) + u(x, t_j)] + O((\Delta t)^2)$ and $f(x, t_{j+1/2}) = \frac{1}{2}[f(x, t_{j+1}) + f(x, t_j)] + O((\Delta t)^2)$. Since error satisfies the semi-discrete difference operator, we obtain

$$L^{\Delta t} e_{j+1}(x) = O((\Delta t)^3), \quad e_{j+1}(0) = 0 = e_{j+1}(1).$$

Hence, by applying the semi-discrete maximum principle, we obtain

$$\|e_{j+1}\| \leq C_1(\Delta t)^3. \quad (16)$$

□

Next, we need to show the bound for the global error of the temporal discretization. Let us denote TE_{j+1} be the global error up to the $(j+1)$ th time step.

Lemma 3.3. *The global discretization error at t_{j+1} is given by*

$$\|TE_{j+1}\| \leq C(\Delta t)^2, \quad \forall j = 1, 2, \dots, M-1. \quad (17)$$

Proof. Using the local error up to the $(j+1)$ th time step given in Lemma 3.2, we obtain the global error bound at the $(j+1)$ th time step as

$$\begin{aligned} \|TE_{j+1}\| &= \left\| \sum_{l=1}^{j+1} e_l \right\| \\ &\leq \|e_1\| + \|e_2\| + \dots + \|e_{j+1}\| \\ &\leq C_1 T (\Delta t)^2, \quad \text{since } (j+1)\Delta t \leq T \\ &= C(\Delta t)^2, \quad C_1 T =: C, \end{aligned}$$

where C is constant independent of c_ε and Δt . □

Next, we set a bound for the derivatives of solution of (12).

Lemma 3.4. *The solution $U^{j+1}(x)$ of the boundary value problems in (12) satisfies the bound*

$$\left| \frac{d^k U_{j+1}(x)}{dx^k} \right| \leq C(1 + c_\varepsilon^{-k} e^{-\frac{p^*(1-x)}{c_\varepsilon}}), \quad x \in \bar{\Omega}, \quad 0 \leq k \leq 4, \quad (18)$$

for $j = 0, 1, 2, \dots, M-1$.

Proof. See in [5]. □

3.2. Discretization in spatial direction. For the spatial direction discretization, we apply a non-standard finite difference method, which depends on the knowledge of the corresponding exact finite difference method, which is stated well in [9]. For the problem of the form in (12), in order to construct exact finite difference scheme, we use the procedures developed in Michens [9]. For the problem in (12) we consider the constant coefficient sub-equations of the form

$$-c_\varepsilon \frac{d^2 U_{j+1}(x)}{dx^2} + p^* \frac{dU_{j+1}(x)}{dx} + 2\theta U_{j+1}(x) = 0, \quad (19)$$

$$-c_\varepsilon \frac{d^2 U_{j+1}(x)}{dx^2} + p^* \frac{dU_{j+1}(x)}{dx} = 0, \tag{20}$$

where $p(x) \geq p^*$ and $d(x) \geq 2\theta$. Thus, (19) has two independent solutions namely $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$, where

$$\lambda_{1,2} = \frac{-p^* \pm \sqrt{(p^*)^2 + 8c_\varepsilon \theta}}{-2c_\varepsilon}. \tag{21}$$

We discretize the spatial domain $[0, 1]$, using uniform mesh length $\Delta x = h$ such that the grid points are given by $\Omega^N = \{x_i = x_0 + ih, i = 1, 2, \dots, N, x_0 = 0, x_N = 1, h = \frac{1}{N}\}$, where N is the number of mesh intervals in spatial discretization. We denoted the approximate solution of $U_{j+1}(x)$ at mesh point x_i by $U_{i,j+1}$. Our objective is to calculate a difference equation which has the same general solution as the differential equation in (19) has at the mesh point x_i which is given by $U_{i,j+1} = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i)$. Using the theory of difference equations for second order linear difference equations in [9], we obtain

$$\begin{vmatrix} U_{i-1,j+1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ U_{i,j+1} & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ U_{i+1,j+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{vmatrix} = 0, \tag{22}$$

Simplifying (22) gives

$$\{\exp(\lambda_2 h) - \exp(\lambda_1 h)\}U_{i-1,j+1} - \{\exp((\lambda_2 - \lambda_1)h) - \exp((\lambda_1 - \lambda_2)h)\}U_{i,j+1} + \{-\exp(-\lambda_2 h) + \exp(-\lambda_1 h)\}U_{i+1,j+1} = 0,$$

substituting the values of λ_1 and λ_2 , we obtain

$$\exp\left(\frac{p^* h}{2c_\varepsilon}\right)U_{i-1,j+1} - 2 \cosh\left(\frac{h\sqrt{(p^*)^2 + 8c_\varepsilon \theta}}{2c_\varepsilon}\right)U_{i,j+1} + \exp\left(-\frac{p^* h}{2c_\varepsilon}\right)U_{i+1,j+1} = 0, \tag{23}$$

is an exact difference scheme for (19). For $c_\varepsilon \rightarrow 0$, we use the approximation $\frac{h\sqrt{(p^*)^2 + 8c_\varepsilon \theta}}{2c_\varepsilon} \approx \frac{p^* h}{2c_\varepsilon}$ in (23). Hence, multiplying both sides of (23) by $\exp\left(\frac{p^* h}{2c_\varepsilon}\right)$ and simplifying gives

$$\exp\left(\frac{p^* h}{c_\varepsilon}\right)U_{i-1,j+1} - \left[\exp\left(\frac{p^* h}{c_\varepsilon}\right) + \exp\left(\frac{p^* h}{2c_\varepsilon} - \frac{p^* h}{2c_\varepsilon}\right)\right]U_{i,j+1} + U_{i+1,j+1} = 0. \tag{24}$$

After few arithmetic adjustment, we obtain

$$-c_\varepsilon \frac{U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}}{\frac{hc_\varepsilon}{p^*}(\exp\left(\frac{p^* h}{c_\varepsilon}\right) - 1)} + p^* \frac{U_{i,j+1} - U_{i-1,j+1}}{h} = 0. \tag{25}$$

From (25), the denominator function for the approximation of second derivative is $\gamma^2 = \frac{hc_\varepsilon}{p^*}(\exp\left(\frac{hp^*}{c_\varepsilon}\right) - 1)$. Generalizing this denominator function for the variable coefficient problem, we write it as

$$\gamma_i^2 = \frac{hc_\varepsilon}{p(x_i)}\left(\exp\left(\frac{hp(x_i)}{c_\varepsilon}\right) - 1\right). \tag{26}$$

Using (26) into (12), we obtain the non-standard finite difference schemes as

$$\begin{cases} L^{h,\Delta t}U_{i,j+1} = g(x_i, t_{j+1}), & i = 1, 2, \dots, N-1, \quad j = 0, 1, \dots, M-1, \\ U_{0,j+1} = \phi(0, t_{j+1}), \\ U_{N,j+1} = \psi(1, t_{j+1}), \\ U_{i,0} = u_0(x_i), \end{cases} \quad (27)$$

where $L^{h,\Delta t}U_{i,j+1} = -\frac{c_\varepsilon}{2} \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{\gamma_i^2} + \frac{p(x_i)}{2} \frac{U_{i,j+1} - U_{i-1,j+1}}{h} + d(x_i)U_{i,j+1}$ and $g(x_i, t_{j+1}) = \frac{f(x_i, t_{j+1}) + f(x_i, t_j)}{2} + \frac{c_\varepsilon}{2} \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\gamma_i^2} - \frac{p(x_i)}{2} \frac{U_{i,j} - U_{i-1,j}}{h} - r(x_i)U_{i,j}$ and γ_i^2 is defined in (26) and it is a function of $c_\varepsilon, p_i = p(x_i)$ and h .

3.3. Uniform convergence analysis. In this section, we want to show that the scheme in (27) satisfies the discrete maximum principle, uniform stability estimates and uniform convergence.

Lemma 3.5. (Discrete maximum principle.) *Let $U_{i,j+1}$ be any mesh function satisfying $U_{0,j+1} \geq 0$, $U_{N,j+1} \geq 0$. Then, $L^{h,\Delta t}U_{i,j+1} \geq 0, \forall i = 1, 2, \dots, N-1$ implies that $U_{i,j+1} \geq 0, \forall i = 0, 1, \dots, N$.*

Proof. Suppose there exist $k \in \{0, 1, \dots, N\}$ such that $U_{k,j+1} = \min_{0 \leq i \leq N} U_{i,j+1}$. Suppose that $U_{k,j+1} < 0$ which implies $k \neq \{0, N\}$. Also we assume that $U_{k+1,j+1} - U_{k,j+1} > 0$ and $U_{k,j+1} - U_{k-1,j+1} < 0$. Using the assumptions made above, we obtain $L^{h,\Delta t}U_{k,j+1} < 0$ for $k = 1, 2, 3, \dots, N-1$. Thus, the supposition $U_{i,j+1} < 0$, for $i = 0, 1, \dots, N$ is wrong. Hence, we obtain $U_{i,j+1} \geq 0, \forall i = 0, 1, \dots, N$. \square

Using this discrete maximum principle, we prove the discrete scheme in (27) satisfies the uniform stability result.

Lemma 3.6. (Discrete uniform stability estimate.) *The solution $U_{i,j+1}$ of the discrete scheme in (27) satisfy the bound*

$$|U_{i,j+1}| \leq \frac{\|L^{h,\Delta t}U\|}{\zeta^*} + \max|u(x_i, t_{j+1})|_{\partial D}, \quad (28)$$

for $d(x) \geq \zeta^*$.

Proof. Let $p = \frac{\|L^{h,\Delta t}U_{i,j+1}\|}{\zeta^*} + \max|u(x_i, t_{j+1})|_{\partial D}$ and we define the barrier functions $\vartheta_{i,j+1}^\pm$ by $\vartheta_{i,j+1}^\pm = p \pm U_{i,j+1}$. At the boundary points, we obtain

$$\begin{aligned} \vartheta_{0,j+1}^\pm &= p \pm U_{0,j+1} = p \pm \phi(0, t_{j+1}) \geq 0, \\ \vartheta_{N,j+1}^\pm &= p \pm U_{N,j+1} = p \pm \psi(1, t_{j+1}) \geq 0. \end{aligned}$$

On the discretized spatial domain $x_i, i = 1, 2, \dots, N - 1$, we obtain

$$\begin{aligned} L^{h,\Delta t} \vartheta_{i,j+1}^\pm &= -c_\varepsilon \left(\frac{p \pm U_{i+1,j+1} - 2(p \pm U_{i,j+1}) + p \pm U_{i-1,j+1}}{2\gamma_i^2} \right) \\ &\quad + p(x_i) \left(\frac{p \pm U_{i,j+1} - p \pm U_{i-1,j+1}}{2h} \right) + d(x_i)(p \pm U_{i,j+1}) \\ &= d(x_i)p \pm L^{h,\Delta t} U_{i,j+1} \\ &= d(x_i) \left(\frac{\|L^{h,\Delta t} U\|}{\zeta^*} + \max|u(x_i, t_{j+1})|_{\partial D} \right) \pm g(x_i, t_{j+1}) \\ &\geq 0, \text{ since } d(x_i) \geq \zeta^* > 0. \end{aligned}$$

Using the discrete maximum principle, we obtain $\vartheta_{i,j+1}^\pm \geq 0, \forall x_i \in \bar{\Omega}^N$. Hence, the required bound is obtained. \square

Now, let us analyse the convergence of the spatial discretization. We proved above the discrete operator $L^{h,\Delta t}$ satisfy the maximum principle and the uniform stability estimate. Let us define the forward and backward finite differences operators in spatial discretization as

$$D^+ V_{j+1}(x_i) = \frac{V_{j+1}(x_{i+1}) - V_{j+1}(x_i)}{h}, \quad D^- V_{j+1}(x_i) = \frac{V_{j+1}(x_i) - V_{j+1}(x_{i-1})}{h}$$

respectively and the second order finite difference operator as

$$D^+ D^- V_{j+1}(x_i) = \frac{D^+ V_{j+1}(x_i) - D^- V_{j+1}(x_i)}{h}.$$

Theorem 3.7. *Let the coefficients functions p, d and the function g in (12) be sufficiently smooth so that $U_{j+1}(x) \in C^4([0, 1])$. Then, the discrete solution $U_{i,j+1}$ of the problem in (12) satisfies*

$$|L^{h,\Delta t}(U_{j+1}(x_i) - U_{i,j+1})| \leq Ch \left(1 + \sup_{x_i \in (0,1)} \frac{\exp(-p^*(1-x_i)/c_\varepsilon)}{c_\varepsilon^3} \right). \tag{29}$$

Proof. Considering the difference of the exact and the approximate solutions in discrete operators, we obtain

$$\begin{aligned} |L^{h,\Delta t}(U_{j+1}(x_i) - U_{i,j+1})| &= |L^{h,\Delta t} U_{j+1}(x_i) - L^{h,\Delta t} U_{i,j+1}| \\ &\leq C \left| -c_\varepsilon \left(\frac{d^2}{dx^2} - \frac{D_x^+ D_x^- h^2}{\gamma_i^2} \right) U_{j+1}(x_i) + p(x_i) \left(\frac{d}{dx} - D_x^- \right) U_{j+1}(x_i) \right| \\ &\leq C c_\varepsilon \left| \left(\frac{d^2}{dx^2} - D_x^+ D_x^- \right) U_{j+1}(x_i) \right| + C c_\varepsilon \left| \left(\frac{h^2}{\gamma_i^2} - 1 \right) D_x^+ D_x^- U_{j+1}(x_i) \right| \\ &\quad + Ch \left| \frac{d^2}{dx^2} U_{j+1}(x_i) \right| \\ &\leq C c_\varepsilon h^2 \left| \frac{d^4}{dx^4} U_{j+1}(x_i) \right| + Ch \left| \frac{d^2}{dx^2} U_{j+1}(x_i) \right| \end{aligned}$$

In the above expression, we used the estimate $c_\varepsilon \left| \frac{h^2}{\gamma_i^2} - 1 \right| \leq Ch$ is depending on the behaviour of denominator function used in non-standard finite difference method. To make it clear let us define $\rho = p(x_i)h/c_\varepsilon$, $\rho \in (0, \infty)$. Then, using the expression for γ_i^2 , we write

$$c_\varepsilon \left| \frac{h^2}{\gamma_i^2} - 1 \right| = p(x_i)h \left| \frac{1}{\exp(\rho) - 1} - \frac{1}{\rho} \right| =: p(x_i)hQ(\rho). \quad (30)$$

Next, we set a bound for $Q(\rho)$ which is given by $Q(\rho) = \frac{\exp(\rho) - 1 - \rho}{\rho(\exp(\rho) - 1)}$ and gives the following bounds

$$\lim_{\rho \rightarrow 0} Q(\rho) = \frac{1}{2}, \quad \lim_{\rho \rightarrow \infty} Q(\rho) = 0. \quad (31)$$

Therefore, $Q(\rho) \leq C_2$, $\forall \rho \in (0, \infty)$. Hence, from (30) and (31) the estimate $c_\varepsilon \left| \frac{h^2}{\gamma_i^2} - 1 \right| \leq Ch$. So, the error bound becomes

$$|L^{h,\Delta t}(U_{j+1}(x_i) - U_{i,j+1})| \leq C_3 c_\varepsilon h^2 \left| \frac{d^4}{dx^4} U_{j+1}(x_i) \right| + C_3 h \left| \frac{d^2}{dx^2} U_{j+1}(x_i) \right|. \quad (32)$$

Using the bound of the derivatives of the solution in Lemma 3.4 into (32) gives

$$\begin{aligned} |L^{h,\Delta t}(U_{j+1}(x_i) - U_{i,j+1})| &\leq C_4 c_\varepsilon h^2 \left| 1 + c_\varepsilon^{-4} \exp\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right) \right| \\ &\quad + C_4 h \left| 1 + c_\varepsilon^{-2} \exp\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right) \right| \\ &\leq C_4 h^2 \left| c_\varepsilon + c_\varepsilon^{-3} \exp\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right) \right| \\ &\quad + C_4 h \left| 1 + c_\varepsilon^{-2} \exp\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right) \right| \\ &\leq Ch \left(1 + \sup_{x_i \in (0,1)} \frac{\exp\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}{c_\varepsilon^3} \right), \text{ since, } c_\varepsilon^3 \leq c_\varepsilon^2. \end{aligned}$$

□

Lemma 3.8. For a fixed mesh N and for $\varepsilon \rightarrow 0$, it holds

$$\lim_{c_\varepsilon \rightarrow 0} \max_{1 \leq i \leq N-1} \frac{\exp\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}{c_\varepsilon^m} = 0, \quad m = 1, 2, 3, \dots \quad (33)$$

where $x_i = ih$, $h = 1/N$, $i = 1, 2, \dots, N-1$.

Proof. Refer in [23].

□

At this stage, we have $|L^{h,\Delta t}U_{j+1}(x_i) - L^{h,\Delta t}U_{i,j+1}| \leq Ch$, by using the discrete maximum principle, we obtain the bound as $|U_{j+1}(x_i) - U_{i,j+1}| \leq CN^{-1}$, where $h = N^{-1}$.

Theorem 3.9. *Under the hypothesis of boundedness of discrete solution, the solution of the discrete schemes in (27) satisfy the parameter uniform bound*

$$\sup_{0 < c_\varepsilon \ll 1} \max |u(x_i, t_{j+1}) - U_{i,j+1}| \leq C(N^{-1} + (\Delta t)^2). \tag{34}$$

Proof. The parameter uniform convergence of the fully discrete scheme, follows from the results of Theorem 1, Lemma 3.8 and the bound from temporal discretization in Lemma 3.3. We justify it as follows

$$\begin{aligned} \sup_{0 < c_\varepsilon \ll 1} \max |u(x_i, t_{j+1}) - U_{i,j+1}| &\leq \sup_{0 < c_\varepsilon \ll 1} \max |u(x_i, t_j) - U_{j+1}(x_i)| \\ &\quad + \sup_{0 < c_\varepsilon \ll 1} \max |U_{j+1}(x_i) - U_{i,j+1}| \tag{35} \\ &\leq C(N^{-1} + (\Delta t)^2). \end{aligned}$$

This completes the proof. □

4. Numerical results and discussion

To validate the established theoretical results, we develop an algorithm and perform experiments using the proposed numerical scheme on the problem of the form in (2).

Example 4.1. Consider the problem

$$\frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + (2 + x + x^2) \frac{\partial u}{\partial x} + \left(\frac{1 + x^2}{2}\right) u(x - \delta, t) = t \sin(\pi x(1 - x)),$$

with initial-boundary conditions $u_0(x) = 0$, on $0 \leq x \leq 1$, $\phi(x, t) = 0$, on $-\delta \leq x \leq 0$, $\psi(1, t) = 0$ and $T = 1$.

Example 4.2. Consider the problem

$$\frac{\partial u}{\partial t} - \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + (2 - x^2) \frac{\partial u}{\partial x} + 2u(x - \delta, t) + (x - 2)u(x, t) = 10t^2 \exp(-t)x(1 - x)$$

with initial-boundary conditions $u_0(x) = 0$, on $0 \leq x \leq 1$, $\phi(x, t) = 0$, on $-\delta \leq x \leq 0$, $\psi(1, t) = 0$ and $T = 3$.

Exact solution is not given for these two examples, hence we find the maximum absolute point-wise error ($E_{\varepsilon, \delta}^{N, M}$), ε -uniform error ($E^{N, M}$), rate of convergence ($r_{\varepsilon, \delta}^{N, M}$) and ε -uniform rate of convergence ($r^{N, M}$) by using the double mesh principle given in [13]. The maximum absolute point-wise error is calculated as

$$E_{\varepsilon, \delta}^{N, M} = \max_{i, j} |U_{i, j}^{N, M} - U_{i, j}^{2N, 2M}|,$$

where N, M are the number of mesh points in x and t direction respectively. Let $U_{i, j}^{N, M}$ stand for the computed solution of the problem using N, M mesh numbers and $U_{i, j}^{2N, 2M}$ are denote the computed solution on double number of mesh points $2N, 2M$ by adding the mid points $x_{i+1/2} = \frac{x_{i+1} + x_i}{2}$ and $t_{j+1/2} = \frac{t_{j+1} + t_j}{2}$ into the mesh points.

TABLE 1. Maximum absolute error and rate of convergence of Example 4.1 at $\delta = 0.5\epsilon$.

ϵ	N=32	64	128	256	512
\downarrow	M=60	120	240	480	960
2^{-6}	2.0006e-03	1.0348e-03	5.2616e-04	2.6529e-04	1.3317e-04
	0.9511	0.9758	0.9879	0.9943	-
2^{-8}	1.9983e-03	1.0336e-03	5.2554e-04	2.6498e-04	1.3304e-04
	0.9511	0.9758	0.9879	0.9940	-
2^{-10}	1.9977e-03	1.0333e-03	5.2539e-04	2.6490e-04	1.3301e-04
	0.9511	0.9758	0.9879	0.9939	-
2^{-12}	1.9976e-03	1.0332e-03	5.2535e-04	2.6488e-04	1.3300e-04
	0.9511	0.9758	0.9879	0.9939	-
2^{-14}	1.9975e-03	1.0332e-03	5.2534e-04	2.6488e-04	1.3299e-04
	0.9511	0.9758	0.9879	0.9939	-
2^{-16}	1.9975e-03	1.0332e-03	5.2534e-04	2.6488e-04	1.3299e-04
	0.9511	0.9758	0.9879	0.9939	-
2^{-18}	1.9975e-03	1.0332e-03	5.2534e-04	2.6488e-04	1.3299e-04
	0.9511	0.9758	0.9879	0.9939	-
2^{-20}	1.9975e-03	1.0332e-03	5.2534e-04	2.6488e-04	1.3299e-04
	0.9511	0.9758	0.9879	0.9939	-
$E^{N,M}$	2.0006e-03	1.0348e-03	5.2616e-04	2.6529e-04	1.3317e-04
$r^{N,M}$	0.9511	0.9758	0.9879	0.9943	-

TABLE 2. Comparison of maximum absolute error of Example 4.1 at $\delta = 0.5\epsilon$ and $M = 2048$.

$\epsilon \downarrow$	Proposed Scheme		$\epsilon \downarrow$	Result in [14]	
	N=32	64		N=32	64
2^{-12}	1.7687e-03	9.1193e-04	2^{-12}	1.7842e-02	6.7266e-03
2^{-13}	1.7687e-03	9.1192e-04	2^{-13}	1.7842e-02	6.7268e-03
2^{-14}	1.7687e-03	9.1192e-04	2^{-14}	1.7842e-02	6.7268e-03
2^{-15}	1.7687e-03	9.1192e-04	2^{-15}	1.7842e-02	6.7269e-03
2^{-16}	1.7687e-03	9.1191e-04	2^{-16}	1.7842e-02	6.7269e-03
2^{-17}	1.7687e-03	9.1191e-04	2^{-17}	1.7842e-02	6.7269e-03
2^{-18}	1.7687e-03	9.1191e-04	2^{-18}	1.7842e-02	6.7269e-03
2^{-19}	1.7687e-03	9.1191e-04	2^{-19}	1.7842e-02	6.7269e-03
2^{-20}	1.7687e-03	9.1191e-04	2^{-20}	1.7842e-02	6.7269e-03

The ϵ -uniform error estimate are calculated using $E^{N,M} = \max_{\epsilon,\delta} (E_{\epsilon,\delta}^{N,M})$. The rate of convergence of the developed scheme is calculated using

$$r_{\epsilon,\delta}^{N,M} = \frac{\ln(E_{\epsilon,\delta}^{N,M}) - \ln(E_{\epsilon,\delta}^{2N,2M})}{\ln 2}$$

TABLE 3. Maximum absolute error and rate of convergence of Example 4.2 at $\delta = 0.5\epsilon$.

ϵ	N= 16	32	64	128	256
\downarrow	M=30	60	120	240	480
2^{-6}	1.1211e-02 0.95423	5.7862e-03 0.97852	2.9365e-03 0.98928	1.4792e-03 0.99457	7.4239e-04 0.99738
2^{-8}	1.1147e-02 0.95427	5.7530e-03 0.97854	2.9196e-03 0.98927	1.4707e-03 0.99460	7.3811e-04 0.99732
2^{-10}	1.1131e-02 0.95428	5.7447e-03 0.97854	2.9154e-03 0.98925	1.4686e-03 0.99461	7.3705e-04 0.99732
2^{-12}	1.1127e-02 0.95429	5.7426e-03 0.97851	2.9144e-03 0.98925	1.4681e-03 0.99464	7.3678e-04 0.99730
2^{-14}	1.1126e-02 0.95428	5.7421e-03 0.97853	2.9141e-03 0.98920	1.4680e-03 0.99466	7.3672e-04 0.99734
2^{-16}	1.1126e-02 0.95431	5.7420e-03 0.97855	2.9140e-03 0.98925	1.4679e-03 0.99460	7.3670e-04 0.99730
2^{-18}	1.1126e-02 0.95431	5.7420e-03 0.97855	2.9140e-03 0.98925	1.4679e-03 0.99460	7.3670e-04 0.99730
2^{-20}	1.1126e-02 0.95431	5.7420e-03 0.97855	2.9140e-03 0.98925	1.4679e-03 0.99460	7.3670e-04 0.99730
$E^{N,M}$	1.1211e-02	5.7862e-03	2.9365e-03	1.4792e-03	7.4239e-04
$r^{N,M}$	0.95423	0.97852	0.98928	0.99457	0.99738

and the ϵ - uniform rate of convergence is calculated using

$$r^{N,M} = \frac{\ln(E^{N,M}) - \ln(E^{2N,2M})}{\ln 2}$$

The solution of the problems in Example 4.1 and 4.2 exhibits a strong boundary layer of thickness $O(c_\epsilon)$ near $x = 1$ as $\epsilon \rightarrow 0$ (see in Figures 2 and 4). In Figures 1 and 3, we show the effect of delay and perturbation parameter on the behaviour of the solution using different values for delay and perturbation parameter to the test problems. The numerical results (the maximum absolute error) in Tables 1 – 3, shows that the proposed method is uniformly convergent (i.e. converges independent of the perturbation parameter) with an order of convergence one. Using the test examples, we confirm that the proposed numerical method is more accurate, stable and parameter uniformly convergent. The result obtained by using the proposed scheme is better than the result given in [14] (see the results in Table 2).

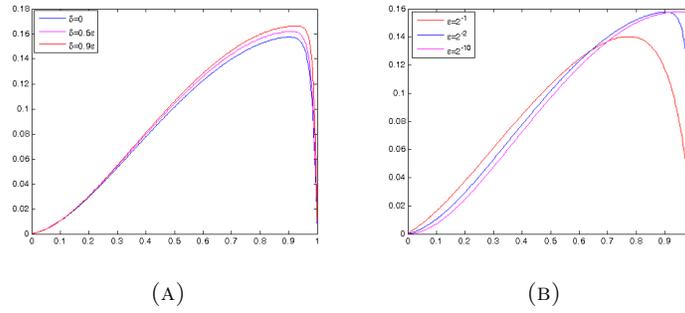


FIGURE 1. Example 4.1, on (A) effect of delay on solution for $\epsilon = 2^{-2}$ on (B) effect of ϵ on the solution at $T = 1$.

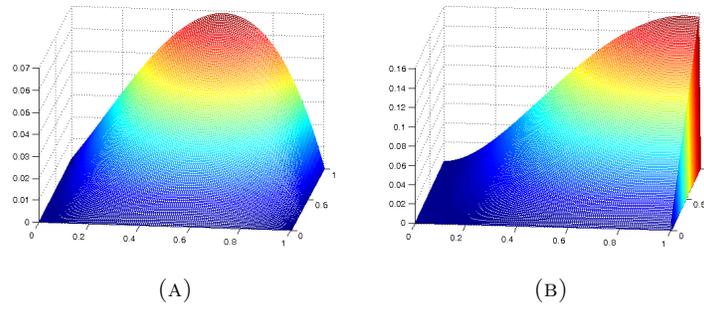


FIGURE 2. Example 4.2, on (A) effect of delay on solution for $\epsilon = 2^{-2}$ on (B) effect of ϵ on the solution at $T=3$.

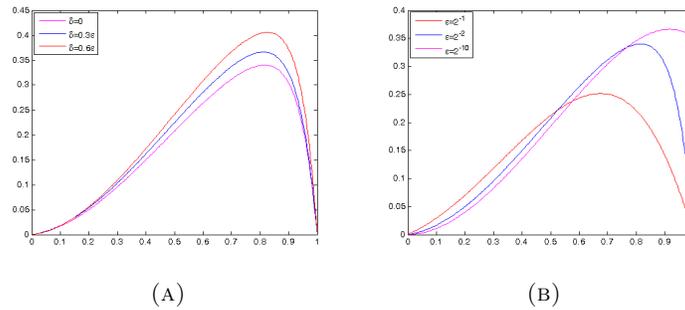


FIGURE 3. Example 4.2, on (A) effect of delay on solution for $\epsilon = 2^{-2}$ on (B) effect of ϵ on the solution at $T=3$.

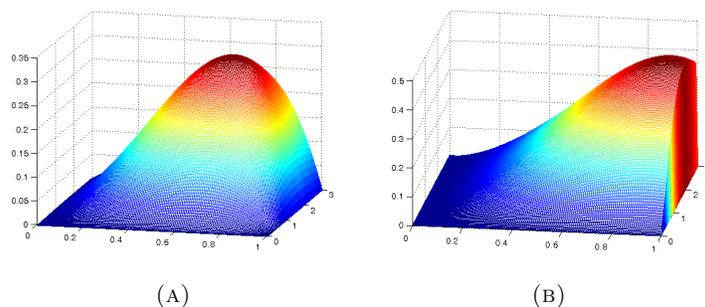


FIGURE 4. 3D view of numerical solution of Example 4.2, on (A) $\varepsilon = 2^{-1}$, on (B) $\varepsilon = 2^{-20}$ for $\delta = 0$.

5. Conclusions

In this paper, a uniformly convergent numerical method is developed for solving a singularly perturbed parabolic delay differential equation with delay on the spatial variable. In the considered case the solution exhibits a boundary layer on the right side of the spatial domain. The developed numerical scheme constitute Crank Nicolson method for in temporal discretization with a non-standard finite difference method in the spatial discretization. The stability and convergence analysis of the proposed scheme is discussed in detail. The applicability of the scheme is investigated by considering two test numerical examples. The effect of the perturbation parameter and the delay parameter on the solution of the problem are discussed and depicted by using figures. The method is shown to be uniformly convergent i.e., converges independent of perturbation parameter with order of convergence $O(N^{-1} + (\Delta t)^2)$. The performance of the proposed scheme is investigated by comparing the results with some published paper. The proposed method gives more accurate and stable numerical results.

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