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EVENTUAL SHADOWING FOR CHAIN TRANSITIVE SETS OF C^1 GENERIC DYNAMICAL SYSTEMS

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Abstract. We show that given any chain transitive set of a C^1 generic diffeomorphism f, if a diffeomorphism f has the eventual shadowing property on the locally maximal chain transitive set, then it is hyperbolic. Moreover, given any chain transitive set of a C^1 generic vector field X, if a vector field X has the eventual shadowing property on the locally maximal chain transitive set, then the chain transitive set does not contain a singular point and it is hyperbolic. We apply our results to conservative systems (volume-preserving diffeomorphisms and divergence-free vector fields).

1. Introduction

A main topic of study in dynamical systems is the stability of a given dynamical system. In 1967, Smale [61] proved that the nonwandering set of an Axiom A diffeomorphism is a disjoint union of transitive invariant closed sets, called *basic sets.* Since Smale's study, this question was satisfactorily answered for hyperbolic systems. In practice, the theory of dynamical systems is motivated by the behavior of the orbits of a given system, which is related to shadowing theory. It is well known that a hyperbolic set has the shadowing property. However, we do not know that if a diffeomorphism has the shadowing property, then it is hyperbolic. Let $f : M \to M$ be a diffeomorphism of a smooth manifold. Abdenur and Diáz [2] suggested the following problem: *the C^1 generic diffeomorphism f has the shadowing property if and only if it is hyperbolic.*

This remains an open problem. However, we can find partial results [2, 3, 44, 46, 47, 51, 57, 59], from which we introduce [2, 3, 44]. Abdenur and Diáz [2] proved that for a locally maximal transitive set Λ of a generic diffeomorphism f, either Λ is hyperbolic or there is a small neighborhood U of Λ such that for any g C^1 close to f, g does not have the shadowing property in U. Ahn *et al.* [3] proved that if a C^1 generic diffeomorphism f has the shadowing property on a

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locally maximal homoclinic class, then it is hyperbolic. Recently, Lee and Lee [44] proved that if a C^1 generic diffeomorphism f has the shadowing property on chain recurrence classes, then it is hyperbolic if the chain recurrence class contains a hyperbolic periodic point, which generalizes the result in [3].

To solve this problem, authors have used several types of shadowing properties (such as limit shadowing [14, 24–26, 33, 48, 52, 57], weak shadowing [10, 26], inverse shadowing [43], orbital shadowing [27, 28, 41], periodic shadowing [29], asymptotic orbital shadowing [39,50], asymptotic average shadowing [30,37,49], average shadowing [37,49], specification properties [8, 11, 34, 38, 60], and ergodic shadowing [31], etc. [35, 40, 42]).

In the literature, a diffeomorphism result can be extended to vector fields. However, it cannot be obtained directly. We say that a diffeomorphism f satisfies a *star condition* if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, every point $p \in P(g)$ is hyperbolic, where $P(g)$ is the set of all periodic points of g. Denote by $\mathcal{F}(M)$ the set of all diffeomorphisms satisfying star conditions. If $f \in \mathcal{F}(M)$, then f satisfies Axiom A without cycles [4, 21]. We say that a flow X^t satisfies a star condition if there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that for any $Y \in \mathcal{U}(X)$, every singularity and every periodic orbit of Y is hyperbolic. If a flow X^t satisfies a star condition, then it is not a hyperbolic nonwandering set, as with the Lorenz attractor [20]. Further, Komuro [22] proved that geometric Lorenz flows do not satisfy the shadowing property.

In this study, we use another type of shadowing property to show that if a C^1 generic diffeomorphism (a vector field) has this shadowing property on closed subsets, then it is hyperbolic. Moreover, we apply this to volume-preserving diffeomorphisms and divergence-free vector fields.

1.1. Diffeomorphisms

Let M be a closed smooth manifold with $\dim M \geq 2$, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1 topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM.

Let $f \in \mathrm{Diff}(M)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}(-\infty \leq a < b \leq \infty)$ in M is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. We say that f has the *shadowing property on* Λ if, for every $\epsilon > 0$, there is a $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. If $\Lambda = M$, then we say that f has the *shadowing property.* We say that a closed invariant set Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of x.

For a given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a finite δ-pseudo-orbit $\{x_i\}_{i=0}^{n}(n \geq 1)$ of f such that $x_0 = x$ and $x_n = y$. For any $x, y \in \Lambda$, we write that $x \rightsquigarrow_\Lambda y$ if $x \rightsquigarrow y$ and $\{x_i\}_{i=0}^{n} \subset \Lambda(n \geq 1)$. We say that the set $C(f)$ is *chain transitive* if for any $x, y \in C(f)$, $x \rightsquigarrow_{C(f)} y$. A closed invariant set Λ is *locally maximal* if there is a neighborhood U of Λ such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U).$$

We say that a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathrm{Diff}(M)$; in this case, $\mathcal{G}$ is dense in $\mathrm{Diff}(M)$. A property $\mathcal{P}$ is said to be C^1*-generic* if $\mathcal{P}$ holds for all diffeomorphisms that belong to some residual subset of $\mathrm{Diff}(M)$.

We say that a closed f-invariant set Λ admits a *dominated splitting* for f if the tangent bundle $T_\Lambda M$ has a continuous Df invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that for all $x \in \Lambda$ and $n \geq 0$, we have

$$||Df^n|_{E(x)}|| \cdot ||Df^{-n}|_{F(f^n(x))}|| \leq C\lambda^n.$$

Abdenur *et al.* [1] proved that C^1-generically, for any chain transitive set $C(f)$, either there is a dominated splitting over $C(f)$ or the set $C(f)$ is contained in the Hausdorff limit of a sequence of periodic sinks or sources of f. Lee [36] proved that if a C^1-generic chain transitive set $C(f)$ is locally maximal, then it admits a dominated splitting. We say that Λ is *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$, then f is said to be *Anosov.*

Lee and Wen [51] proved that if a C^1-generic diffeomorphism f has the shadowing property on a locally maximal chain transitive set $C(f)$, then it is hyperbolic. In this study, we use another type of shadowing (eventual shadowing property), which is a general result for the result [51].

We say that f has the *eventual shadowing property* on Λ if for all $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, there exist $y \in M$ and $N \in \mathbb{Z}$ such that

$$d(f^i(y), x_i) < \epsilon \text{ for all } i \geq N \text{ and}$$

$$d(f^i(y), x_i) < \epsilon \text{ for all } i \leq -N.$$

If $\Lambda = M$, then we say that f has the *eventual shadowing property.* Note that if a diffeomorphism f has the shadowing property, then it has the eventual shadowing property. However, the converse is not true (see [19, Example 5]). It is not difficult to show that a diffeomorphism f has the eventual shadowing property if and only if f^k has the eventual shadowing property, for all $k \in \mathbb{Z} \setminus \{0\}$. This proof is similar to the proof of the shadowing property (see [5]). Let Λ be a closed invariant set of f. Then, clearly, if f has the eventual shadowing property, then f has the eventual shadowing property on Λ.

In this paper, we assume that a chain transitive set $C(f)$ is *nontrivial*, which means that $C(f)$ is not reduced to orbit. The following is a main theorem of the paper:

Theorem A. *There is a residual set $\mathcal{G}$ in* $\mathrm{Diff}(M)$*, which is such that if $f \in \mathcal{G}$ has the eventual shadowing property on a locally maximal chain transitive set $C(f)$, it is hyperbolic on $C(f)$.*

1.2. Vector fields

Let M be a closed smooth manifold with $\dim M \geq 3$. Denote by $\mathfrak{X}(M)$ the set of C^1 vector fields on M endowed with the C^1 topology. Then, every $X \in \mathfrak{X}(M)$ generates a C^1 flow $X^t : M \times \mathbb{R} \to M$; that is a C^1 map such that $X^t : M \to M$ is a diffeomorphism satisfying $X^0(x) = x$ and $X^{t+s}(x) = X^t(X^s(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$. For any $\delta > 0$, a sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1, \text{ and } -\infty \leq a < i < b \leq \infty\}$ is a δ-pseudo-orbit of X if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for any $a \leq i \leq b-1$.

An increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ is called a *reparameterization* of $\mathbb{R}$. Denote by $\mathrm{Rep}(\mathbb{R})$ the set of reparameterizations of $\mathbb{R}$. Fix $\epsilon > 0$ and define $\mathrm{Rep}(\epsilon)$ as follows:

$$\mathrm{Rep}(\epsilon) = \big\{h \in \mathrm{Rep}(\mathbb{R}) : \big|\frac{h(t)}{t} - 1\big| < \epsilon\big\}.$$

For a closed X^t-invariant set $\Lambda \subset M$, we say that X has the *shadowing property* on Λ if for any $\epsilon > 0$, there is $\delta > 0$ satisfying the following property: given any δ-pseudo-orbit $\xi = \{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\} \subset \Lambda$, there are a point $y \in M$ and an increasing homeomorphism $h \in \mathrm{Rep}(\epsilon)$ such that $d(X^{h(t)}(y), X^{t-s_i}(x_i)) < \epsilon$ for any $s_i < t < s_{i+1}$, where s_i is defined as

$$s_i = \begin{cases} t_0 + t_1 + \cdots + t_{i-1}, & \text{if } i > 0, \\ 0, & \text{if } i = 0, \\ -t_{-1} - t_{-2} - \cdots - t_i, & \text{if } i < 0. \end{cases}$$

The point $y \in M$ is said to be a *shadowing point* of ξ. If $\Lambda = M$, then we say that X has the shadowing property.

We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that $\bigcap_{t\in\mathbb{R}} X^t(U) = \Lambda$.

Let Λ be a closed X^t-invariant set. We say that the set Λ is *transitive* if there is a point $x \in \Lambda$ such that its positive orbit $\{X^t(x) : t \geq 0\}$ is dense in Λ. We say that an X^t-invariant set $C(X)$ is *chain transitive* if, for any $x, y \in C(X)$ and $\delta > 0$, there is a δ-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1 \text{ for } 0 \leq i \leq n\} \subset C(X)$ with $x_0 = x, x_n = y$. It is known that if a flow is transitive, then it is chain transitive. However, the converse is not true. We say that a subset $\mathcal{G} \subset \mathfrak{X}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathfrak{X}(M)$; in this case, $\mathcal{G}$ is dense in $\mathfrak{X}(M)$. A property $\mathcal{P}$ is said to be C^1*-generic* if $\mathcal{P}$ holds for all vector fields that belong to some residual subset of $\mathfrak{X}(M)$.

Let X^t be the flow of $X \in \mathfrak{X}(M)$, and let Λ be an X^t-invariant compact set. The set Λ is called *hyperbolic* for X if there are constants $C > 0, \lambda > 0$ and a splitting $T_xM = E^s_x \oplus \langle X(x)\rangle \oplus E^u_x$ such that the tangent flow $DX^t : TM \to TM$

leaves invariant the continuous splitting and

$$\|DX^t|_{E^s_x}\| \le Ce^{-\lambda t} \text{ and } \|DX^{-t}|_{E^u_x}\| \le Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$. We say that $X \in \mathfrak{X}(M)$ is *Anosov* if M is hyperbolic for X.

Let $Sing(X) = \{x \in M : X(x) = 0\}$ be the set of singularities of X and $Per(X) = \{x \in M : \text{there is } T > 0 \text{ such that } X^T(x) = x\}$ be the set of periodic orbits of X. Denote by $Crit(X) = Sing(X) \cup Per(X)$.

In [57], Ribeiro proved that, C^1-generically, if a flow X^t has the shadowing property in a locally maximal chain transitive set $C(X)$, then it is transitive hyperbolic.

We say that $x \in M$ is a *regular point* if $x \in M \setminus Sing(X)$. Denote by $R(M)$ the set of all regular points of M.

Let $x \in R(M)$, and let $N \subset TM$ be the subbundle such that the fiber N_x at $x \in M$ is the orthogonal linear subspace of $\langle X(x)\rangle$ in T_xM; i.e., $N_x = \langle X(x)\rangle^{\perp}$. Here $\langle X(x)\rangle$ is the linear subspace spanned by $X(x)$ for $x \in M$. Let $\pi : TN \to N$ be the projection along X, and let

$$P_{x,t}(v) = \pi(D_xX^t(v))$$

for $v \in N_x$ and $x \in M$. It is well known that $P_t : N \to N$ is a one-parameter transformation group.

We say that Λ is *hyperbolic* if the bundle N_Λ has a P^X_t-invariant splitting $\Delta^s \oplus \Delta^u$ and there exists an $l > 0$ such that

$$\|P^X_l|_{\Delta^s_x}\| \le \frac{1}{2} \text{ and } \|P^X_{-l}|_{\Delta^u_{X^l(x)}}\| \le \frac{1}{2}$$

for all $x \in \Lambda$. Doering [16] showed the following, which is a method of proving hyperbolicity.

Remark 1.1. Let $\Lambda \subset M$ be a compact invariant set of X^t. Λ is a hyperbolic set of X^t if and only if the linear Poincaré flow restriction on Λ has a hyperbolic splitting $N_\Lambda = \Delta^s \oplus \Delta^u$, where $N = \bigcup_{x \in M_X} N_x$.

From Ribeiro's result [57], we consider a locally maximal chain transitive set that has another type of shadowing property for C^1-generic vector fields. Now, we introduce a flow version of the concepts described in the previous section. We say that X has the *eventual shadowing property* on Λ if, for any $\epsilon > 0$, there is $\delta > 0$ such that for any $(\delta, 1)$-pseudo-orbit $\{(x_i, t_i) : t_i \ge 1, i \in \mathbb{Z}\} \subset \Lambda$ there exist $y \in M, t_n(n \ge 1) \in \mathbb{R}$ and $h \in Rep(\epsilon)$ such that

$$d(X^{h(t)}(y), X^{t-s_{n+i}}(x_{n+i})) < \epsilon,\ s_{n+i} < t < s_{n+i+1} \text{ and}$$

$$d(X^{h(t)}(y), X^{t-s_{-n-i}}(x_{-n-i})) < \epsilon,\ s_{-n-i} < t < s_{-n-i+1},$$

where $s_i = t_0 + t_1 + \cdots + t_i (i \ge 0)$, $s_0 = 0$, and $s_{-i} = -t_0 - t_{-1} - \cdots - t_{-i} (i \ge 0)$. If $\Lambda = M$, then we say that X has the eventual shadowing property. The following is an extension of the result in Theorem A.

Theorem B. *There is a residual set $\mathcal{R}$ in $\mathfrak{X}(M)$ such that if $X \in \mathcal{R}$ has the eventual shadowing property on a locally maximal chain transitive set $C(X)$, then $C(X) \cap Sing(X) = \emptyset$ and $C(X)$ is hyperbolic.*

2. Proof of Theorem A

The following lemma was obtained by Crovisier (see [15, Theorem 2]).

Lemma 2.1. *There is a residual set $\mathcal{G}_1 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_1$ and any chain transitive set $C(f)$, there is a sequence $Orb(p_n)$ of periodic orbits of f such that $\lim_{n\to\infty} Orb(p_n) = C(f)$, in the sense of the Hausdorff metric.*

We also recall that the Hausdorff distance between two compact subsets A and B of M is given by

$$d_H(A,B) = \max\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\}.$$

Lemma 2.2. *Given any chain transitive set $C(f)$ of $f \in \mathcal{G}_1$, if $C(f)$ is locally maximal, then $C(f) \cap P(f) \neq \emptyset$.*

Proof. Let $f \in \mathcal{G}_1$ and let a chain transitive set $C(f)$ of f be locally maximal in U. Suppose, by contradiction, that $C(f) \cap P(f) = \emptyset$. Since $C(f)$ is compact, there is $\epsilon > 0$ such that $C(f) \subset B_\epsilon(C(f)) \subset U$. By Lemma 2.1, there is a periodic orbit sequence $Orb(p_n)$ of f such that for sufficiently large n, we have

$$d_H(Orb(p_n), C(f)) < \frac{\epsilon}{2}.$$

It is clear that $Orb(p_n) \subset B_\epsilon(C(f)) \subset U$. Since $C(f)$ is locally maximal in U, for all $i \in \mathbb{Z}$,

$$f^i(Orb(p_n)) \subset f^i(U).$$

Thus, if $C(f)$ is locally maximal, then $C(f)\cap P(f) \neq \emptyset$, which is a contradiction. □

Lemma 2.3. *Let Λ be a compact f-invariant set of f. If f has the eventual shadowing property on a locally maximal Λ, then the eventual shadowing points are take from Λ.*

Proof. Let U be a locally maximal neighborhood of Λ. Since Λ is compact, there is $\epsilon > 0$ such that $\Lambda \subset B_\epsilon(\Lambda) \subset U$. Let $0 < \delta \le \epsilon$ be the number of the eventual shadowing property, and let $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ be a δ-pseudo-orbit of f. By the eventual shadowing property on Λ, there are $y \in M$ and $N \in \mathbb{Z}$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \ge N$ and $d(f^{-i}(y), x_{-i}) < \epsilon$ for all $-i \le -N$. Then, we have that for all $i \ge N$, $f^i(y) \in B_\epsilon(\Lambda)$ and for all $-i \le -N$, $f^{-i}(y) \in B_\epsilon(\Lambda)$ and so,

$$f^i(f^N(y)) \in B_\epsilon(\Lambda) \text{ and } f^{-i}(f^{-N}(y)) \in B_\epsilon(\Lambda).$$

Since Λ is locally maximal, we know that

$$\bigcap_{n\in\mathbb{Z}} f^n(f^{N+i}(y)) \in \bigcap_{n\in\mathbb{Z}} f^n(B_\epsilon(\Lambda)) \subset \bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda.$$

Then we have $f^{N+i}(y) \in \Lambda$. Since Λ is an f-invariant set, $y \in f^{-N-i}(\Lambda) = \Lambda$. Thus the eventual shadowing point y is take from Λ. □

It is known that if p is a hyperbolic periodic point of f with period k, then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \text{ and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. Let p be a hyperbolic periodic point of f. Then there exists an $\epsilon = \epsilon(p) > 0$ such that

$$W^s_\epsilon(p) = \{x \in M : d(f^i(x), f^i(p)) \le \epsilon \text{ as } i \ge 0\} \text{ and}$$
$$W^u_\epsilon(p) = \{x \in M : d(f^i(x), f^i(p)) \le \epsilon \text{ as } i \le 0\}.$$

Then the set $W^s_{\epsilon(p)}(p)$ is called the *local stable manifold* of p and the set $W^u_{\epsilon(p)}(p)$ is called the *local unstable manifold* of p. Note that if a closed f-invariant set Λ is hyperbolic, then there is $\eta > 0$ such that for any $0 < \epsilon \le \eta$, the above sets are C^1-embedded disks.

Lemma 2.4. *If f has the eventual shadowing property on a locally maximal $C(f)$, then for any hyperbolic $p, q \in C(f) \cap P(f)$, we have $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$.*

Proof. Since $C(f)$ is a chain transitive set of f (by [32, Lemma 2.1]), f does not contain sinks or sources. Thus, every periodic point in $C(f)$ is saddle. Let p and q be hyperbolic periodic points of f. Take $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$ and let $0 < \delta \le \epsilon$ be the number of the eventual shadowing property for f. For simplicity, we may assume that $f(p) = p$ and $f(q) = q$. Since f is chain transitive, there is a finite δ-pseudo-orbit $\{x_i\}_{i=0}^n (n \ge 1) \subset C(f)$ such that $x_0 = p$, $x_n = q$, and $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, \dots, n-1$. Put $x_i = f^i(p)$ for all $i \le 0$ and $x_{n+i} = f^i(q)$ for all $i \ge 0$. Then, the sequence

$$\{\dots, p(= x_{-1}), p(= x_0), x_1, x_2, \dots, q(= x_n), q(= x_{n+1}), \dots\} = \{x_i\}_{i\in\mathbb{Z}} \subset C(f)$$

is a δ-pseudo-orbit of f. By the eventual shadowing property on $C(f)$, there are $z \in M$ and $N \in \mathbb{Z}$ such that

$$d(f^i(y), x_i) < \epsilon \text{ for } i \ge N \text{ and } d(f^i(y), x_i) < \epsilon \text{ for } i \le -N.$$

Since $x_{-i} = p = f^{-i}(p)$ for $i \ge 0$ and $f^{n+i}(q) = q = x_{n+i}$ for $i \ge 0$, if $N \ge n$, then we know

$$f^{-N}(y) \in B_\epsilon(x_{-N}) = B_\epsilon(p)$$

and

$$f^N(y) \in B_\epsilon(x_N) = B_\epsilon(q).$$

Thus for all $i \ge N$

$$f^{i+N}(y) = f^i(f^N(y)) \in B_\epsilon(x_{N+i}) = B_\epsilon(q) \tag{1}$$

and for all $-i \leq -N$,

$$f^{-N-i}(y) \in B_\epsilon(x_{-N-i}) = B_\epsilon(p). \tag{2}$$

By (1), we have $d(f^i(f^N(y)), q) < \epsilon$ for all $i \geq 0$, and by (2), we have $d(f^{-i}(f^{-N}(y)), p) < \epsilon$ for all $i \geq 0$. Then $f^N(y) \in W^s_\epsilon(q)$ and $f^{-N}(y) \in W^u_\epsilon(p)$, and so $y \in f^N(W^u_\epsilon(p))$ and $y \in f^{-N}(W^s_\epsilon(q))$. Since $f^N(W^u_\epsilon(p)) \subset W^u(p)$ and $f^{-N}(W^s_\epsilon(q) \subset W^s(q)$, we have $y \in W^u(p) \cap W^s(q)$. Thus, $W^u(p) \cap W^s(q) \neq \emptyset$. The other case is similar. □

Let q be a hyperbolic periodic point of f. We say that p and q are *homoclinically related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \text{ and } W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

It is clear that if $p \sim q$, then index(p) = index(q). That is, $\dim W^s(p) = \dim W^s(q)$.

A diffeomorphism $f \in \text{Diff}(M)$ is said to be *Kupka-Smale* if the periodic points of f are hyperbolic, and if $p, q \in P(f)$, then $W^s(p)$ is transversal to $W^u(q)$. Denote by $\mathcal{KS}$ the set of all Kupka-Smale diffeomorphisms. It is known that the set of all Kupka-Smale diffeomorphisms is C^1-residual in Diff(M) (see [56]).

Lemma 2.5. *There is a residual set $\mathcal{G}_2 \subset \text{Diff}(M)$ such that given any chain transitive set $C(f)$ of $f \in \mathcal{G}_2$, if f has the eventual shadowing property on locally maximal $C(f)$, then for any $q \in C(f) \cap P(f)$, we have* index(p) = index(q).

Proof. Let $f \in \mathcal{G}_2 = \mathcal{G}_1 \cap \mathcal{KS}$ and let $C(f)$ be a locally maximal chain transitive set of f. Suppose that f has the eventual shadowing property on $C(f)$. Since $C(f)$ is locally maximal of f, then by Lemma 2.2, we know $C(f) \cap P(f) \neq \emptyset$. In this proof, we will derive a contradiction. We assume that there are two hyperbolic periodic points $p, q \in C(f)$ such that index$(p) \neq$ index(q). Since index$(p) \neq$ index(q), we know $\dim W^s(p) + \dim W^u(q) < \dim M$ or $\dim W^u(p) + \dim W^s(q) < \dim M$. Then, we consider the case in which $\dim W^s(p) + \dim W^u(q) < \dim M$ (the other case is similar). Since $f \in \mathcal{KS}$ and $\dim W^s(p) + \dim W^u(q) < \dim M$, we know that $W^s(p) \cap W^u(q) = \emptyset$. This is a contradiction. Because f has the eventual shadowing property on $C(f)$, by Lemma 2.4, $W^s(p) \cap W^u(q) \neq \emptyset$. Thus, if $f \in \mathcal{G}_2$ has the eventual shadowing property on a locally maximal chain transitive set $C(f)$, then for any $q \in C(f) \cap P(f)$, we have index(p) = index(q). □

We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set of points $\{x \in M : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{CR}(f)$. The chain recurrence class of f is the set of equivalent classes $\leftrightsquigarrow$ on $\mathcal{CR}(f)$. Denote by $C(p, f) = \{x \in M : x \rightsquigarrow p \text{ and } p \rightsquigarrow x\}$, which is a closed invariant set. Let q be a hyperbolic periodic point of f. We say that p and q are *homoclinically related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \text{ and } W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

It is clear that if $p \sim q$, then $\text{index}(p) = \text{index}(q)$. That is, $\dim W^s(p) = \dim W^s(q)$. Denote by $H(p, f) = \overline{\{p \sim q\}}$. It is known that $H(p, f) \subset C(p, f)$ (see [59]).

Lemma 2.6. *There is a residual set $\mathcal{G}_3 \subset \text{Diff}(M)$ such that every $f \in \mathcal{G}_3$ satisfies*:

(a) *A locally maximal transitive set Λ is locally maximal $H(p, f)$ for some periodic point $p \in \Lambda$ (see* [1]*).*
(b) *$H(p, f) = C(p, f)$ for some hyperbolic periodic point p (see* [12]*).*
(c) *If $C_f(p)$ is locally maximal, then $C_f(p)$ is robustly isolated. That is, there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of $C_f(p)$ such that for any $g \in \mathcal{U}(f)$, $C_g(p_g) = \mathcal{CR}(g) \cap U = \Lambda_g(U)(= \bigcap_{n\in\mathbb{Z}} g^n(U))$ (see* [13]*).*
(d) *Any chain transitive $C(f)$ of f is a transitive Λ of f (see* [15]*).*

The following lemma is called Franks' lemma [18].

Lemma 2.7. *Let $\mathcal{U}(f)$ be any given C^1 neighborhood of f. Then, there exist $\epsilon > 0$ and a C^1 neighborhood $\mathcal{V}(f) \subset \mathcal{U}(f)$ of f such that for a given $g \in \mathcal{V}(f)$, a finite set $\{x_1, x_2, \ldots, x_k\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_k\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le k$, there exists $\widetilde{g} \in \mathcal{U}(f)$ such that $\widetilde{g}(x) = g(x)$ if $x \in \{x_1, x_2, \ldots, x_k\} \cup (M \setminus U)$ and $D_{x_i}\widetilde{g} = L_i$ for all $1 \le i \le k$.*

For any $\delta > 0$, we say that a hyperbolic periodic point p of f with period $\pi(p)$ is a δ *weak hyperbolic periodic point* if there is an eigenvalue λ of $Df^{\pi(p)}(p)$ such that

$$(1-\delta)^{\pi(p)} < |\lambda| < (1+\delta)^{\pi(p)}.$$

Lemma 2.8. *There is a residual set $\mathcal{G}_4 \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G}_4$ and any $\delta > 0$, if a chain transitive set $C(f)$ of f is locally maximal and $C(f)$ contains a δ weak hyperbolic periodic point p, then there is g C^1 close to f such that g has two hyperbolic periodic points $p, q \in C(g)$ such that* $\text{index}(p) \neq \text{index}(q)$, *where $C(g)$ is the chain transitive set of g.*

Proof. Let $f \in \mathcal{G}_4 = \mathcal{G}_2 \cap \mathcal{G}_3$ and let U be a locally maximal neighborhood of $C(f)$. Suppose that there is $p \in C(f) \cap P(f)$ such that for any $\delta > 0$, p is a δ weak hyperbolic periodic point. Since $f \in \mathcal{G}_3$ and $C(f)$ is locally maximal, $C(f)$ is a transitive set Λ of f, and so, $C(f) = H_f(p) = C_f(p)$ and $C(f)$ is robustly isolated. For simplicity, we may assume that $f^{\pi(p)}(p) = f(p) = p$. $p \in C(f) \cap P(f)$ is a δ weak hyperbolic periodic point, for any $\delta > 0$ there is an eigenvalue λ of D_pf such that

$$(1-\delta) < |\lambda| < (1+\delta).$$

By Lemma 2.7, there is g C^1 close to f such that $g(p) = f(p) = p$ and D_pg has an eigenvalue λ such that $|\lambda| = 1$. Note that by Lemma 2.7, there is g_1 C^1 close to f such that D_pg_1 has only one eigenvalue λ with $|\lambda| = 1$. Denote by

E_p^c the eigenspace corresponding to λ. In this proof, we consider two cases: (i) λ is real, and (ii) λ is complex.

First, we may assume that $\lambda \in \mathbb{R}$ (the other case is similar). By Lemma 2.7, there are $\alpha > 0$, $B_\alpha(p) \subset U$ and h C^1 close to g ($h \in \mathcal{U}(f)$) such that

- $h(p) = g(p) = p$,
- $h(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x)$ for $x \in B_\alpha(p)$, and
- $h(x) = g(x)$ for $x \notin B_{4\alpha}(p)$.

Let $\eta = \alpha/4$. Take a nonzero vector $v \in \exp_p(E_p^c(\alpha))$ that corresponds to λ such that $\|v\| = \eta$. Here, $E_p^c(\alpha)$ is the α-ball in E_p^c with its center at $\overrightarrow{0_p}$. Then, we have

$$h(\exp_p(v)) = \exp_p \circ D_p g \circ \exp_p^{-1}(\exp_p(v)) = \exp_p(v).$$

Put $\mathcal{J}_p = \exp_p(\{tv : -\eta/4 \le t \le \eta/4\})$. Then, $\mathcal{J}_p$ is centered at p and $h(\mathcal{J}_p) = \mathcal{J}_p$. Since $B_\alpha(p) \subset U$, we know that $\mathcal{J}_p \subset \Lambda_h(U) = \bigcap_{n\in\mathbb{Z}} h^n(U)$. Since $h(\mathcal{J}_p) = \mathcal{J}_p$, take two endpoints q, r of $\mathcal{J}_p$. Then, we know that

$$D_q h|_{E_p^c} = D_r h|_{E_p^c} = 1.$$

By Lemma 2.7, there is ϕ C^1 close to h ($\phi \in \mathcal{U}(f)$) such that $\mathrm{index}(q_\phi) \neq \mathrm{index}(r_\phi)$, where q_ϕ and r_ϕ are hyperbolic points in U with respect to ϕ. Thus, $q_\phi, r_\phi \in C(\phi) = \Lambda_\phi(U) = \bigcap_{n\in\mathbb{Z}} \phi^n(U)$, where $C(\phi)$ is the chain transitive set of ϕ.

Finally, we consider $\lambda \in \mathbb{C}$. For simplicity, we assume that $f(p) = p$. As in the proof of the case in which $\lambda \in \mathbb{R}$, by Lemma 2.7, there are $\alpha > 0, B_\alpha(p) \subset U$ and $g \in \mathcal{U}(f)$ such that

$$g(p) = f(p) = p \text{ and } g(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x)$$

for $x \in B_\alpha(p)$. Since $\lambda = 1$, there is $n > 0$ such that $D_p g^n(v) = v$ for any $v \in \exp_p^{-1}(E_p^c(\alpha))$. Let $v \in \exp_p(E_p^c(\alpha))$ such that $\|v\| = \alpha/4$. Then, we have a small arc

$$\exp_p(\{tv : 0 \le t \le 1 + \alpha/4\}) = \mathcal{I}_p \subset \Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$$

such that

(i) $g^i(\mathcal{I}_p) \cap g^j(\mathcal{I}_p) = \emptyset$ if $0 \le i \neq j \le n-1$,
(ii) $g^n(\mathcal{I}_p) = \mathcal{I}_p$, and
(iii) $g^n_{|\mathcal{I}_p} : \mathcal{I}_p \to \mathcal{I}_p$ is the identity map.

Then, we take two points $q, r \in \mathcal{I}_p$ such that the points are the endpoints of $\mathcal{I}_p$. As in the previous arguments, there is g_1 C^1 close to g such that $\mathrm{index}(q_{g_1}) \neq \mathrm{index}(r_{g_1})$, where q_{g_1} and r_{g_1} are hyperbolic with respect to g_1. Thus, $q_{g_1}, r_{g_1} \in C_{g_1}(p_{g_1}) = \Lambda_{g_1}(U) = \bigcap_{n\in\mathbb{Z}} g_1^n(U) = C(g_1)$, where $C(g_1)$ is the chain transitive set of g_1. This completes the proof of the lemma. □

Lemma 2.9 ([45, Lemma 2.2]). *There is a residual set $\mathcal{G}_5 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_5$, if for any C^1 neighborhood $\mathcal{U}(f)$ of f there is $g \in \mathcal{U}(f)$*

such that g has two periodic points p and q with $\mathrm{index}(p) \neq \mathrm{index}(q)$*, then f has two periodic points p_f and q_f with* $\mathrm{index}(p_f) \neq \mathrm{index}(q_f)$

Lemma 2.10. *There is a residual set $\mathcal{G}_6 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_6$, if f has the eventual shadowing property on a locally maximal $C(f)$, then there is $\delta > 0$ such that for any $p \in C(f) \cap P(f)$, p is not a δ weak hyperbolic periodic point of f.*

Proof. Let $f \in \mathcal{G}_6 = \mathcal{G}_4 \cap \mathcal{G}_5$ and let $C(f)$ be a locally maximal chain transitive set of f. Suppose, by contradiction, that for any $\delta > 0$, there is $p \in C(f) \cap P(f)$ such that p is a δ weak hyperbolic periodic point of f. Since $f \in \mathcal{G}_3$ and $C(f)$ is locally maximal, $C(f)$ is robustly isolated. Since $f \in \mathcal{G}_4$ and $p \in C(f) \cap P(f)$ is a δ weak hyperbolic periodic point of f, by Lemma 2.8, there is g C^1 close to f such that g has two hyperbolic periodic points $q, r \in C(g)$ with $\mathrm{index}(q) \neq \mathrm{index}(r)$. Since $f \in \mathcal{G}_5$, f has two hyperbolic periodic points $q_f, r_f \in C(f)$ with $\mathrm{index}(q_r) \neq \mathrm{index}(r_f)$. This is a contradiction, since f has the eventual shadowing property on $C(f)$ by Lemma 2.5, $\mathrm{index}(p) = \mathrm{index}(q)$ for every $p, q \in C(f) \cap P(f)$. □

We say that f satisfies a *star condition on $C(f)$* if there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of C such that for any $g \in \mathcal{U}(f)$, every $q \in \Lambda_g \cap P(g)$ is hyperbolic. Denote by $\mathcal{F}(C(f))$ the set of all diffeomorphisms that satisfy the local star condition on $C(f)$.

Lemma 2.11 ([6, Lemma 5.1(2)])**.** *There is a residual set $\mathcal{G}_7 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_7$, for any $\delta > 0$ and any C^1 neighborhood $\mathcal{U}(f)$ of f, if there are $g \in \mathcal{U}(f)$ and a hyperbolic $p \in P(g)$ such that p is a δ weak hyperbolic periodic point, then there is a hyperbolic $p_f \in P(f)$ with 2δ weak hyperbolic periodic points.*

Proposition 2.12. *There is a residual set $\mathcal{G}_8 \subset \mathrm{Diff}(M)$ such that given any chain transitive set $C(f)$ of $f \in \mathcal{G}_8$, if f has the eventual shadowing property on locally maximal $C(f)$, then $f \in \mathcal{F}(C(f))$.*

Proof. Let $f \in \mathcal{G}_8 = \mathcal{G}_6 \cap \mathcal{G}_7$ and let $C(f)$ be a locally maximal chain transitive set of f. Suppose, by contradiction, that $f \notin \mathcal{F}(C(f))$. Then, there is g C^1 close to f such that for any $\delta > 0$, g has a $\delta/2$ weak hyperbolic periodic point $p \in C(g)$. Since $f \in \mathcal{G}_7$, there is $p_f \in C(f) \cap P(f)$ such that p_f is a δ weak hyperbolic periodic point. This is a contradiction; since f has the eventual shadowing property on $C(f)$, by Lemma 2.10 every periodic point in $C(f)$ is not a δ weak hyperbolic periodic point. Thus, if f has the eventual shadowing property on $C(f)$, then $f \in \mathcal{F}(C(f))$. □

The following result is from Lee and Wen [51, Proposition 2.1].

Proposition 2.13. *Given any chain transitive set $C(f)$ of $f \in \mathcal{G}_8$, if $C(f)$ is locally maximal and $f \in \mathcal{F}(C(f))$, then there exist constants $m > 0$ and $0 < \lambda < 1$ such that for any $p \in \Lambda \cap P(f)$, we have the following*:

(a)

$$\prod_{i=0}^{\pi(p)-1} \|Df^m|_{E^s(f^{im}(p))}\| < \lambda^{\pi(p)} \quad and,$$

$$\prod_{i=0}^{\pi(p)-1} \|Df^{-m}|_{E^u(f^{-im}(p))}\| < \lambda^{\pi(p)}.$$

(b) $\|Df^m|_{E^s(p)}\| \cdot \|Df^{-1}|_{E^u(f^m(p))} < \lambda^2$,

where $\pi(p)$ denotes the period of p.

In [53], Mañé gave a result on the approximation of periodic orbit from a theoretical viewpoint. We say that a point $x \in M$ is *well-closable* for $f \in \mathrm{Diff}(M)$ if, for any $\epsilon > 0$, there are $g \in \mathrm{Diff}(M)$ with $d_1(f,g) < \epsilon$ and $p \in M$ such that $d(f^n(x), g^n(p)) < \epsilon$ for any $0 \le n \le \pi(p)$, where $\pi(p)$ is the period of p, and d_1 is the C^1 metric. Let Σ_f denote the set of well-closable points of f. In [53], Mañé showed that for any f-invariant Borel probability measure μ on M, $\mu(\Sigma_f) = 1$. Let $\mathcal{M}$ be the space of all Borel measures μ on M with the weak*-topology. Then, we know that for any ergodic measure $\mu \in \mathcal{M}$ of f, μ is supported on a periodic orbit $Orb(p) = \{p, f(p), \ldots, f^{\pi(p)-1}(p)\}$ if and only if

$$\mu = \frac{1}{\pi(p)} \sum_{i=0}^{\pi(p)-1} \delta_{f^i(p)},$$

where δ_x is the atomic measure respecting x.

Lemma 2.14 ([1, Theorem 3.8])**.** *There is a residual set $\mathcal{G}_9 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_9$, any ergodic measure μ_n of f, there is a sequence of periodic orbit $Orb(p_n)$ such that $\mu_n \to \mu$ in weak* topology and $Orb(p_n) \to Supp(\mu)$ in the Hausdorff metric.*

Lemma 2.15 ([54, Lemma 1.5])**.** *Let $\Lambda \subset M$ be a closed f-invariant set and $E \subset T_\Lambda M$ be a continuous invariant subbundle. If there exists $m > 0$ such that*

$$\int \log \|Df^m|_E\| d\mu < 0$$

for any ergodic $\mu \in \mathcal{M}(f^m|_\Lambda)$, then E is contracting, where $\mathcal{M}(f^m|_\Lambda)$ is the set of invariant probabilities on the Borel σ-algebra of Λ.

Proof of Theorem A. Let $f \in \mathcal{G} = \mathcal{G}_8 \cap \mathcal{G}_9$ and let $C(f)$ be a locally maximal chain transitive set of f. Suppose that f has the eventual shadowing property on $C(f)$. Since $f \in \mathcal{G}$, and $C(f)$ is locally maximal, we know that $C(f) = H_f(p)$ for some hyperbolic periodic point p. Then, by Proposition 2.12, $f \in \mathcal{F}(C(f)) = \mathcal{F}(H_f(p))$. Thus, by Proposition 2.13, $T_{C(f)(=H_f(p))}M = E \oplus F$ with $\dim E = \mathrm{index}(p)$. Suppose, by contradiction, that E is not contracting (the other case is similar). Let $\mu \in \mathcal{M}(f|_{H_f(p)})$ such that μ is an ergodic measure supported on $H_f(p)$. Take $p_n \in Orb(p_n)$ with period $\pi(p_n)$. For simplicity, we assume that $f^{\pi(p_n)}(p_n) = f(p_n) = p_n$. Then, by Lemma 2.14, we have

$$\int \|Df|_E\| d\mu = \lim_{n\to\infty} \int \|Df|_{E^s(p_n)}\| d\mu_{p_n} < 0.$$

Thus, by Lemma 2.15, E is contracting. This is a contradiction. Thus, if f has the eventual shadowing property on $C(f)$, then $C(f)$ is hyperbolic. □

3. Proof of Theorem B

We define the strong stable and unstable manifolds of a hyperbolic periodic point p respectively as follows:

$$W^{ss}(p) = \{x \in M : d(X^t(x), X^t(p)) \to 0 \text{ as } t \to \infty\}$$

and

$$W^s(Orb_X(p)) = \bigcup_{t\in\mathbb{R}} W^{ss}(X^t(p)),$$

where $Orb_X(p)$ is the orbit of p. If $\epsilon > 0$, the local strong stable manifold is defined as

$$W^{ss}_{\epsilon(p)}(p) = \{x \in M : d(X^t(x), X^t(p)) < \epsilon \text{ as } t \geq 0\}.$$

By the stable manifold theorem, there is an $\epsilon = \epsilon(p) > 0$ such that

$$W^{ss}(p) = \bigcup_{t\geq 0} X^{-t}(W^{ss}_{\epsilon(p)}(X^t(p))).$$

We can define this similarly for the unstable manifolds.

If σ is a hyperbolic singularity of X, then there exists an $\epsilon = \epsilon(\sigma) > 0$ such that

$$W^s_\epsilon(\sigma) = \{x \in M : d(X^t(x), \sigma) \leq \epsilon \text{ as } t \geq 0\} \text{ and}$$

$$W^s(\sigma) = \bigcap_{t\geq 0} X^t(W^s_\epsilon(\sigma)).$$

Analogous definitions hold for unstable manifolds.

Lemma 3.1. *If $X \in \mathfrak{X}(M)$ has the eventual shadowing property on a locally maximal set $C(X)$, then for any hyperbolic $\gamma, \eta \in C(X) \cap Crit(X)$, we have $W^s(\gamma) \cap W^u(\eta) \neq \emptyset$ and $W^u(\gamma) \cap W^s(\eta) \neq \emptyset$.*

Proof. Let $\gamma, \eta \in C(X) \cap Crit(X)$ be hyperbolic. Then, we show three cases for the orbits.

Case 1. We consider that $\gamma, \eta \in C(X) \cap Per(X)$ are hyperbolic. Let $p \in \gamma$ and $q \in \eta$. Take $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$. Let $0 < \delta = \delta(\epsilon) \leq \epsilon$ be the number of the eventual shadowing property for X. Since $C(X)$ is a chain transitive set of X, there is a finite $(\delta, 1)$-pseudo-orbit of X such that $x_0 = p$ and $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for $t_i \geq 1, i = 0, \ldots, n-1$ and $x_n = q$. Then, we construct a $(\delta, 1)$-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\} \subset C(X)$ such that (i) $x_i = X^i(p)$, $t_i = 1, i \leq 0$,

(ii) $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for $t_i \geq 1$ and $i = 0, \ldots, n-1$ and (iii) $x_i = X^i(q)$, $t_i = 1$, $i \geq n$. Then, $\{(x_i, t_i) : t_i = 1, i \in \mathbb{Z}\} = \{\ldots, p, x_0(= p), x_1, x_2, \ldots, x_n(= q), q, q, \ldots\} \subset C(X)$ is a $(\delta, 1)$-pseudo-orbit of X. Since X has the eventual shadowing property on a locally maximal set $C(X)$, there is a point $z \in C(X)$, $t_n \in \mathbb{R}$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that

$$(3) \qquad d(X^{h(t)}(z), X^{t-s_{-n-i}}(x_{-n-i})) < \epsilon,\ s_{-n-i} < t < s_{-n-i+1},\ \text{and}$$

$$(4) \qquad d(X^{h(t)}(z), X^{t-s_{n+i}}(x_{n+i})) < \epsilon,\ s_{n+i} < t < s_{n+i+1},$$

where $s_{-n} = -t_0 - t_{-1} - \cdots - t_{-n}, s_0 = 0$, and $s_n = t_0 + t_1 + \cdots + t_n$. Since $t_i = 1$ for $i \leq 0$, $x_{-n} = X^{-n}(p)$ and therefore,

$$X^{t-s_{-n}}(x_{-n}) = X^{t+n}(X^{-n}(p)) = X^t(p).$$

Then, we have $d(X^{h(t)}(z), X^t(p)) < \epsilon$ for all $t < 0$. Since $x_n = q$, and $x_{n+i} = X^{t_i}(q) = X^i(q) = q$ for $t_i = 1$ and $i \geq 0$, we have $X^{t-s_n}(x_n) = X^{t-s_n}(q)$, and so $X^{t-s_{n+i}}(x_{n+i}) = X^{t-s_n}(X^i(q))$ for all $i \geq 0$. By (2) and (3), we have

$$Orb(z) \cap W^u(p) \cap W^s(q) \neq \emptyset.$$

Thus, $W^u(\gamma) \cap W^s(\eta) \neq \emptyset$. Similarly, we have $W^s(\gamma) \cap W^u(\eta) \neq \emptyset$.

Case 2. We consider that $\sigma_1, \sigma_2 \in C(X) \cap Sing(X)$ are hyperbolic. Since $\sigma_1, \in \sigma_2 \in C(X) \cap Sing(X)$ are hyperbolic, there are $\epsilon(\sigma_1) > 0$ and $\epsilon(\sigma_2) > 0$ such that $W^u_{\epsilon(\sigma_1)}(\sigma_1)$ and $W^s_{\epsilon(\sigma_2)}(\sigma_2)$ are well defined. Take $\epsilon = \min\{\epsilon(\sigma_1), \epsilon(\sigma_2)\}$ and let $0 < \delta \leq \epsilon$ be the number of the eventual shadowing property for X. Since $\sigma_1, \sigma_2 \in C(X) \cap Sing(X)$, there is a finite $(\delta, 1)$-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, i = 0, \ldots, k-1\} \subset C(X)$ such that $x_0 = \sigma_1$, $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for $i = 0, \ldots, k-1$ and $x_k = \sigma_2$. Construct the following sequence:

(i) $x_i = X^i(\sigma_1)$ for $t_i = 1$, $i \leq 0$,
(ii) $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for $t_i \geq 1$, $i = 0, \ldots, k-1$,
(iii) $x_i = X^i(\sigma_2)$ for $t_i = 1$, $i \geq k$.

Then, the sequence $\{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\} \subset C(X)$ is a $(\delta, 1)$-pseudo-orbit of X. Since X has the eventual shadowing property on a locally maximal set $C(X)$, there are a point $z \in C(X)$, $t_k \in \mathbb{R}$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that

$$(5) \qquad d(X^{h(t)}(z), X^{t-s_{-k-i}}(x_{-k-i})) < \epsilon,\ s_{-k-i} < t < s_{-k-i+1},\ \text{and}$$

$$(6) \qquad d(X^{h(t)}(z), X^{t-s_{k+i}}(x_{k+i})) < \epsilon,\ s_{k+i} < t < s_{k+i+1},$$

where $s_{-n} = -t_0 - t_{-1} - \cdots - t_{-n} - \cdots, s_0 = 0$, and $s_n = t_0 + t_1 + \cdots + t_n + \cdots$.

Note that by the construction of the $(\delta, 1)$-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\}$, we know $s_{k+i} = s_k + i$ and $s_{-k-i} = -k - i$ for $i \geq 0$. Since $X^{t-s_{-k-i}}(x_{-k-i}) = X^{t+k+i}(X^{-k-i}(\sigma_1)) = X^t(\sigma_1) = \sigma_1$, by (4), we know that

$$d(X^{h(t)}(z), X^{t-s_{-k-i}}(x_{-k-i})) = d(X^{h(t)}(z), X^t(\sigma_1)) = d(X^{h(t)}(z), \sigma_1) < \epsilon$$

for all $t < 0$. Then, we have $Orb(z) \cap W^u(\sigma_1) \neq \emptyset$.

Since $X^{t-s_{k+i}}(x_{k+i}) = X^{t-s_k-i}(X^i(\sigma_2)) = X^{t-s_k}(\sigma_2) = \sigma_2$, by (5), we know that

$$d(X^{h(t)}(z), X^{t-s_{k+i}}(x_{k+i})) = d(X^{h(t)}(z), X^{t-s_k}(\sigma_2)) = d(X^{h(t)}(z), \sigma_2) < \epsilon$$

for all $t > k$. Then, we have $Orb(z) \cap W^s(\sigma_2) \neq \emptyset$. Thus, $Orb(z) \cap W^u(\sigma_1) \cap W^u(\sigma_2) \neq \emptyset$. Similarly, we have $W^s(\sigma_1) \cap W^u(\sigma_2) \neq \emptyset$.

Case 3. We consider that $\sigma \in C(X) \cap Sing(X)$ and $p \in \gamma \in C(X) \cap Per(X)$ are hyperbolic. This proof is similar to those of Cases 1 and 2. Thus, we have $W^s(\sigma) \cap W^u(\gamma) \neq \emptyset$ and $W^u(\sigma) \cap W^s(\gamma) \neq \emptyset$. □

Lemma 3.2 ([57, Lemma 7]). *There is a residual set $\mathcal{R}_0 \subset \mathfrak{X}(M)$ such that given any chain transitive set $C(X)$ of $X \in \mathcal{R}_0$, if $C(X)$ is locally maximal and $W^s(\gamma) \cap W^u(\eta) \neq \emptyset$ for any hyperbolic $\gamma, \eta \in C(X) \cap Crit(X)$, then $C(X) \cap Sing(X) = \emptyset$.*

Lemma 3.3 ([57, Theorem 9]). *There is a residual set $\mathcal{R}_1 \subset \mathfrak{X}(M)$ such that given any chain transitive set $C(X)$ of $X \in \mathcal{R}_1$, if $C(X)$ is locally maximal and $W^s(\gamma) \cap W^u(\eta) \neq \emptyset$ for any $\gamma, \eta \in P(X)$, then $C(X)$ is a transitive hyperbolic set.*

We say that $X \in \mathfrak{X}(M)$ is *Kupka-Smale* if every $p \in Crit(X)$ is hyperbolic and its invariant manifolds intersect transversely. Denote by $\mathcal{KS}$ the set of Kupka-Smale vector fields. It is known that $\mathcal{KS}$ is a residual set of $\mathfrak{X}(M)$ (see [23]).

Proof of Theorem B. Let $X \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{KS}$ and $C(X)$ be a locally maximal chain transitive set of X. Suppose that X has the eventual shadowing property on $C(X)$. Since $X \in \mathcal{KS}$ and $C(X)$ is locally maximal, every critical point in $C(X)$ is hyperbolic. Since X has the eventual shadowing property on $C(X)$, by Lemmas 3.1, 3.2, and 3.3, we know $C(X) \cap Sing(X) = \emptyset$, and $C(X)$ is transitive hyperbolic. □

4. Conservative systems

4.1. Volume-preserving diffeomorphisms

Let M be a closed smooth manifold with $\dim M \geq 3$, let μ denote the Lebesgue measure induced by the Riemannian volume form on M, and let $\mathrm{Diff}_\mu(M)$ denote the set of volume-preserving diffeomorphisms defined on M. Consider this space endowed with the C^1 Whitney topology. For a point $x \in M$, we say that x is a *nonwandering point* if, for any neighborhood U of x, there is $n \in \mathbb{Z}$ such that $f^n(U) \cap U \neq \emptyset$. Denote by $\Omega(f)$ the set of all nonwandering points of f. It is clear that $\overline{P(f)} \subset \Omega(f)$, where $P(f)$ is the set of periodic points of f, and $\overline{P(f)}$ is the closure of $P(f)$. We say that f satisfies *Axiom A* if $\Omega(f) = \overline{P(f)}$ is hyperbolic. In the volume-preserving case, by the Poincaré Recurrence Theorem, we have $\Omega(f) = M$. Thus, if f satisfies Axiom A, then f is Anosov.

Lemma 4.1 ([12, Theorem 1.3]). *There is a residual set $\mathcal{T}_1 \subset \mathrm{Diff}_\mu(M)$ such that for any $f \in \mathcal{T}_1$, f is transitive.*

We say that $f \in \mathrm{Diff}_\mu(M)$ is *Kupka-Smale* if every periodic point is hyperbolic and its invariant manifolds intersect transversely. Robinson [58] showed that the set of Kupka-Smale volume-preserving diffeomorphisms is a C^1-residual subset of $\mathrm{Diff}_\mu(M)$. Denote by $\mathcal{K}_\mu$ the Kupka-Smale volume-preserving diffeomorphisms. The following lemma was proved by Bessa, Lee, and Wen [11].

Lemma 4.2 ([11, Proposition 2.4]). *There is a residual set $\mathcal{T}_2 \subset \mathrm{Diff}_\mu(M)$ such that for any $f \in \mathcal{T}_2$, if there is g C^1 close to f such that g has two hyperbolic periodic points p, q with different indices, then f has two hyperbolic periodic points p_f, q_f with different indices.*

Lemma 4.3. *There is a residual set $\mathcal{T}_3 \subset \mathrm{Diff}_\mu(M)$ such that for any $f \in \mathcal{T}_3$, if f has the eventual shadowing property, then there is $\delta > 0$ such that for any $p \in P(f)$, p is not a δ weak hyperbolic periodic point.*

Proof. Let $f \in \mathcal{T}_3 = \mathcal{T}_1 \cap \mathcal{T}_2 \cap \mathcal{K}_\mu$ have the eventual shadowing property. Suppose, by contradiction, that for any $\delta > 0$, there is $p \in P(f)$ such that p is a δ weak hyperbolic periodic point. Since $f \in \mathcal{T}_1$, f is transitive, and so it is chain transitive. As in the proof of Lemma 2.8, there is g C^1 close to f such that g has two hyperbolic periodic points p and q with $\mathrm{index}(p) \neq \mathrm{index}(q)$. Since $f \in \mathcal{T}_3$, by Lemma 4.2, f has two hyperbolic periodic points p_f and q_f with $\mathrm{index}(p_f) \neq \mathrm{index}(q_f)$. Since f is chain transitive and f has the eventual shadowing property, as in the previous section for Lemma 2.4, we have $W^s(s) \cap W^u(r) \neq \emptyset$ and $W^u(s) \cap W^s(r) \neq \emptyset$ for any $s, r \in P(f)$. Since $f \in \mathcal{K}_\mu$, $\mathrm{index}(s) = \mathrm{index}(r)$ for all $s, r \in P(f)$. This is a contradiction. Thus, if $f \in \mathcal{T}_3$ has the eventual shadowing property, then every $p \in P(f)$ is not a δ weak hyperbolic periodic point. □

Lemma 4.4 ([11, Lemma 2.8]). *There is a residual set $\mathcal{T}_4 \subset \mathrm{Diff}_\mu(M)$ such that for any $f \in \mathcal{T}_4$, for any $\delta > 0$, if any C^1 neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}_\mu(M)$, there is $g \in \mathcal{U}(f)$ and a hyperbolic $p \in P(g)$ such that p is not a δ weak hyperbolic periodic point, then there is a hyperbolic $p_f \in P(f)$ such that p_f is not a 2δ weak hyperbolic periodic point.*

We say that $f \in \mathrm{Diff}_\mu(M)$ is a *star* if there is a C^1 neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}_\mu(M)$ such that for any $g \in \mathcal{U}(f)$, every $p \in P(g)$ is hyperbolic. Denote by $\mathcal{F}_\mu(M)$ the set of all star diffeomorphisms. Newhouse [55] proved that if $f \in \mathcal{F}_\mu(M)$ and $\dim M + 2$, then f is Anosov. For any dimensional case, Arbieto and Catalan [7] proved that if $f \in \mathrm{Diff}_\mu(M)$ is a star, then it is Anosov.

Theorem 4.5 ([7, Theorem 1.1]). *Let $f \in \mathrm{Diff}_\mu(M)$. If $f \in \mathcal{F}_\mu(M)$, then f is Anosov.*

Theorem C. *For C^1-generic $f \in \mathrm{Diff}_\mu(M)$, if f has the eventual shadowing property, then it is Anosov.*

Proof of Theorem C. Let $f \in \mathcal{T}_3 \cap \mathcal{T}_4$ have the eventual shadowing property. If $f \in \mathrm{Diff}_\mu(M)$ is a star, then it is Anosov. To prove this, it is enough to show that $f \in \mathcal{F}_\mu(M)$. By contradiction, we may assume that $f \notin \mathcal{F}_\mu(M)$. Then, for any $\delta > 0$, there is g C^1 close to f such that g has a periodic point p that is a $\delta/2$ weak hyperbolic periodic point. Since $f \in \mathcal{T}_4$, f has a periodic point p_f that is a δ weak hyperbolic periodic point. This is a contradiction, since $f \in \mathcal{T}_3 \cap \mathcal{T}_4$ has the eventual shadowing property. By Lemma 4.3, every $p \in P(f)$ is not a δ weak hyperbolic periodic point. Thus, if $f \in \mathcal{T}_3 \cap \mathcal{T}_4$ has the eventual shadowing property, then it is Anosov. □

4.2. Divergence-free vector fields

Let M be a closed smooth manifold with $\dim M \geq 4$ and let μ denote the Lebesgue measure induced by the Riemannian volume form on M. Consider this space endowed with the C^1 Whitney topology. Given a C^r ($r \geq 1$) vector field $X\colon M \to TM$, the solution of the equation $x' = X(x)$ generates a C^r flow X^t; by the other side, given a C^r flow, we can define a C^{r-1} vector field by considering $X(x) = \frac{dX^t(x)}{dt}|_{t=0}$. We say that X is *divergence-free* if its divergence is equal to zero, that is, $\nabla \cdot X = 0$ or equivalently, if the measure μ is invariant for the associated flow.

Let $\mathfrak{X}_\mu(M)$ denote the space of C^1 divergence-free vector fields and consider the usual C^1 Whitney topology on this space. Bessa *et al.* [11] proved that C^1-generically, if a divergence-free vector field X is expansive, then it is Anosov.

Lemma 4.6 ([9, Theorem 1.1])**.** *There is a residual set $\mathcal{S}_1 \subset \mathfrak{X}_\mu(M)$ such that for any $X \in \mathcal{S}_1$, X is transitive. Moreover, it is mixing.*

We say that $X \in \mathfrak{X}_\mu(M)$ is *Kupka-Smale* if any element of $Crit(X)$ is hyperbolic and its invariant manifolds intersect transversely. Robinson [58] showed that the set of Kupka-Smale divergence-free vector fields is a C^1-residual subset of $\mathfrak{X}_\mu(M)$. Denote by $\mathcal{KS}_\mu$ the Kupka-Smale divergence-free vector fields.

Lemma 4.7. *There is a residual set $\mathcal{S}_2 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{S}_2$, if X has the eventual shadowing property, then $Sing(X) = \emptyset$.*

Proof. Let $X \in \mathcal{S}_2 = \mathcal{S}_1 \cap \mathcal{KS}_\mu$ have the eventual shadowing property. Since $X \in \mathcal{S}_1$, X is transitive, and so X is chain transitive. Since $X \in \mathcal{KS}_\mu$, as in the proofs of Lemma 3.1 and [57, Lemma 7], for any $p, q \in Crit(X)$, we have $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$, and therefore $Sing(X) = \emptyset$. □

Lemma 4.8. *For $X \in \mathcal{S}_2$, if X has the eventual shadowing property, then for any $\gamma, \eta \in Per(X)$,*

$$\mathrm{index}(\gamma) = \mathrm{index}(\eta).$$

Proof. Let $X \in \mathcal{S}_2$ have the eventual shadowing property. Since X is chain transitive, as in the proof of Lemma 3.1, for any hyperbolic $\gamma, \eta \in Per(X)$, we have

$$W^s(\gamma)\cap W^u(\eta)\neq\emptyset, \text{ and } W^u(\gamma)\cap W^u(\eta)\neq\emptyset.$$

Since $X\in\mathcal{KS}_\mu$, $W^s(\gamma)\pitchfork W^u(\eta)\neq\emptyset$, and $W^u(\gamma)\pitchfork W^u(\eta)\neq\emptyset$, and so index$(\gamma)=$ index(η). □

Lemma 4.9 ([11, Lemma 4.4]). *There is a residual set $\mathcal{S}_3\subset\mathfrak{X}_\mu(M)$ such that for any $X\in\mathcal{S}_3$, if for any C^1 neighborhood $\mathcal{U}(X)$ there are $Y\in\mathcal{U}(X)$ and two hyperbolic periodic orbits $\gamma,\eta\in Per(Y)$ such that* index$(\gamma)\neq$ index(η)*, then X has two hyperbolic periodic orbits $\gamma_X,\eta_X\in Per(X)$ such that* index$(\gamma_X)\neq$ index(η_X).

Lemma 4.10. *There is a residual set $\mathcal{S}_4\subset\mathfrak{X}_\mu(M)$ such that for any $X\in\mathcal{S}_4$, if X has the eventual shadowing property, then there is $\delta>0$ such that every $p\in\gamma\in Per(X)$ is not a δ weak hyperbolic periodic point.*

Proof. Let $X\in\mathcal{S}_4=\mathcal{S}_2\cap\mathcal{S}_3$ have the eventual shadowing property. Suppose, by contradiction, that for any $\delta>0$ there is a point $p\in\gamma\in Per(X)$ such that p is a δ weak hyperbolic periodic point. Then, by [11, Lemma 4.6], there is Y C^1 close to X such that Y has two orbits $\gamma,\eta\in Per(Y)$ with index$(\gamma)\neq$ index(η). Since $X\in\mathcal{S}_3$, X has two orbits $\gamma_X,\eta_X\in Per(X)$ with index$(\gamma_X)\neq$ index(η_X). Since X has the eventual shadowing property, by Lemma 4.8, index$(\gamma_X)=$ index(η_X). This is a contradiction. □

Lemma 4.11 ([11, Lemma 4.9]). *There is a residual set $\mathcal{S}_5\subset\mathfrak{X}_\mu(M)$ such that for any $X\in\mathcal{S}_5$, if for any C^1 neighborhood $\mathcal{U}(X)$ of X, there are $Y\in\mathcal{U}(X)$ and $p\in\gamma\in Per(Y)$ such that p is a δ weak hyperbolic periodic point, then there is a $p\in\gamma_f\in Per(X)$ such that p_f is a 2δ weak hyperbolic periodic point.*

A divergence-free vector field $X\in\mathfrak{X}_\mu(M)$ is said to be a *star* if there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that for any $Y\in\mathcal{U}(X)$, every $p\in Crit(X)$ is hyperbolic. The set of star divergence-free vector fields is denoted by $\mathcal{G}^*_\mu(M)$. Ferreira [17] proved the following:

Theorem 4.12 ([17, Theorem 1]). *Let $X\in\mathfrak{X}_\mu(M)$. If $X\in\mathcal{G}^*_\mu(M)$, then $Sing(X)=\emptyset$ and X is Anosov.*

Theorem D. *For C^1-generic $X\in\mathfrak{X}_\mu(M)$, if X has the eventual shadowing property, then it is Anosov.*

Proof of Theorem D. Let $X\in\mathcal{S}_4\cap\mathcal{S}_5$ have the eventual shadowing property. Suppose, by contradiction, that $X\notin\mathcal{G}^*_\mu(M)$. Then, for any $\delta>0$, there is Y C^1 close to X such that Y has a $\delta/2$ weak hyperbolic periodic point $p\in\gamma\in Per(Y)$. Since $X\in\mathcal{S}_5$, by Lemma 4.11, $p_f\in\gamma_f\in Per(X)$ is a δ weak hyperbolic periodic point. Since $X\in\mathcal{S}_4$ and X has the eventual shadowing property, by Lemma 4.10, this is a contradiction. Thus, if $X\in\mathcal{S}_4\cap\mathcal{S}_5$ has the eventual shadowing property, then by Lemma 4.7 and Theorem 4.12, X is transitive Anosov. □

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