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GROUND STATE SIGN-CHANGING SOLUTIONS FOR A CLASS OF SCHRÖDINGER-POISSON-KIRCHHOFF TYPE PROBLEMS WITH A CRITICAL NONLINEARITY IN $\mathbb{R}^3$

Aixia Qian and Mingming Zhang

Abstract. In the present paper, we are concerned with the existence of ground state sign-changing solutions for the following Schrödinger-Poisson-Kirchhoff system

$$\begin{cases} -(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\triangle u+V(x)u+k(x)\phi u=\lambda f(x)u+|u|^4u, & \text{in } \mathbb{R}^3,\\ -\triangle\phi=k(x)u^2, & \text{in } \mathbb{R}^3,\end{cases}$$

where $b > 0$, $V(x)$, $k(x)$ and $f(x)$ are positive continuous smooth functions; $0 < \lambda < \lambda_1$ and λ_1 is the first eigenvalue of the problem $-\triangle u + V(x)u = \lambda f(x)u$ in H. With the help of the constraint variational method, we obtain that the Schrödinger-Poisson-Kirchhoff type system possesses at least one ground state sign-changing solution for all $b > 0$ and $0 < \lambda < \lambda_1$. Moreover, we prove that its energy is strictly larger than twice that of the ground state solutions of Nehari type.

1. Introduction

In this paper, we consider the existence of ground state sign-changing solutions for the following Schrödinger-Poisson-Kirchhoff system

$$\text{(SKP)}\quad \begin{cases} -(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\triangle u+V(x)u+k(x)\phi u=\lambda f(x)u+|u|^4u, & \text{in } \mathbb{R}^3,\\ -\triangle\phi=k(x)u^2, & \text{in } \mathbb{R}^3,\end{cases}$$

where $b > 0$ and $V(x)$, $k(x)$, $f(x)$ are nonnegative, $0 < \lambda < \lambda_1$ and λ_1 is the first eigenvalue of the problem $-\triangle u + V(x)u = \lambda f(x)u$ in H (see (1.2) for the definition of the space H). For more mathematical and physical interpretation of the problem (SKP), we refer to [1, 8, 10, 12–14, 16, 23] and the references therein.

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When $b \equiv 0$ and $k(x) \equiv 0$ in the system (SKP), it reduces to the classic semilinear elliptic problem. Weth, Bartsch and Willem [2] obtained a ground state sign-changing solution. Remarkably, the system (SKP) is nonlocal because of the term $k(x)\phi u$ and $b(\int_{\mathbb{R}_3} |\nabla u|^2 dx)\triangle u$, which make the problem more complicated. This phenomenon provokes some mathematical difficulties, but it makes the study of the system (SKP) particulary interesting.

For the following Schrödinger-Poisson-Kirchhoff system

$$\begin{cases} -(1+b\int_{\mathbb{R}^3} |\nabla u|^2 dx)\triangle u + u + k(x)\phi u = g(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle\phi = k(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

Wang and Zhou [20] obtained a sign-changing solution in the case when $b=0$, k is a constant, and the nonlinearity $g(u) = |u|^{p-1}u$ satisfies 4-superlinear, subcritical growth condition on u. Huang, Rocha and Chen [11] studied the existence of sign-changing solutions for (1.1) with $b=0$, $k(x)=0$ and critical growth condition. If $b=0$, $k(x)\neq 0$, Zhong and Tang [26] studied the existence of ground state sign-changing solutions for the system (1.1) with critical growth. If $b\neq 0$, $k(x)=0$, Chen and Tang [10] studied the existence and the asymptotic behavior of ground state sign-changing solutions for the system (1.1). However, the above work obtained a sign-changing solution only in the case that the system (1.1) does not involve the nonlocal term or the nonlinearity does not satisfy the critical growth condition.

Comparing with the previous works, we investigate the existence of ground state sign-changing solutions for (SKP) with a critical nonlinearity. In this paper, we denote $\phi(u) = \int_{\mathbb{R}^3} \phi_u u^2 dx$, therefore the system (SKP) possesses two nonlocal terms $k(x)\phi_u u$ and $b(\int_{\mathbb{R}^3} |\nabla|u|)\triangle u$. We adopt an idea from [26] and [10]. However, [26] and [10] has only one non-local term and [10] is not the case of critical growth, so our construction methods are different from them. Moreover, regarding the existence of the ground state and sign-changing solutions for the Schrödinger-Poisson-Kirchhoff systems with critical growth, to the best of our knowledge, few works concern on this up to now.

To avoid much details for checking the compactness, for $V(x)$ not being a constant, we always assume that the potential function $V(x)$ satisfies
(V) $V \in \mathrm{C}(\mathbb{R}^3, \mathbb{R}^+)$ such that $H \subset H^1(\mathbb{R}^3)$ and for $2 < s < 6$ the embedding $H \hookrightarrow L^s(\mathbb{R}^3)$ is compact, where

$$H := \begin{cases} H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|)\}, & \text{if } V(x) \text{ is a constant}, \\ \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx < \infty\}, & \text{if } V(x) \text{ is not a constant}, \end{cases} \tag{1.2}$$

with norm

$$\|u\| := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx \right)^{\frac{1}{2}};$$
$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

with norm

$$\|u\|_{\mathrm{H}^1} := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx\right)^{\frac{1}{2}};$$

$$\mathrm{D}^{1,2}(\mathbb{R}^3) := \{u \in L^{2^*}(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

with norm

$$\|u\|_{\mathrm{D}^{1,2}} := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

As usual, for $1 \leq s < +\infty$,

$$\|u\|_s := \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{\frac{1}{s}}, \quad u \in L^s(\mathbb{R}^3).$$

We first give assumptions about $f(x)$ and $k(x)$.

(K) $k \in L^p(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \setminus \{0\}$ for some $p \in [2, +\infty)$ and $k(x) > 0$ for $\forall x \in \mathbb{R}^3$.

(f_1) $f \in L^{\frac{3}{2}}(\mathbb{R}^3) \setminus \{0\}$ and $f(x) > 0$ for $\forall x \in \mathbb{R}^3$.

(f_2) there exist $\rho > 0$ and $\alpha > 0$ such that $f(x) \geq C|x|^{-\alpha}$ for $|x| < \rho$.

Let S be the best Sobolev constant for the embedding of $\mathrm{D}^{1,2}(\mathbb{R}^3)$ in $L^{2^*}(\mathbb{R}^3)$. Particularly,

$$S := \inf_{\mathrm{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{(\int_{\mathbb{R}^3} u^6 dx)^{\frac{1}{3}}}, \quad |u|_6^2 \leq S^{-1} \int_{\mathbb{R}^3} |\nabla u|^2 dx. \tag{1.3}$$

It is well known that S is achieved by the Talenti function [22]

$$u_\epsilon = \frac{(\epsilon)^{\frac{1}{4}}}{(\epsilon + |x|^2)^{\frac{1}{2}}} \in \mathrm{D}^{1,2}(\mathbb{R}^3). \tag{1.4}$$

We can also get that $\int_{\mathbb{R}^3} |\nabla u_\epsilon|^2 dx = |u_\epsilon|_6^6 = S^{\frac{3}{2}}$.

As is well known to us, for $u \in H$, Lax-Milgram theorem implies that there exists a unique $\phi_u \in \mathrm{D}^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v = \int_{\mathbb{R}^3} k(x) u^2 v dx \quad \text{for } \forall v \in \mathrm{D}^{1,2}(\mathbb{R}^3), \tag{1.5}$$

that is, ϕ_u is the weak solution of $-\Delta\phi = k(x)u^2$. Moreover we have

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{k(x)u^2(y)}{4\pi|x-y|} dy,$$

$$T_{\phi_u}(u) = \int_{\mathbb{R}^3} k(x)\phi_u u^2 dx = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{k(x)k(y)u^2(x)u^2(y)}{|x-y|} dxdy, \tag{1.6}$$

clearly $\phi_u(x) \geq 0$ for any $x \in \mathbb{R}^3$.

Define the energy functional I_λ on the space H by

$$\begin{aligned} I_\lambda(u) = {} & \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3} k(x)\phi_u u^2 dx \\ & - \frac{\lambda}{2}\int_{\mathbb{R}^3} f(x)u^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} u^6 dx. \end{aligned} \tag{1.7}$$

Then the I_λ is well defined on H and belongs to $C^1(H,\mathbb{R})$. For each $u, v \in H$, we have

$$\begin{aligned}\langle I'_\lambda(u), v\rangle = &(1+b|\nabla u|_2^2)\int_{\mathbb{R}^3}\nabla u\nabla v dx + \int_{\mathbb{R}^3}V(x)uvdx + \int_{\mathbb{R}^3}k(x)\phi_u uvdx\\ &-\lambda\int_{\mathbb{R}^3}f(x)uvdx - \int_{\mathbb{R}^3}|u|^4uvdx.\end{aligned}\tag{1.8}$$

Critical points of the functional I_λ correspond to the weak solutions for nonlocal problem (SKP). Furthermore, if $u \in H$ is a critical point of I_λ, (u, ϕ_u) is a solution of the system (SKP). Since $\phi_u(x) \geq 0$, then (u, ϕ_u) is a sign-changing solution of the system (SKP) if and only if u is a critical point of I_λ and $u^\pm \neq 0$, where

$$u^+(x) = \max\{u(x), 0\}, \quad u^-(x) = \min\{u(x), 0\}.$$

Next we give an essential decomposition, which is useful in finding the ground state sign-changing solutions of the system (SKP). Firstly, it follows from (1.8) and Fubini theorem that

$$T_{\phi_{u^+}}(u^-) = \int_{\mathbb{R}^3}k(x)\phi_{u^+}|u^-|^2dx = \int_{\mathbb{R}^3}k(x)\phi_{u^-}|u^+|^2dx = T_{\phi_{u^-}}(u^+),$$

$$T_{\phi_u}(u) = \int_{\mathbb{R}^3}k(x)\phi_u u^2dx = T_{\phi_{u^+}}(u^+) + T_{\phi_{u^-}}(u^-) + 2T_{\phi_{u^+}}(u^-).\tag{1.9}$$

Then by a simple calculation, we can obtain

$$I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-) + \frac{b}{2}\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2 + \frac{1}{2}T_{\phi_{u^+}}(u^-),\tag{1.10}$$

$$\langle I'_\lambda(u), u^+\rangle = \langle I'_\lambda(u^+), u^+\rangle + T_{\phi_{u^+}}(u^-) + b\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2,\tag{1.11}$$

$$\langle I'_\lambda(u), u^-\rangle = \langle I'_\lambda(u^-), u^-\rangle + T_{\phi_{u^+}}(u^-) + b\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2.\tag{1.12}$$

Clearly, when $b = 0$ and $k(x) = 0$, the system (SKP) doesn't depend on the nonlocal term $\int_{\mathbb{R}^3}|\nabla u|^2dx\triangle u$ and $k(x)\phi_u u$ any more, that is, it becomes

$$\begin{cases} -\triangle u + V(x)u = \lambda f(x)u + |u|^4u, & \text{in } \mathbb{R}^3,\\ u \in H,\end{cases}\tag{1.13}$$

with energy functional

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^3}f(x)u^2dx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6dx, \quad \forall u \in H.\tag{1.14}$$

It is well defined on H and is of C^1 with

$$I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-),\ \langle I'_\lambda(u), u^+\rangle = \langle I'_\lambda(u^+), u^+\rangle,$$

$$\langle I'_\lambda(u), u^-\rangle = \langle I'_\lambda(u^-), u^-\rangle.\tag{1.15}$$

From (1.10), (1.11), (1.12) and (1.15), we can see that there are some essential differences in studying the sign-changing solutions of the problem (SKP) between $b = 0$, $k(x) = 0$ and $b \neq 0$, $k(x) \neq 0$, because of the so called nonlocal term $\int_{\mathbb{R}^3}|\nabla u|^2dx\triangle u$ and $k(x)\phi_u u$. Therefore, the methods of finding sign-changing solutions for the problem (1.13) seem to be not applicable to the system (SKP). In other words, our methods are different from those used in Zhang [24] and Huang etc [11]. Similarly, Zhong and Tang [26] studied the special case of (SKP), their methods of finding sign-changing solutions is also not applicable to the present problem.

In order to obtain the existence results of sign-changing solutions, we will consider the following minimization problem

$$M_\lambda := \{u \in H : u^\pm \neq 0, \langle I'_\lambda(u), u^+\rangle = \langle I'_\lambda(u), u^-\rangle = 0\},$$
$$m_\lambda := \inf\{I_\lambda(u) : u \in M_\lambda\}.$$

We will show that the minimizer of M_λ is corresponding to a sign-changing solution for the system (SKP). If (u, ϕ_u) is a sign-changing solution of the system (SKP), one can get $u \in M_\lambda$, and it is easier to study I_λ on M_λ.

Another aim of the article is to prove that the energy of any sign-changing solution in H of the system (SKP) is strictly lager than two times of the ground state solutions of (SKP). This property is called energy doubling by Weth in [21]. Motivated by [26] and [10], we can get the ground state solutions of the problem (SKP) as minimizers of the corresponding energy functional I_λ on the following manifold

$$N_\lambda := \{u \in H, u \neq 0, \langle I'_\lambda(u), u\rangle = 0\}$$

with

$$c_\lambda := \inf\{I_\lambda(u) : u \in N_\lambda\},$$

which play an active role in finding the ground state solutions of Nehari type for (SKP).

Our main results can be stated by the following theorems.

Theorem 1.1. *If the assumptions* (V), (K) *and* (f_1), (f_2) *hold,* $b > 0$, $0 < \lambda < \lambda_1$ *and* $\frac{3}{2} < \alpha < 2$, *then the system* (SKP) *has at least one ground state sign-changing solution which has precisely two nodal domains.*

Theorem 1.2. *Under the assumptions of Theorem 1.1, we can get that* $I_\lambda(u_\lambda) > 2c_\lambda$, *where* u_λ *is the ground state sign-changing solution obtained in Theorem 1.1. Especially* c_λ *is achieved either by a positive or a negative function.*

In fact, Theorem 1.2 indicates that the energy of any sign-changing solution for (SKP) is strictly larger than twice of the least energy.

Remark 1.3. (i) Under the assumption (f_1), we can obtain that $\lambda_1 > 0$.

(ii) (f_2) can help us estimate the least energy level m_λ of the functional I_λ on M_λ and make I_λ meet the $(PS)_{m_\lambda}$-condition.

We organize this paper as follows. In Section 2, we present some notations and prove some useful preliminary lemmas which provide the way for getting one ground state sign-changing solution. In Section 3, we finish the proof of Theorem 1.1 and Theorem 1.2.

2. Some preliminaries

In this section, we give some preliminary lemmas which are essential to demonstrate our results.

Lemma 2.1 ([3, 7, 18, 25]). *For each $u \in H$, there exists a unique element $\phi_u \in H$ such that $-\triangle\phi_u = k(x)u^2$, especially ϕ_u has the following properties.*

(i) *Suppose that $k \in L^\infty(\mathbb{R}^3)$, then for each $u \in H$, there exists $C > 0$ such that $\|\phi_u\| \le C\|u\|^2$ and $T_{\phi_u}(u) = \int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx \le C\|u\|^4_{\frac{12}{5}} \le C\|u\|^4$;*

(ii) *$\phi_u \ge 0$ for a.e $x \in \mathbb{R}^3$, Moreover, $\phi_u > 0$ in $\mathbb{R}^3$ when $u \ne 0$;*

(iii) *$\phi_{tu} = t^2\phi_u$, $\forall t > 0$;*

(iv) *ϕ maps bounded sets into bounded sets;*

(v) *If $u_n \rightharpoonup u$ weakly in $\mathrm{H}^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ weakly in $\mathrm{D}^{1,2}(\mathbb{R}^3)$; $\phi_{u_n} \to \phi_u$ strongly in $L^6_{loc}(\mathbb{R}^3)$.*

Lemma 2.2. *If hypotheses (f_1) and (K) hold, then*

(i) *the functional $F : u \in H \to \int_{\mathbb{R}^3} f(x)u^2 dx$ is weakly continuous; for any $u \in H$, $G : u \in H \to \int_{\mathbb{R}^3} f(x)uvdx$ is weakly continuous;*

(ii) *the functional $K : u \in H \to \int_{\mathbb{R}^3} k(x)\phi_u u^2 dx$ is weakly continuous; $Q : u \in H \to \int_{\mathbb{R}^3} k(x)\phi_u uvdx$ is weakly continuous.*

Proof. Since part (i) is easier than (ii), we omit it. For part 2, in view of the Sobolev embedding theorems and (v) of Lemma 2.1, by $u_n \rightharpoonup u$ weakly in H, we can deduce that

$$\begin{array}{llll} (1) & u_n \rightharpoonup u \text{ in } L^6(\mathbb{R}^3), & (2) & u_n^2 \to u^2 \text{ in } L^3_{loc}(\mathbb{R}^3), \\ (3) & \phi_{u_n} \rightharpoonup \phi_u \text{ in } \mathrm{D}^{1,2}(\mathbb{R}^3), & (4) & \phi_{u_n} \to \phi_u \text{ in } L^6_{loc}(\mathbb{R}^3). \end{array} \tag{2.1}$$

Hence, given $\epsilon > 0$, by (2.1)(3), one has that for n large enough

$$\left| \int_{\mathbb{R}^3} k(x)u^2(\phi_{u_n} - \phi_u)dx \right| \le \epsilon. \tag{2.2}$$

For each fixed v, by (2.1)(1), there holds

$$\left| \int_{\mathbb{R}^3} k(x)\phi_u v(u_n - u)dx \right| < \epsilon. \tag{2.3}$$

Especially, considering (2.1)(2) and (2.1)(4) respectively, we can claim that for each $\epsilon > 0, \rho > 0$ and for n large enough, there hold

$$|u_n^2 - u^2|_{3,B_\rho(0)} < \epsilon, \tag{2.4}$$

$$|\phi_{u_n} - \phi_u|_{6,B_\rho(0)} < \epsilon. \tag{2.5}$$

On the other hand, since $\{u_n\}$ is bounded in H and the Sobolev embedding $\mathrm{D}^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ is continuous, $\{\phi_{u_n}\}$ is bounded in $\mathrm{D}^{1,2}(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$ by (iv) of Lemma 2.1. Moreover, by (K), ku_n^2 and ku^2 belong to $L^{\frac{6}{5}}(\mathbb{R}^3)$, so for each $\epsilon > 0$, there exists $\bar\rho = \bar\rho(\epsilon) > 0$ such that

$$|k|_{2,\mathbb{R}^3\setminus B_\rho(0)} < \epsilon, \quad \forall\rho \ge \bar\rho. \tag{2.6}$$

Therefore, by (2.2), (2.4), (2.6), we have that for n large enough

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} k(x)\phi_{u_n}u_n^2dx - \int_{\mathbb{R}^3} k(x)\phi_u u^2dx\right| \\
\leq &\left|\int_{\mathbb{R}^3} k(x)\phi_{u_n}(u_n^2-u^2)dx + \int_{\mathbb{R}^3} k(x)u^2(\phi_{u_n}-\phi_u)dx\right| \\
\leq &\int_{\mathbb{R}^3} |k(x)\phi_{u_n}(u_n^2-u^2)|dx + \left|\int_{\mathbb{R}^3} k(x)u^2(\phi_{u_n}-\phi_u)dx\right| \\
\leq &|\phi_{u_n}|_6\left(\int_{\mathbb{R}^3}|k(x)(u_n^2-u^2)|^{\frac{6}{5}}dx\right)^{\frac{5}{6}}+\epsilon \\
\leq &C\left[\left(\int_{\mathbb{R}^3\setminus B_\rho(0)}|k(x)(u_n^2-u^2)|^{\frac{6}{5}}dx\right)^{\frac{5}{6}}+\left(\int_{B_\rho(0)}|k(x)(u_n^2-u^2)|^{\frac{6}{5}}dx\right)^{\frac{5}{6}}\right]+\epsilon \\
\leq &C\left(|k|^{\frac{6}{5}}_{2,\mathbb{R}^3\setminus B_\rho(0)}\cdot|u_n^2-u^2|_3^{\frac{6}{5}}+|k|_2^{\frac{6}{5}}\cdot|u_n^2-u^2|^{\frac{6}{5}}_{3,B_\rho(0)}\right)^{\frac{5}{6}}+\epsilon \\
\leq &C'\epsilon.
\end{aligned}$$

Similarly, by (2.3), (2.5), (2.6), one has that for n large enough

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} k(x)\phi_{u_n}u_n v dx - \int_{\mathbb{R}^3} k(x)\phi_u uv dx\right| \\
\leq &\int_{\mathbb{R}^3} |k(x)u_n v(\phi_{u_n}-\phi_u)|dx + \left|\int_{\mathbb{R}^3} k(x)\phi_u v(u_n-u)dx\right| \\
\leq &|u_n|_6\cdot|v|_6\left(\int_{\mathbb{R}^3}|k(x)(\phi_{u_n}-\phi_u)|^{\frac{3}{2}}dx\right)^{\frac{2}{3}}+\epsilon \\
\leq &C\left(|k|^{\frac{3}{2}}_{2,\mathbb{R}^3\setminus B_\rho(0)}\cdot|\phi_{u_n}-\phi_u|_6^{\frac{3}{2}}+|k|_2^{\frac{3}{2}}\cdot|\phi_{u_n}-\phi_u|^{\frac{3}{2}}_{6,B_\rho(0)}\right)^{\frac{2}{3}}+\epsilon \\
\leq &\tilde{C}\epsilon.
\end{aligned}$$

□

Lemma 2.3. *For each $\bar{s},\bar{t}>0$, if $\mu\in[\frac{1}{2},1]$, then the following system*

$$\begin{cases}\Phi(\bar{s},\bar{t})=\bar{s}-aS\mu^{-\frac{1}{3}}(\bar{s}+\bar{t})^{\frac{1}{3}}=0,\\ \Psi(\bar{s},\bar{t})=\bar{t}-bS^2\mu^{-\frac{2}{3}}(\bar{s}+\bar{t})^{\frac{2}{3}}=0\end{cases}$$

has a unique solution $(\bar{s}_0,\bar{t}_0)$. Moreover, if $\Phi(\bar{s},\bar{t})\geq 0$, $\Psi(\bar{s},\bar{t})\geq 0$, then $\bar{s}\geq\bar{s}_0$, $\bar{t}\geq\bar{t}_0$.

Proof. Similar to the proof of Lemma 3.6 [23], if $\Phi(\bar{s}_0,\bar{t}_0)=\Psi(\bar{s}_0,\bar{t}_0)=0$, then $\bar{s}_0+\bar{t}_0=\frac{\mu\bar{s}_0^3}{a^3S^3}$, where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$. So it follows from the second equality of system that

$$\left(\frac{\mu\bar{s}_0^3-a^3S^3\bar{s}_0}{a^3S^3}\right)^3=b^3\mu^{-2}S^6\left(\frac{\mu\bar{s}_0^3}{a^3S^3}\right)^2,$$

which implies that

$$\bar{s}_0 = \frac{abS^3 + a\sqrt{b^2S^6 + 4a\mu S^3}}{2\mu}$$

and

$$\bar{t}_0 = \frac{b^2S^3\sqrt{b^2S^6 + 4a\mu S^3}}{2\mu^2} + \frac{b^3S^6}{2\mu^2} + \frac{abS^3}{\mu}.$$

If $\Phi(\bar{s},\bar{t}) \geq 0,\ \Psi(\bar{s},\bar{t}) \geq 0$, then $(\bar{s}+\bar{t}) \geq aS\mu^{-\frac{1}{3}}(\bar{s}+\bar{t})^{\frac{1}{3}} + bS^2\mu^{-\frac{2}{3}}(\bar{s}+\bar{t})^{\frac{2}{3}}$. Define

$$e(l) = l - aS\mu^{-\frac{1}{3}}l^{\frac{1}{3}} - bS^2\mu^{-\frac{2}{3}}l^{\frac{2}{3}},\ l > 0.$$

Then $e(l)$ has a unique zero point $l_0 > 0$ and $e(l) \geq 0$, hence, $l \geq l_0$. Namely $\bar{s}+\bar{t} \geq \bar{s}_0 + \bar{t}_0$. If $\bar{s} < \bar{s}_0$, then

$$\Phi(\bar{s},\bar{t}) = \bar{s} - aS\mu^{-\frac{1}{3}}(\bar{s}+\bar{t})^{\frac{1}{3}} < \bar{s}_0 - aS\mu^{-\frac{1}{3}}(\bar{s}_0+\bar{t}_0)^{\frac{1}{3}},$$

which contradicts with $\Phi(\bar{s},\bar{t}) \geq 0$, so $\bar{s} \geq \bar{s}_0$. Similarly, $\bar{t} \geq \bar{t}_0$. □

Lemma 2.4. *If $0 < \lambda < \lambda_1, b > 0,\ u \in H$ with $u^{\pm} \neq 0$, then there exists a unique pair of positive numbers (s_u, t_u) such that $s_u u^+ + t_u u^- \in M_\lambda$. Moreover,*

$$I_\lambda(s_u u^+ + t_u u^-) = \max_{s,t\geq 0} I_\lambda(su^+ + tu^-).$$

Proof. We prove the theorem by three steps.

Step 1. Let $u \in H$ with $u^{\pm} \neq 0$. We will first prove the existence of (s_u, t_u). Define

$$\begin{aligned} g_u(s,t) = {} & s^2\|u^+\|^2 + bs^4\|\nabla u^+\|_2^4 + bs^2t^2\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2 \\ & + s^4T_{\phi_{u^+}}(u^+) + s^2t^2T_{\phi_{u^+}}(u^-) \\ & - \lambda s^2\int_{\mathbb{R}^3} f(x)|u^+|^2dx - s^6\int_{\mathbb{R}^3}|u^+|^6dx, \end{aligned} \tag{2.7}$$

$$\begin{aligned} e_u(s,t) = {} & t^2\|u^-\|^2 + bt^4\|\nabla u^-\|_2^4 + bs^2t^2\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2 \\ & + t^4T_{\phi_{u^-}}(u^-) + s^2t^2T_{\phi_{u^+}}(u^-) \\ & - \lambda t^2\int_{\mathbb{R}^3} f(x)|u^-|^2dx - t^6\int_{\mathbb{R}^3}|u^-|^6dx. \end{aligned} \tag{2.8}$$

For any fixed $t \geq 0$, it is easy to see that $g_u(0,t) = 0$, $g_u(s,s) > 0$, $e_u(s,s) > 0$ for $s > 0$ small enough and $g_u(t,t) < 0$, $e_u(t,t) < 0$ for $t > 0$ large enough. Then there exists $0 < y < Y$, such that

$$g_u(y,y) > 0,\quad e_u(y,y) > 0,\quad g_u(Y,Y) < 0,\quad e_u(Y,Y) < 0. \tag{2.9}$$

Thus we can obtain from (2.7), (2.8), (2.9) that

$$g_u(y,t) > 0,\quad g_u(Y,t) < 0,\quad \forall t \in [y,Y]. \tag{2, 10}$$

$$e_u(s,y) > 0,\quad e_u(s,Y) < 0,\quad \forall s \in [y,Y]. \tag{2.11}$$

By Miranda's theorem [19], there exists a unique pair (s_u, t_u) of positive numbers with $y < s_u, t_u < Y$ such that

$$g_u(s_u, t_u) = 0, \quad e_u(s_u, t_u) = 0,$$

which confirms that $s_u u^+ + t_u u^- \in M_\lambda$.

Step 2. We prove that the (s_u, t_u) is unique. Suppose $(\bar{s}, \bar{t})$ is another pair of positive numbers such that $\bar{s}u^+ + \bar{t}u^- \in M_\lambda$. Let $m = su^+ + tu^- \in M_\lambda$. Then $m^+ = su^+$, $m^- = tu^-$. Similarly, if $v = s_u u^+ + t_u u^-$, then $v^+ = s_u u^+$, $v^- = t_u u^-$. Therefore $w = \bar{s}u^+ + \bar{t}u^- = av^+ + dv^-$, where $a = \frac{\bar{s}}{s_u}$, $d = \frac{\bar{t}}{t_u}$. In view of the definition of the M_λ, we have that

$$\begin{cases} g_m(s,t) = \langle I'_\lambda(m), m^+\rangle = \langle I'_\lambda(m), (su^+) = 0\rangle, \\ e_m(s,t) = \langle I'_\lambda(m), m^-\rangle = \langle I'_\lambda(m), (tu^-) = 0\rangle. \end{cases} \tag{2.12}$$

By (2.12), we can deduce that

$$\begin{cases} g_v(1,1) = 0, \\ e_v(1,1) = 0. \end{cases} \tag{2.13}$$

Similarly there holds

$$\begin{cases} g_v(a,d) = g_w(1,1) = 0, \\ e_v(a,d) = e_w(1,1) = 0. \end{cases} \tag{2.14}$$

By (2.13), (2.14), we have that

$$\begin{cases} g_v(a,d) = g_w(1,1) = g_v(1,1) = 0, \\ e_v(a,d) = e_w(1,1) = e_v(1,1) = 0. \end{cases} \tag{2.15}$$

So in order to show the uniqueness, we only need to prove that $a = d = 1$. Suppose that $0 < a \le d$, by (2.7), (2.8), we have that

$$\begin{cases} g_v(a,a) \le g_v(a,d) = g_w(1,1) = 0, \\ e_v(d,d) \ge e_v(a,d) = e_v(1,1) = 0, \end{cases}$$

i.e.,

$$\begin{cases} \quad a^2\|v^+\|^2 + a^4[b\|\nabla v^+\|_2^4 + b\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2 + T_{\phi_v}(v^+)] \\ \le \lambda a^2 \int_{\mathbb{R}^3} f(x)|v^+|^2 dx + a^6 \int_{\mathbb{R}^3} |v^+|^6 dx, \\ \quad d^2\|v^-\|^2 + d^4[b\|\nabla v^-\|_2^4 + b\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2 + T_{\phi_v}(v^-)] \\ \ge \lambda d^2 \int_{\mathbb{R}^3} f(x)|v^-|^2 dx + d^6 \int_{\mathbb{R}^3} |v^-|^6 dx. \end{cases} \tag{2.16}$$

By a calculation, one has that

$$a^2 \geq \frac{b[\|\nabla v^+\|_2^4+\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2]+T_{\phi_v}(v^+)}{2\int_{\mathbb{R}^3}|v^+|^6dx} + \frac{\sqrt{[b\|\nabla v^+\|_2^4+b\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2+T_{\phi_v}(v^+)]^2+4\int_{\mathbb{R}^3}|v^+|^6dx(\|v^+\|^2-\lambda\int_{\mathbb{R}^3}f(x)|v^+|^2dx)}}{2\int_{\mathbb{R}^3}|v^+|^6dx}, \tag{2.17}$$

$$d^2 \leq \frac{b[\|\nabla v^-\|_2^4+\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2]+T_{\phi_v}(v^-)}{2\int_{\mathbb{R}^3}|v^-|^6dx} + \frac{\sqrt{[b\|\nabla v^-\|_2^4+b\|\nabla v^+\|_2^2\|\nabla v^-\|_2^2+T_{\phi_v}(v^-)]^2+4\int_{\mathbb{R}^3}|v^-|^6dx(\|v^-\|^2-\lambda\int_{\mathbb{R}^3}f(x)|v^-|^2dx)}}{2\int_{\mathbb{R}^3}|v^-|^6dx}. \tag{2.18}$$

From the fact $g_v(1,1) = e_v(1,1) = 0$, we can derive that

$$\begin{cases} \int_{\mathbb{R}^3}|v^+|^6dx = (\|v^+\|^2 - \lambda\int_{\mathbb{R}^3}f(x)|v^+|^2dx) \\ \qquad + b[\|\nabla v^+\|_2^4 + \|\nabla v^+\|_2^2\|\nabla v^-\|_2^2] + T_{\phi_v}(v^+), \\ \int_{\mathbb{R}^3}|v^-|^6dx = (\|v^-\|^2 - \lambda\int_{\mathbb{R}^3}f(x)|v^-|^2dx) \\ \qquad + b[\|\nabla v^-\|_2^4 + \|\nabla v^+\|_2^2\|\nabla v^-\|_2^2] + T_{\phi_v}(v^-). \end{cases} \tag{2.19}$$

It follows from (2.17), (2.18), and (2.19) that $a^2 \geq 1$, $d^2 \leq 1$. Together with the condition of $0 < a \leq d$, we can get that $a = b = 1$.

Step 3. We prove that $I_\lambda(s_u u^+ + t_u u^-) = \max_{s,t\geq 0} I_\lambda(su^+ + tu^-)$. Since

$$\begin{aligned} I_\lambda(su^+ + tu^-) = &\frac{s^2}{2}\|u^+\|^2 + \frac{s^4}{4}[b\|\nabla u^+\|_2^4 + T_{\phi_{u^+}}(u^+)] \\ &- \frac{\lambda}{2}s^2\int_{\mathbb{R}^3}f(x)|u^+|^2dx - \frac{1}{6}s^6\int_{\mathbb{R}^3}|u^+|^6dx \\ &+ \frac{t^2}{2}\|u^-\|^2 + \frac{t^4}{4}[b\|\nabla u^-\|_2^4 + T_{\phi_{u^-}}(u^-)] \\ &- \frac{\lambda}{2}t^2\int_{\mathbb{R}^3}f(x)|u^-|^2dx - \frac{1}{6}t^6\int_{\mathbb{R}^3}|u^-|^6dx \\ &+ \frac{s^2t^2}{2}T_{\phi_{u^+}}(u^-) + \frac{s^2t^2}{2}b\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2, \end{aligned}$$

let $I_\lambda(su^+ + tu^-) = h(s,t)$. It is easy to see that $h(s,t) > 0$ as $|(s,t)| \to 0$; $h(s,t) < 0$ as $|(s,t)| \to \infty$, which means the maximum point cannot be achieved on the boundary of $\mathbb{R}^2_+$. Without loss of generality, we only prove that $I_\lambda(s_u u^+) < I_\lambda(s_u u^+ + t_u u^-)$ or $I_\lambda(t_u u^-) < I_\lambda(s_u u^+ + t_u u^-)$. Here we prove that $I_\lambda(s_u u^+) < I_\lambda(s_u u^+ + t_u u^-)$. In fact

$$\frac{\partial}{\partial t}h(s_u, t) = t\|u^-\|^2 + t^3[b\|\nabla u^-\|_2^4 + T_{\phi_{u^-}}(u^-)] - \lambda t\int_{\mathbb{R}^3}f(x)|u^-|^2dx$$

$$- t^5 \int_{\mathbb{R}^3} |u^-|^6 dx + s_u^2 t[T_{\phi_{u^+}}(u^-) + b\|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2] > 0$$

with $t > 0$ small enough, implies that $h(s_u, t) = I_\lambda(s_u u^+ + tu^-)$ is an increasing function with respect to $t \in [0, \epsilon)$ for ϵ small enough. As a result $I_\lambda(s_u u^+) < h(s_u, t)$. So (s_u, t_u) is a positive maximum point for $h(s, t)$.

In view of above fact, we can know that $\frac{\partial}{\partial t_u} h(s_u, t_u) = \frac{\partial}{\partial s_u} h(s_u, t_u)$ is equivalent to $g_u(s_u, t_u) = e_u(s_u, t_u) = 0$. By the definition of M_λ, one has that $s_u u^+ + t_u u^- \in M_\lambda$. Together with the uniqueness of the (s_u, t_u), we can obtain that

$$I_\lambda(s_u u^+ + t_u u^-) = \max_{s,t \geq 0} I_\lambda(su^+ + tu^-).$$

□

Lemma 2.5. *If $0 < \lambda < \lambda_1$, $b > 0$, then for $\forall u \in H$ with $u^\pm \neq 0$, we have*

(i) *if $g_u(1,1) \leq 0$, $e_u(1,1) \leq 0$, then there exists unique pair $(s_u, t_u) \in (0,1] \times (0,1]$, such that $s_u u^+ + t_u u^- \in M_\lambda$.*

(ii) *if $g_u(1,1) \geq 0$, $e_u(1,1) \geq 0$, then there exists unique pair $(s_u, t_u) \in [1, +\infty] \times [1, +\infty)$, such that $s_u u^+ + t_u u^- \in M_\lambda$, where $g_u(s,t)$, $e_u(s,t)$ are given as* (2.7) *and* (2.8).

Proof. (i) If $u \in H$, $u^\pm \neq 0$, we can get that there exists a unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in M_\lambda$. Assume that $0 < s_u \leq t_u$, then one has that

$$\begin{aligned}
&t_u^2 \|u^-\|^2 + t_u^4 [b(\|\nabla u^-\|_2^4 + \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2) + T_{\phi_u}(u^-)] \\
&\geq t_u^2 \|u^-\|^2 + t_u^4 [b\|\nabla u^-\|_2^4 + T_{\phi_{u^-}}(u^-)] + s_u^2 t_u^2 [b\|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2 + T_{\phi_{u^+}}(u^-)] \\
&= \lambda t_u^2 \int_{\mathbb{R}^3} f(x)|u^-|^2 dx + t_u^6 \int_{\mathbb{R}^3} |u^-|^6 dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
t_u^2 \leq\ & \frac{b[\|\nabla u^-\|_2^4 + \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2] + T_{\phi_u}(u^-)}{2\int_{\mathbb{R}^3} |u^-|^6 dx} \\
&+ \frac{\sqrt{[b\|\nabla u^-\|_2^4 + b\|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2 + T_{\phi_u}(u^-)]^2 + 4\int_{\mathbb{R}^3} |u^-|^6 dx (\|u^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx)}}{2\int_{\mathbb{R}^3} |u^-|^6 dx}.
\end{aligned}$$

Since $g_u(1,1) \leq 0$, $e_u(1,1) \leq 0$, we have that

$$\begin{aligned}
&\|u^-\|^2 - \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2 dx \\
&\leq \int_{\mathbb{R}^3} |u^-|^6 dx - b[\|\nabla u^-\|_2^4 + \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2] - T_{\phi_u}(u^-),
\end{aligned}$$

thus $t_u^2 \leq 1$. Together with $0 < s_u \leq t_u$, we have that $0 < s_u \leq t_u \leq 1$.

Similarly, we can prove (ii). □

Lemma 2.6. *Define functions s, t: $H \to (0, +\infty)$ by $s(u) = s_u$, $t(u) = t_u$. Under the assumption of Lemma 2.5, we have*

(i) *s, t are continuous in H.*

(ii) *if* $u_n^+ \to 0$ *strongly in* H *as* $n \to \infty$*, one has* $s_{u_n} \to +\infty$*; if* $u_n^- \to 0$ *strongly in* H *as* $n \to \infty$*, one has* $t_{u_n} \to +\infty$.

(iii) *if* $\{u_n\} \subset M_\lambda$*,* $\lim\limits_{n\to\infty} I_\lambda(u_n) = m_\lambda$*, then* $m_\lambda > 0$ *and* $C' \leq \|u_n\| \leq C''$ *for some* $C', C'' \geq 0$.

Proof. (i) By Lemma 2.4, there exist $(\bar{s}_{u_n}, \bar{t}_{u_n})$ and $(\bar{s}_u, \bar{t}_u)$, such that $\bar{s}_{u_n} u_n^+ + \bar{t}_{u_n} u_n^- \in M_\lambda$, $\bar{s}_u u^+ + \bar{t}_u u^- \in M_\lambda$. Then we have that

$$\begin{aligned} &\|u_n^+\|^2 + \bar{s}_{u_n}^2[b\|\nabla u_n^+\|_2^4 + T_{\phi_{u_n^+}}(u_n^+)] \\ &+ \bar{t}_{u_n}^2[b\|\nabla u_n^+\|_2^2\|\nabla u_n^-\|_2^2 + T_{\phi_{u_n^+}}(u_n^-)] \\ &= \lambda \int_{\mathbb{R}^3} f(x)|u_n^+|^2 dx + \bar{s}_{u_n}^4 \int_{\mathbb{R}^3} |u_n^+|^6 dx, \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\|u_n^-\|^2 + \bar{t}_{u_n}^2[b\|\nabla u_n^-\|_2^4 + T_{\phi_{u_n^-}}(u_n^-)] \\ &+ \bar{s}_{u_n}^2[b\|\nabla u_n^+\|_2^2\|\nabla u_n^-\|_2^2 + T_{\phi_{u_n^+}}(u_n^-)] \\ &= \lambda \int_{\mathbb{R}^3} f(x)|u_n^-|^2 dx + \bar{t}_{u_n}^4 \int_{\mathbb{R}^3} |u_n^-|^6 dx. \end{aligned} \tag{2.21}$$

Take a sequence $\{u_n\}$ in H, such that $u_n \to u$ strongly in H, then $u_n^\pm \to u^\pm$ strongly in H. We claim that $\{\bar{s}_{u_n}\}$, $\{\bar{t}_{u_n}\}$ are bounded in $\mathbb{R}^+$. Otherwise, if $\bar{s}_{u_n} \to +\infty$ as $n \to +\infty$, let

$$\begin{aligned} y(u_n) = &\|u_n^+\|^2 + \bar{s}_{u_n}^2[b\|\nabla u_n^+\|_2^4 + T_{\phi_{u_n^+}}(u_n^+)] \\ &+ \bar{t}_{u_n}^2[b\|\nabla u_n^+\|_2^2\|\nabla u_n^-\|_2^2 + T_{\phi_{u_n^+}}(u_n^-)] \\ &- \lambda \int_{\mathbb{R}^3} f(x)|u_n^+|^2 dx = \bar{s}_{u_n}^4 \int_{\mathbb{R}^3} |u_n^+|^6 dx := h(u_n), \end{aligned} \tag{2.22}$$

together with $u_n^\pm \to u^\pm \neq 0$ in H, we can get that

$$\frac{y(u_n)}{\bar{s}_{u_n}^4} \to 0, \qquad \frac{h(u_n)}{\bar{s}_{u_n}^4} \to C,$$

as $\bar{s}_{u_n} \to +\infty$, which leads to a contradiction. So going if necessary to a subsequence, still denoted by $\{\bar{s}_{u_n}\}$ and $\{\bar{t}_{u_n}\}$, suppose that there is a pair of nonnegative number $(\bar{s}, \bar{t})$ such that

$$\lim_{n\to\infty} \bar{s}_{u_n} = \bar{s}, \qquad \lim_{n\to\infty} \bar{t}_{u_n} = \bar{t}.$$

Passing to the limit in (2.20) and (2.21), we get that

$$\begin{aligned} &\|u^+\|^2 + \bar{s}^2[b\|\nabla u^+\|_2^4 + T_{\phi_{u^+}}(u^+)] \\ &+ \bar{t}^2[b\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2 + T_{\phi_{u^+}}(u^-)] \\ &= \lambda \int_{\mathbb{R}^3} f(x)|u^+|^2 dx + \bar{s}^4 \int_{\mathbb{R}^3} |u^+|^6 dx, \end{aligned} \tag{2.23}$$

$$\|u^-\|^2 + \bar{t}^2[b\|\nabla u^-\|_2^4 + T_{\phi_{u^-}}(u^-)]$$

$$+ \bar{s}^2[b\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2 + T_{\phi_{u^+}}(u^-)]$$
$$= \lambda \int_{\mathbb{R}^3} f(x)|u^-|^2dx + \bar{t}^4 \int_{\mathbb{R}^3} |u^-|^6dx. \tag{2.24}$$

It follows from $b > 0$, $0 < \lambda < \lambda_1$ and $u^\pm \neq 0$, that $\bar{s} > 0$, $\bar{t} > 0$. Therefore, $\bar{s}u^+ + \bar{t}u^- \in M_\lambda$. By the uniqueness of (s_u, t_u), we have that $s_u = \bar{s}$, $t_u = \bar{t}$. Namely the functions s, t are continuous in H.

(ii) We only need to show that $t_{u_n} \to +\infty$, if $u_n^- \to 0$ in H as $n \to \infty$. Similarly, we can prove that $s_{u_n} \to +\infty$, if $u_n^+ \to 0$ in H. On the contrary, if there exists $M > 0$, such that $t_{u_n} \leq M$, then it follows from the Sobolev inequality that

$$o(\|u^-\|^2) = {t_{u_n}}^4 \int_{\mathbb{R}^3} |u^-|^6dx \leq C\|u_n^-\|^6.$$

By (2.21), $b > 0$ and $0 < \lambda < \lambda_1$, we deduce that

$$\begin{aligned}
&\|u_n^-\|^2 + t_{u_n}^2[b\|\nabla u_n^-\|_2^4 + T_{\phi_{u_n^-}}(u_n^-)] + s_{u_n}^2[b\|\nabla u_n^+\|_2^2\|\nabla u_n^-\|_2^2 + T_{\phi_{u_n^+}}(u_n^-)] \\
&- \lambda \int_{\mathbb{R}^3} f(x)|u_n^-|^2dx - t_{u_n}^4 \int_{\mathbb{R}^3} |u_n^-|^6dx \\
\geq\ & (1 - \frac{\lambda}{\lambda_1})\|u_n^-\|^2 - o(\|u_n^-\|^2) > 0,
\end{aligned}$$

which contradicts with the fact that $s_{u_n}u_n^+ + t_{u_n}u_n^- \in M_\lambda$, so $t_{u_n} \to +\infty$.

(iii) Since $\{u_n\} \subset M_\lambda$, we have

$$\begin{aligned}
&\|u_n^\pm\|^2 + [b\|\nabla u_n^\pm\|_2^4 + \|\nabla u_n^+\|_2^2\|\nabla u_n^-\|_2^2] + T_{\phi_{u_n}}(u_n^\pm) \\
=\ & \lambda \int_{\mathbb{R}^3} f(x)|u_n^\pm|^2dx + \int_{\mathbb{R}^3} |u_n^\pm|^6dx.
\end{aligned}$$

By $0 < \lambda < \lambda_1$, $b > 0$, Sobolev's inequality and Hölder's inequality, one has that

$$\|u_n^\pm\|^2 \leq \lambda \int_{\mathbb{R}^3} f(x)|u_n^\pm|^2dx + \int_{\mathbb{R}^3} |u_n^\pm|^6dx \leq \frac{\lambda}{\lambda_1}\|u_n^\pm\|^2 + S^{-3}\|u_n^\pm\|^6,$$

then

$$\|u_n^\pm\| \geq [S^3(1 - \frac{\lambda}{\lambda_1})]^{\frac{1}{4}} > 0. \tag{2.25}$$

Furthermore, it follows from $\{u_n\} \subset M_\lambda \subset N_\lambda$ that

$$\begin{aligned}
&m_\lambda + o(1) \\
&= I_\lambda(u_n) = I_\lambda(u_n) - \frac{1}{6}\langle I'_\lambda(u_n), (u_n)\rangle \\
&= \frac{1}{3}(\|u_n\|^2 - \lambda \int_{\mathbb{R}^3} f(x)u_n^2dx) + \frac{1}{12}[b\|\nabla u_n\|_2^2 \int_{\mathbb{R}^3} |\nabla u_n|^2dx + T_{\phi_{u_n}}(u_n^\pm)] \\
&\geq \frac{1}{3}(1 - \frac{\lambda}{\lambda_1}))\|u_n\|^2,
\end{aligned} \tag{2.26}$$

which means that $m_\lambda > 0$ and $\{u_n\}$ is bounded in H. By (2.25) and (2.26), we have that $C' \le \|u_n\| \le C''$ for some $C', C'' > 0$. □

Inspired by [5] and [22], with the help of Nehari manifold and Minimax methods in critical point theory, the following results hold.

Lemma 2.7. *Suppose* (f_1), (f_2), (K), (V) *hold and let* $b > 0$, *we have that*

(i) *for each* $u \in H \setminus \{0\}$, *there exists a unique* $\tilde{t_u} > 0$, *such that* $\tilde{t_u} u \in N_\lambda$ *and* $I_\lambda(\tilde{t_u} u) = \max_{t \ge 0} I_\lambda(tu)$;

(ii) *if* $0 < \lambda < \lambda_1$, $\frac{3}{2} < \alpha < 2$, *the system* (SKP) *has a positive ground state solution* $u_0 \in N_\lambda$, *and* $I_\lambda(u_0) = c_\lambda$, *then* $c_\lambda < c^* = \frac{b}{4}S^3 + \frac{[b^2S^4+4S]^{\frac{3}{2}}}{24} + \frac{b^3S^6}{24}$.

Proof. Since the proof of (i) is standard, we omit it here (you can see [11]). Next we give the proof for (ii).

Recall that S is attained by the Tulenti function $u_\epsilon = \frac{\epsilon^{\frac{1}{4}}}{(\epsilon+|x|^2)^{\frac{1}{2}}}$. Define $w_\epsilon = \varphi \circ u_\epsilon$, where $\varphi \in C_0^\infty(\mathbb{R}^3) : \mathbb{R}^3 \to [0,1]$ satisfies

$$\varphi(x) = \begin{cases} 1, & x \in B_{2R}(0), \\ 0, & x \in \mathbb{R}^3 \backslash B_{2R}(0). \end{cases} \tag{2.27}$$

Similar to the calculation of [4], we have the following estimate as $\epsilon \to 0$,

$$\int_{\mathbb{R}^3} |\nabla w_\epsilon|^2 dx = K_1 + O(\epsilon^{\frac{1}{2}}), \quad \int_{\mathbb{R}^3} |w_\epsilon|^6 dx = K_2 + O(\epsilon^{\frac{1}{2}}), \quad S = \frac{K_1}{K_2^{\frac{1}{3}}}, \tag{2.28}$$

$$\int_{\mathbb{R}^3} |w_\epsilon|^s dx = \begin{cases} O(\epsilon^{\frac{s}{4}}), & s \in [2,3), \\ O(\epsilon^{\frac{3}{4}}|\ln \epsilon|), & s = 3, \\ O(\epsilon^{\frac{6-s}{4}}), & s \in (3,6), \end{cases} \tag{2.29}$$

where K_1,K_2 are positive constants. According to (2.28) and (2.29) we have

$$\frac{\int_{\mathbb{R}^3} |\nabla w_\epsilon|^2 dx}{(\int_{\mathbb{R}^3} |w_\epsilon|^6 dx)^{\frac{1}{3}}} = S + O(\epsilon^{\frac{1}{2}}). \tag{2.30}$$

In fact, for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $t_\epsilon w_\epsilon \in N_\lambda$. Furthermore, $\{t_\epsilon\}_{\epsilon>0}$ has a positive lower bound. Otherwise, there exists a subsequence ϵ_n, such that $t_{\epsilon_n} \to 0$ as $n \to \infty$. By the definition of c_λ, we have that $0 < c_\lambda \le \lim_{n\to\infty} I_\lambda(t_{\epsilon_n} w_{\epsilon_n}) = 0$, which leads to a contradiction.

Therefore, in view of the definition of c_λ and (i), we have $c_\lambda \le \max_{t\ge 0} I_\lambda(tw_\epsilon)$. Define a function

$$e(t) = \frac{1}{2}t^2\|w_\epsilon\|_2^2 + \frac{b}{4}t^4 \left(\int_{\mathbb{R}^3} |\nabla w_\epsilon|^2 dx \right)^2 - \frac{1}{6}t^6 \int_{\mathbb{R}^3} |w_\epsilon|^6 dx.$$

By $e'(t) = 0$, it is easy to see that $e(t)$ attains its maximum at

$$t_0 = \left[\frac{b(\int_{\mathbb{R}^3} |\nabla w_\epsilon|^2 dx)^2 + \sqrt{b^2(\int_{\mathbb{R}^3} |\nabla w_\epsilon|^2 dx)^4 + 4\|w_\epsilon\|^2 \int_{\mathbb{R}^3} |w_\epsilon|^6 dx}}{2\int_{\mathbb{R}^3} |w_\epsilon|^6 dx} \right]^{\frac{1}{2}}$$

and

(2.31)

$$e(t_0) = \frac{b\|w_\epsilon\|^2 \|\nabla w_\epsilon\|_2^4}{4\int_{\mathbb{R}^3} |w_\epsilon|^6 dx} + \frac{[b^2\|\nabla w_\epsilon\|_2^8 + 4\|w_\epsilon\|^2 \int_{\mathbb{R}^3} |w_\epsilon|^6 dx]^{\frac{3}{2}} + b^3\|\nabla w_\epsilon\|_2^{12}}{24(\int_{\mathbb{R}^3} |w_\epsilon|^6 dx)^2}.$$

(2.29) and (2.30) imply that

$$e(t_0) = \frac{b}{4} S^3 + \frac{[b^2 S^4 + 4S]^{\frac{3}{2}} + b^3 S^6}{24} + O(\epsilon^{\frac{1}{2}}) = c^* + O(\epsilon^{\frac{1}{2}}).$$

By (i), we have that

$$\begin{aligned} I_\lambda(tw_\epsilon) \le\ & \max_{t\ge 0} I_\lambda(tw_\epsilon) \\ =\ & \max_{t\ge 0} \Big[\frac{1}{2}t^2\|w_\epsilon\|^2 + \frac{b}{4}t^4\|\nabla w_\epsilon\|_2^4 - \frac{1}{6}t^6 \int_{\mathbb{R}^3} |w_\epsilon|^6 dx + \frac{1}{4}t^4 T_{\varphi_{w_\epsilon}}(w_\epsilon) \\ & - \frac{1}{2}t^2\lambda \int_{\mathbb{R}^3} f(x)|w_\epsilon|^2 dx\Big]. \end{aligned}$$

By (2.30) and (2.31), it is not difficult to get that as $\epsilon \to 0$,

$$\begin{aligned} & \max_{t\ge 0} \Big[\frac{1}{2}t^2\|w_\epsilon\|^2 + \frac{b}{4}t^4\|\nabla w_\epsilon\|_2^4 - \frac{1}{6}\int_{\mathbb{R}^3} |w_\epsilon|^6 dx\Big] \\ & = e(t_0) = \frac{b}{4}S^3 + \frac{[b^2 S^4 + 4S]^{\frac{3}{2}} + b^3 S^6}{24} + O(\epsilon^{\frac{1}{2}}). \end{aligned} \tag{2.32}$$

By Hölder's inequality, the boundedness of s and t, (2.29) and Lemma 1.2(i), we can deduce that as $\epsilon \to 0$,

$$\frac{1}{4}t^4 T_{\varphi_{w_\epsilon}}(w_\epsilon) \le \frac{1}{4}t^4 |\varphi_{w_\epsilon}|_6 |w_\epsilon|^2_{\frac{12}{5}} \le C|w_\epsilon|^4_{\frac{12}{5}} \le C\|w_\epsilon\|^4 \le C'\epsilon. \tag{2.33}$$

In view of (f_2), if $0 < \lambda < \lambda_1$, $\epsilon \in (0, \tau^2]$ and $\frac{3}{2} < \alpha < 2$, we can have

$$\begin{aligned} \frac{1}{2}\lambda \int_{\mathbb{R}^3} f(x)|w_\epsilon|^2 dx &\ge C\epsilon^{\frac{1}{2}} \int_{|x|\le\tau} \frac{|x|^{-\alpha}}{\epsilon + |x|^2} dx + \frac{1}{2}\lambda \int_{|x|\ge\tau} f(x)|w_\epsilon|^2 dx \\ &\ge C\epsilon^{\frac{1}{2}} \int_0^\tau \frac{r^2}{r^\alpha(\epsilon + r^2)} dr \\ &= C\epsilon^{1-\frac{\alpha}{2}} \int_0^{\tau\epsilon^{-\frac{1}{2}}} \frac{\rho^2}{\rho^\alpha(1+\rho^2)} d\rho \\ &\ge C'\epsilon^{1-\frac{\alpha}{2}} \int_0^1 \frac{\rho^2}{2\rho^\alpha} d\rho \\ &= C'\epsilon^{1-\frac{\alpha}{2}}. \end{aligned} \tag{2.34}$$

It follows from (2.32), (2.33), (2.34) and $\frac{3}{2} < \alpha < 2$ that, as $\epsilon \to 0$,

$$\begin{aligned} I_\lambda(tw_\epsilon) &\le \max_{t\ge 0} I_\lambda(tw_\epsilon) \le e(t_0) + C'(\epsilon - \epsilon^{1-\frac{\alpha}{2}}) \\ &= \frac{b}{4}S^3 + \frac{[b^2S^4+4S]^{\frac{3}{2}}}{24} + \frac{+b^3S^6}{24} + O(\epsilon^{\frac{1}{2}}) + C'(\epsilon - \epsilon^{1-\frac{\alpha}{2}}). \end{aligned} \tag{2.35}$$

That is

$$\max_{t\ge 0} I_\lambda(tw_\epsilon) < \frac{b}{4}S^3 + \frac{[b^2S^4+4S]^{\frac{3}{2}}}{24} + \frac{+b^3S^6}{24} = c^*.$$

Combining (2.35), we deduce that $c_\lambda < c^*$. □

3. Proof of the main results

In this section, we will claim the existence of sign-changing solutions for the system (SKP). Since the problem (SKP) involves bi-nonlocal terms and critical nonlinearity, we need to construct a sign-changing $(PS)_{m_\lambda}$-sequence. Inspired by the method of [6], we give some definitions.

Let P denote the cone of nonnegative functions in H, $Q = [0,1]\times[0,1]$ and Σ be the set of continuous map δ with s, $t \in [0,1]$, namely

$$\Sigma = \begin{cases} \delta \in C(Q,H); (a) : \delta(s,0) = 0, \delta(0,t) \in P, \delta(1,t) \in -P; \\ (b) : I_\lambda(\delta(s,1)) \le 0, f(\delta(s,1)) \ge 2, \end{cases}$$

where

$$f(\delta(s,1)) = \frac{\int_{\mathbb{R}^3} |\delta(s,1)|^6 dx}{\|\delta(s,1)\|^2 - \lambda\int_{\mathbb{R}^3} f(x)|\delta(s,1)|^2 dx + T_{\phi_{\delta(s,1)}}\delta(s,1) + b\|\nabla\delta(s,1)\|_2^2 \int_{\mathbb{R}^3}|\nabla\delta(s,1)|^2 dx}.$$

Choosing $u \in H$ with $u^\pm \ne 0$. Let $\delta(s,t) = mt(1-s)u^+ + mtsu^-$, where $m > 0$, $s,t \in [0,1]$. It is easy to see that $\delta \in \Sigma$ for $m > 0$ large enough, which means $\Sigma \ne \emptyset$. Define

$$\Gamma_\lambda(u,v) = \begin{cases} \frac{\int_{\mathbb{R}^3}|u|^6 dx}{\|u\|^2 - \lambda\int_{\mathbb{R}^3} f(x)|u|^2dx + T_{\phi_u}(u) + T_{\phi_v}(u) + b[\|\nabla u\|_2^4 + \|\nabla u\|_2^2\|\nabla v\|_2^2]}, & u \ne 0, \\ 0, & u = 0. \end{cases}$$

Clearly, $\Gamma_\lambda(u,v) > 0$ if $0 < \lambda < \lambda_1$, $b > 0$ and $u \ne 0$; $u \in M_\lambda$ if and only if $\Gamma_\lambda(u^+,u^-) = \Gamma_\lambda(u^-,u^+) = 1$. Set

$$\Theta_\lambda = \{u \in H : |\Gamma_\lambda(u^+,u^-) - 1| < \frac{1}{2}, |\Gamma_\lambda(u^-,u^+) - 1| < \frac{1}{2}\}.$$

3.1. Main lemmas and theirs proof

Lemma 3.1. $\inf_{\delta\in\Sigma}\sup_{u\in\delta(Q)} I_\lambda(u) = \inf_{u\in M_\lambda} I_\lambda(u) = m_\lambda$.

Proof. On the one hand, for any $u \in M_\lambda$, there exists $\delta(s,t) = mt(1-s)u^+ + mtsu^- \in \Sigma$ for $m > 0$ large enough. By Lemma 2.4, we can deduce that

$$I_\lambda(u) = \max_{s,t\ge 0} I_\lambda(su^+ + tu^-) \ge \sup_{u\in\delta(Q)} I_\lambda(u) \ge \inf_{\delta\in\Sigma}\sup_{u\in\delta(Q)} I_\lambda(u).$$

Consequently,

$$\inf_{u\in M_\lambda} I_\lambda(u) \geq \inf_{\delta\in\Sigma}\sup_{u\in\delta(Q)} I_\lambda(u).$$

On the other hand, from the definition of Σ, we claim that for each $\delta \in \Sigma$, there exists $u_\delta \in \delta(Q)\bigcap M_\lambda$, which implies that

$$\sup_{u\in\delta(Q)} I_\lambda(u) \geq I_\lambda(u_\delta) \geq \inf_{u\in M_\lambda} I_\lambda(u).$$

Therefore

$$\inf_{\delta\in\Sigma}\sup_{u\in\delta(Q)} I_\lambda(u) \geq \inf_{u\in M_\lambda} I_\lambda(u).$$

In fact, for each $\delta \in \Sigma$, $t \in [0,1]$, it is easy to check that $\delta(0,t) \in P$, $\delta(1,t) \in -P$, so we have that

$$\Gamma_\lambda(\delta^+(0,t),\delta^-(0,t))-\Gamma_\lambda(\delta^-(0,t),\delta^+(0,t)) = \Gamma_\lambda(\delta^+(0,t),\delta^-(0,t)) \geq 0, \tag{3.1}$$

$$\Gamma_\lambda(\delta^+(1,t),\delta^-(1,t))-\Gamma_\lambda(\delta^-(1,t),\delta^+(1,t)) = -\Gamma_\lambda(\delta^-(1,t),\delta^+(1,t)) \leq 0. \tag{3.2}$$

Meanwhile, we derive from the definition of Σ that for all $\delta \in \Sigma$ and $s \in [0,1]$,

$$\Gamma_\lambda(\delta^+(s,1),\delta^-(s,1)) + \Gamma_\lambda(\delta^-(s,1),\delta^+(s,1)) \geq f(\delta^+(s,1)) \geq 2,$$

which follows from $\frac{d}{c} + \frac{f}{e} \geq \frac{d+f}{c+e}$ for all $c,d,e,f > 0$. Consequently

$$\Gamma_\lambda(\delta^+(s,1),\delta^-(s,1)) + \Gamma_\lambda(\delta^-(s,1),\delta^+(s,1)) - 2 \geq 0, \tag{3.3}$$

$$\Gamma_\lambda(\delta^+(s,0),\delta^-(s,0)) + \Gamma_\lambda(\delta^-(s,0),\delta^+(s,0)) - 2 \leq 0. \tag{3.4}$$

So by (3.1), (3.2), (3.3), (3.4) and Miranda's theorem [15], one has that there exists $(\hat{s},\hat{t}) \in Q$ such that

$$\Gamma_\lambda(\delta^+(\hat{s},\hat{t}),\delta^-(\hat{s},\hat{t})) - \Gamma_\lambda(\delta^-(\hat{s},\hat{t}),\delta^+(\hat{s},\hat{t})) = 0,$$

$$\Gamma_\lambda(\delta^+(\hat{s},\hat{t}),\delta^-(\hat{s},\hat{t})) + \Gamma_\lambda(\delta^-(\hat{s},\hat{t}),\delta^+(\hat{s},\hat{t})) = 2.$$

Then

$$\Gamma_\lambda(\delta^+(\hat{s},\hat{t}),\delta^-(\hat{s},\hat{t})) = \Gamma_\lambda(\delta^-(\hat{s},\hat{t}),\delta^+(\hat{s},\hat{t})) = 1,$$

which means that for all $\delta \in \Sigma$, there exists $u_\delta = \delta(\hat{s},\hat{t}) \in \delta(Q)\bigcap M_\lambda$. Therefore,

$$\inf_{\delta\in\Sigma}\sup_{u\in\delta(Q)} I_\lambda(u) = \inf_{u\in M_\lambda} I_\lambda(u) = m_\lambda.$$

□

Lemma 3.2. *Suppose that* $0 < \lambda < \lambda_1$, $b > 0$. *Then there exists a* $(PS)_{m_\lambda}$-*sequence* $\{u_n\} \subset \Theta_\lambda$ *for* I_λ.

Proof. Firstly, we find a $(PS)_{m_\lambda}$-sequence $\{u_n\} \subset H$ for I_λ. Consider a minimizing sequence $\{v_n\} \subset M_\lambda$, $\delta_n \in \Sigma$, where $\delta_n(s,t) = mt(1-s)v_n^+ + mtsv_n^- \in \Sigma$, then

$$\lim_{n\to+\infty}\max_{v\in\delta_n(Q)} I_\lambda(v) = \lim_{n\to\infty} I_\lambda(v_n).$$

By classical deformation lemma [17], we can derive that there exists $\{u_n\} \subset H$ such that

$$I_\lambda(u_n) \to m_\lambda, \quad I'_\lambda(u_n) \to 0, \quad dist(u_n, \delta_n(Q)) \to 0 \tag{3.5}$$

as $n \to \infty$.

Suppose the thesis is false. Then it is possible to find an $r > 0$, such that $\delta_n(Q) \bigcap V_r = \emptyset$ for n large enough, where

$$V_r = \{u \in H : \exists w \in H \text{ s.t. } \|w - u\| \le r, \|I'_\lambda(w)\| \le r, |I_\lambda(w) - m_\lambda| \le r\}.$$

By deformation lemma in [17,22], there exists a continuous map $\eta : [0,1] \times H \to H$ satisfying, for some $\epsilon \in (0, \frac{m_\lambda}{2})$ and each $t \in [0,1]$,

(a) $\eta(0,u) = u; \eta(t,-u) = -\eta(t,u)$;

(b) $\eta(t,u) = u; \ \forall u \in I_\lambda^{m_\lambda-\epsilon} \bigcup (H \backslash I_\lambda^{m_\lambda+\epsilon})$;

(c) $\eta(1, I_\lambda^{m_\lambda+\frac{\epsilon}{2}} \backslash V_r) \subset I_\lambda^{m_\lambda-\frac{\epsilon}{2}}$;

(d) $\eta(1, (I_\lambda^{m_\lambda+\frac{\epsilon}{2}} \bigcap P) \backslash V_r) \subset I_\lambda^{m_\lambda-\frac{\epsilon}{2}} \bigcap P$, where $I_\lambda^d = \{u \in H : I_\lambda(u) \le d\}$.

Since $\lim\limits_{n\to+\infty} \max\limits_{v\in\delta_n(Q)} I_\lambda(v) = m_\lambda$, we can select n such that

$$\delta_n(Q) \subset I_\lambda^{m_\lambda+\frac{\epsilon}{2}}, \quad \delta_n(Q) \bigcap V_r = \emptyset. \tag{3.6}$$

Define $\bar{\delta}_n : Q \to H$ by $\bar{\delta}_n(s,t) = \eta(1, \delta_n(s,t))$ for $\forall (s,t) \in Q$. Then similar proof as that of [26], $\bar{\delta}_n \in \Sigma$. By (3.6) and property (c) of η, we can derive that $\tilde{\delta}_n(Q) \subset I_\lambda^{m_\lambda-\frac{\epsilon}{2}}$, which leads to a contradiction,

$$m_\lambda = \inf_{\delta\in\Sigma} \sup_{v\in\delta(Q)} I_\lambda(v) \le \max_{v\in\delta_n(Q)} I_\lambda(v) \le m_\lambda - \frac{\epsilon}{2}.$$

So we can find a $(PS)_{m_\lambda}$-sequence $\{u_n\}$ for I_λ.

Next, we prove that $\{u_n\} \subset \Theta_\lambda$ for n large enough. We only need to show that $u_n^\pm \ne 0$, namely that $\Gamma_\lambda(u_n^+, u_n^-) \to 1, \Gamma_\lambda(u_n^-, u_n^+) \to 1$. By (3.5), there exists a sequence $\{w_n\}$, such that

$$w_n = \gamma_n v_n^+ + \beta_n v_n^- \in \delta_n(Q), \quad I_\lambda(w_n) \to m_\lambda, \quad \|w_n - u_n\| \to 0. \tag{3.7}$$

So in order to get $u_n^\pm \ne 0$, we can prove that $\gamma_n v_n^+ \ne 0$ and $\beta_n v_n^- \ne 0$ for n large enough. By $\{v_n\} \subset M_\lambda$ and Lemma 2.6(iii), we only need to show that $\lim\limits_{n\to\infty} \gamma_n \ne 0, \lim\limits_{n\to\infty} \beta_n \ne 0$. Otherwise, if $\lim\limits_{n\to\infty} \gamma_n \ne 0, \beta_n \to 0$, it follows from the continuity of I_λ and (3.7) that

$$m_\lambda = \lim_{n\to\infty} I_\lambda(w_n) = \lim_{n\to\infty} I_\lambda(\gamma_n v_n^+ + \beta_n v_n^-) = \lim_{n\to\infty} I_\lambda(\gamma_n v_n^+).$$

Let $t_n^4 = (1 - \frac{\lambda}{\lambda_1}) \frac{\|v_n^-\|^2}{\int_{\mathbb{R}^3} |v_n^-|^6 dx}$, by Lemma 2.4, $b > 0$ and $0 < \lambda < \lambda_1$, we have that

$$\begin{aligned} m_\lambda = & \lim_{n\to\infty} I_\lambda(v_n) = \liminf_{n\to\infty} \max_{s,t>0} I_\lambda(s v_n^+ + t v_n^-) \\ \ge & \liminf_{n\to\infty} I_\lambda(\gamma_n v_n^+ + t_n v_n^-) \end{aligned}$$

$$
\begin{aligned}
&= \liminf_{n\to\infty}\Big[I_\lambda(\gamma_n v_n^+) + \frac{t_n^2}{2}\|v_n^-\|^2 + \frac{t_n^4}{4}(b\|\nabla v_n^-\|_2^4 + T_{\phi_{v_n^-}}(v_n^-)) \\
&\qquad - \frac{\lambda}{2}t_n^2\int_{\mathbb{R}^3} f(x)|v_n^-|^2dx - \frac{1}{6}t_n^6\int_{\mathbb{R}^3}|v_n^-|^6dx \\
&\qquad + \frac{\gamma_n^2 t_n^2}{2}(b\|\nabla v_n^+\|_2^2\|\nabla v_n^-\|_2^2 + T_{\phi_{v_n^+}}(v_n^-))\Big] \\
&\geq \liminf_{n\to\infty}\Big[\frac{t_n^2}{2}(1-\frac{\lambda}{\lambda_1})\|v_n^-\|^2 - \frac{1}{6}t_n^6\int_{\mathbb{R}^3}|v_n^-|^6dx + I_\lambda(\gamma_n v_n^+)\Big] \\
&= \liminf_{n\to\infty}\Big[\frac{1}{3}(1-\frac{\lambda}{\lambda_1})^{\frac{3}{2}}\frac{\|v_n^-\|^3}{(\int_{\mathbb{R}^3}|v_n^-|^6dx)^{\frac{1}{2}}} + I_\lambda(\gamma_n v_n^+)\Big] \\
&\geq \liminf_{n\to\infty} I_\lambda(\gamma_n v_n^+) + \frac{1}{3}S^{\frac{3}{2}}(1-\frac{\lambda}{\lambda_1})^{\frac{3}{2}} \\
&= \lim_{n\to\infty} I_\lambda(\gamma_n v_n^+) + \frac{1}{3}S^{\frac{3}{2}}(1-\frac{\lambda}{\lambda_1})^{\frac{3}{2}} \\
&= m_\lambda + \frac{1}{3}S^{\frac{3}{2}}(1-\frac{\lambda}{\lambda_1})^{\frac{3}{2}},
\end{aligned}
$$

which leads to a contradiction. Thus $\{u_n\} \subset \Theta_\lambda$ for n large enough. □

Lemma 3.3. *Under conditions (f_1), (K) and $b > 0$, if $0 < \lambda < \lambda_1$, then we have that any bounded sequence $\{u_n\} \subset \Theta_\lambda \subset H$ such that*

$$I_\lambda(u_n) \to d \in (0, c_\lambda + c^*), \quad I'_\lambda(u_n) \to 0,$$

contains a convergent subsequence. In other words, there exists $u \in H$ with $u \neq 0$, such that $I'_\lambda(u) = 0$ and $I_\lambda(u) = d$, where $c^* = \frac{b}{4}S^3 + \frac{[b^2S^4+4S]^{\frac{3}{2}}}{24} + \frac{+b^3S^6}{24}$.

Proof. Since $\{u_n\}$ is bounded in H, up to a subsequence, still denoted by $\{u_n\}$. We can suppose that there exists $u \in H$ such that

$$
\begin{aligned}
&u_n \rightharpoonup u \text{ weakly in } H, \\
&u_n \to u \text{ strongly in } L^s_{loc}(\mathbb{R}^3),\ s \in [1,6), \qquad (3.8)\\
&u_n(x) \to u(x) \text{ a.e on } \mathbb{R}^3.
\end{aligned}
$$

It follows from Lemma 2.2 that $I'_\lambda(u) = 0$. Setting $w_n = u_n - u$, then $w_n \rightharpoonup 0$ weakly in H. By the well-known Brézis-Lieb Lemma [22], one has that

$$
\begin{aligned}
\|u_n\|^2 &= \|w_n\|^2 + \|u\|^2 + o(1), \\
\|\nabla u_n\|_2^2 &= \|\nabla w_n\|_2^2 + \|\nabla u\|_2^2 + o(1), \qquad (3.9)\\
\|\nabla u_n\|_2^4 &= \|\nabla w_n\|_2^4 + \|\nabla u\|_2^4 + 2\|\nabla w_n\|_2^2\|\nabla u\|_2^2 + o(1), \\
\|u_n\|_6^6 &= \|w_n\|_6^6 + \|u\|_6^6 + o(1).
\end{aligned}
$$

It follows from Lemma 2.2 that

$$F(w_n) = \int_{\mathbb{R}^3} f(x)|w_n|^2dx \to 0, \quad K(w_n) = \int_{\mathbb{R}^3} k(x)\phi_{w_n}w_n^2dx \to 0. \qquad (3.10)$$

Then by (3.9), (3.10), and $I'_\lambda(u)=0$, we have

$$\begin{aligned} 0 &= \langle I'_\lambda(u_n,u_n)\rangle + o(1) \\ &= \|u_n\|^2 + b\|\nabla u_n\|_2^4 + T_{\phi_{u_n}}(u_n) - \lambda\int_{\mathbb{R}^3} f(x)|u_n|^2dx - \int_{\mathbb{R}^3}|u_n|^6dx + o(1) \\ &= \langle I'_\lambda(u,u)\rangle + \|w_n\|^2 + b[\|\nabla w_n\|_2^4 + 2\|\nabla w_n\|_2^2\|\nabla u\|_2^2] - \int_{\mathbb{R}^3}|w_n|^6dx + o(1) \\ &= \|w_n\|^2 + b[\|\nabla w_n\|_2^4 + 2\|\nabla w_n\|_2^2\|\nabla u\|_2^2] - \int_{\mathbb{R}^3}|w_n|^6dx + o(1), \end{aligned}$$

i.e.,

$$\|w_n\|^2 + b[\|\nabla w_n\|_2^4 + 2\|\nabla w_n\|_2^2\|\nabla u\|_2^2] - \int_{\mathbb{R}^3}|w_n|^6dx = o(1). \tag{3.11}$$

Up to a subsequence, we can assume that there exist $l_i>0 (i=1,2,3)$ such that

$$\|w_n\|^2 \to l_1, \quad b(\|\nabla w_n\|_2^4 + 2\|\nabla w_n\|_2^2\|\nabla u\|_2^2) \to l_2, \quad \int_{\mathbb{R}^3}|w_n|^6dx \to l_3.$$

If $l_1=0$, namely that $w_n\to 0$ strongly in H, then the conclusion holds. So we may assume that $l_1>0$, then $l_1+l_2=l_3$. If $l_1>0$, then $l_2,\ l_3>0$. In view of (3.8), (3.10), and (3.11), we can deduce that

$$\begin{aligned} d &= I_\lambda(u_n) \\ &= I_\lambda(u) + \frac{1}{2}\|w_n\|^2 + \frac{1}{4}b[\|\nabla w_n\|_2^4 + 2\|\nabla w_n\|_2^2\|\nabla u\|_2^2] - \frac{1}{6}\int_{\mathbb{R}^3}|w_n|^6dx + o(1) \\ &= I_\lambda(u) + \frac{1}{3}\|w_n\|^2 + \frac{1}{12}b[\|\nabla w_n\|_2^4 + 2\|\nabla w_n\|_2^2\|\nabla u\|_2^2] + o(1). \end{aligned}$$

Letting $n\to\infty$, we have

$$d + o(1) = I_\lambda(u_n) = I_\lambda(u) + \frac{1}{3}l_1 + \frac{1}{12}l_2 + o(1). \tag{3.12}$$

Note that, by the Sobolev imbedding inequality, we have

$$\int_{\mathbb{R}^3}|\nabla w_n|^2dx \geq S\left(\int_{\mathbb{R}^3}|w_n|^6dx\right)^{\frac{1}{3}}, b\left(\int_{\mathbb{R}^3}|\nabla w_n|^2dx\right)^2 \geq bS^2\left(\int_{\mathbb{R}^3}|w_n|^6dx\right)^{\frac{2}{3}},$$

then

$$l_1 \geq S\left(l_1+l_2\right)^{\frac{1}{3}},\ l_2 \geq bS^2\left(l_1+l_2\right)^{\frac{2}{3}}.$$

By Lemma 2.1, we have

$$\begin{aligned} \frac{1}{3}l_1 + \frac{1}{12}l_2 &\geq \frac{1}{3}\left(\frac{bS^3+\sqrt{b^2S^6+4S^3}}{2}\right) + \frac{1}{12}\left(\frac{b^2S^3\sqrt{b^2S^6+4S^3}+b^3S^6+2bS^3}{2}\right) \\ &= \frac{b}{4}S^3 + \frac{[b^2S^4+4S]^{\frac{3}{2}}}{24} + \frac{+b^3S^6}{24} \\ &= c^*. \end{aligned}$$

Therefore

$$d = I_\lambda(u) + \frac{1}{3}l_1 + \frac{1}{12}l_2 + o(1) \geq c_\lambda + c^*,$$

which contradicts with our assumption $d \in (0, c_\lambda + c^*)$. Hence, $l_1 = 0$, i.e., $\|w_n\|^2 = o(1)$, therefore $u_n \to u$ in H. □

Lemma 3.4. *Under conditions (V), (f_1), (f_2) and (K), if $b > 0$, $0 < \lambda < \lambda_1$, $\frac{3}{2} < \alpha < 2$, then we have $m_\lambda < c_\lambda + c^*$.*

Proof. What we need to do is to find an element in M_λ such that the value of I_λ is strictly less that $c_\lambda + c^*$ at that point. We can suppose that u_0 is the ground state positive solution of the system (SKP). By elliptic estimates, similar to that proof of Theorem 1.2 [9], it can be proved that $u_0 \in L^\infty(\mathbb{R}^3)$. We will prove Lemma 3.4 by two steps in the following statement.

Step 1. we claim that there exist $\bar{s_\epsilon}$, $\bar{t_\epsilon} > 0$ such that $\bar{s_\epsilon}u_0 - \bar{t_\epsilon}w_\epsilon \in M_\lambda$, where w_ϵ is defined by the Talent function in Lemma 2.7(ii). In fact, denote $\varphi(a) = \frac{1}{a}u_0 - w_\epsilon$ with $a > 0$; define

$$a_1 = \sup\{a \in \mathbb{R}^+ : \varphi^+(a) \neq 0\}, \quad a_2 = \inf\{a \in \mathbb{R}^+ : \varphi^-(a) \neq 0\}.$$

Because u_0 is positive, by the definition of a_1, it is obvious that $a_1 = +\infty$ and $0 < a_2 < a_1$.

On the one hand, if $a \to a_2^+$, $\varphi^-(a) \to 0$, then it follows from Lemma 2.6(ii) that $\bar{t}(\varphi(a)) \to +\infty$; while by (2.20), $\{\bar{s}(\varphi(a))\}$ is bounded in $\mathbb{R}^+$. Therefore

$$\bar{s}(\varphi(a)) - \bar{t}(\varphi(a)) \to -\infty. \tag{3.13}$$

On the other hand, if $a \to a_1 = +\infty$, $\frac{1}{a}u_0 \to 0$ in H, then $\varphi^+(a) \to 0$. By Lemma 2.6(ii) and (2.21), we have that $\bar{s}(\varphi(a)) \to +\infty$ and $\{\bar{t}(\varphi(a))\}$ is bounded in $\mathbb{R}^+$. Thus

$$\bar{s}(\varphi(a)) - \bar{t}(\varphi(a)) \to +\infty. \tag{3.14}$$

Moreover, by Lemma 2.6(i), (3.13) and (3.14), there exists $a_\epsilon \in (a_2, a_1)$ such that

$$\bar{s}(\varphi(a_\epsilon)) - \bar{t}(\varphi(a_\epsilon)) = 0, \ i.e., \ \bar{s}(\varphi(a_\epsilon)) = \bar{t}(\varphi(a_\epsilon)).$$

Hence, let $\bar{s_\epsilon} = \frac{1}{a_\epsilon}s(\varphi(a_\epsilon))$, $\bar{t_\epsilon} = t(\varphi(a_\epsilon))$, one has that

$$\bar{s}(\varphi(a_\epsilon))\varphi(a_\epsilon) = \bar{s}(\varphi(a_\epsilon))\varphi^+(a_\epsilon) + \bar{s}(\varphi(a_\epsilon))\varphi^-(a_\epsilon)$$

$$= \bar{s}(\varphi(a_\epsilon))\varphi^+(a_\epsilon) + \bar{t}(\varphi(a_\epsilon))\varphi^-(a_\epsilon) = \bar{s_\epsilon}u_0 - \bar{t_\epsilon}w_\epsilon \in M_\lambda.$$

Step 2. We will prove that $m_\lambda < c_\lambda + c^*$. It follows from Lemma 2.4 that $I_\lambda(\bar{s_\epsilon}u_0 - \bar{t_\epsilon}w_\epsilon) = \sup_{s,t>0} I_\lambda(\bar{s}u_0 - \bar{t}w_\epsilon)$, which means we only need to prove that

$$m_\lambda \leq \sup_{s,t>0} I_\lambda(\bar{s}u_0 - \bar{t}w_\epsilon) < c_\lambda + c^*.$$

Since $I_\lambda(\bar{s}u_0-\bar{t}w_\epsilon)<0$ if s or t is sufficiently large, in other worlds, we need to consider the case that $\bar{s}$, $\bar{t}$ are contained in a bounded interval. By a calculation, we can get that

$$I_\lambda(\bar{s}u_0-\bar{t}w_\epsilon)=I_\lambda(\bar{s}u_0)+A+B-L-D-E+F-G,$$

where

$$A=\frac{1}{2}\|\bar{t}w_\epsilon\|^2+\frac{b}{4}\|\nabla\bar{t}w_\epsilon\|_2^4-\frac{1}{6}\int_{\mathbb{R}^3}|\bar{t}w_\epsilon|^6dx,\quad B=\frac{3}{2}b\|\nabla\bar{s}u_0\|_2^2\|\nabla\bar{t}w_\epsilon\|_2^2,$$

$$L=b(\|\nabla\bar{s}u_0\|_2^3\|\nabla\bar{t}w_\epsilon\|_2+\|\nabla\bar{s}u_0\|_2\|\|\nabla\bar{t}w_\epsilon\|_2^3),\quad D=\frac{\lambda}{2}\int_{\mathbb{R}^3}f(x)|\bar{t}w_\epsilon|^2dx,$$

$$E=\bar{s}\bar{t}\int_{\mathbb{R}^3}[\nabla u_0\nabla w_\epsilon+V(x)u_0w_\epsilon-\lambda\int_{\mathbb{R}^3}f(x)u_0w_\epsilon]dx,$$

$$F=\frac{1}{4}[T_{\phi_{(\bar{s}u_0-\bar{t}w_\epsilon)}}(\bar{s}u_0-\bar{t}w_\epsilon)-T_{\phi_{\bar{s}u_0}}(\bar{s}u_0)],$$

$$G=\frac{1}{6}\int_{\mathbb{R}^3}(|\bar{s}u_0-\bar{t}w_\epsilon|^6-|\bar{s}u_0|^6-\bar{t}w_\epsilon^6)dx.$$

For A, it follows from (2.32) that as $\epsilon\to 0$,

$$A=\frac{b}{4}S^3+\frac{[b^2S^4+4S]^{\frac{3}{2}}}{24}+\frac{+b^3S^6}{24}+O(\epsilon^{\frac{1}{2}})=c^*+O(\epsilon^{\frac{1}{2}}). \tag{3.15}$$

For B, by (2.28) we have that

$$\begin{aligned}B&=\frac{3}{2}b\|\nabla\bar{s}u_0\|_2^2\|\nabla\bar{t}w_\epsilon\|_2^2\\&=\frac{3}{2}b\bar{s}^2\bar{t}^2\|\nabla u_0\|_2^2\|\nabla w_\epsilon\|_2^2\le C(K_1+O(\epsilon^{\frac{1}{2}}))\le C'\epsilon^{\frac{1}{2}}.\end{aligned} \tag{3.16}$$

For L, similar to the calculation of [25], it follows from (2.28), (2.29) that as $\epsilon\to 0$

$$\begin{aligned}L&=b(\|\nabla\bar{s}u_0\|_2^3\|\nabla\bar{t}w_\epsilon\|_2+\|\nabla\bar{s}u_0\|_2\|\nabla\bar{t}w_\epsilon\|_2^3)\\&=b[\bar{s}^3\bar{t}\|u_0\|_{D^{1,2}(\mathbb{R}^3)}^3\|\nabla w_\epsilon\|_2+\bar{s}\bar{t}^3\|u_0\|_{D^{1,2}(\mathbb{R}^3)}^3\|\nabla w_\epsilon\|_2^3]\\&\le C[\|\nabla w_\epsilon\|_2+\|\nabla w_\epsilon\|_2^3]\\&\le C[O(\epsilon^{\frac{1}{4}})+O(\epsilon^{\frac{3}{4}})]\\&\le C'\epsilon^{\frac{1}{4}}.\end{aligned} \tag{3.17}$$

For D, by (2.34) we have

$$D=\frac{\lambda}{2}\int_{\mathbb{R}^3}f(x)|\bar{t}w_\epsilon|^2dx\ge C'\epsilon^{1-\frac{\alpha}{2}}. \tag{3.18}$$

Since u_0 is the ground state positive solution, $I'_\lambda(u_0) = 0 = \langle I'_\lambda(u_0), w_\epsilon\rangle$, hence for E, one has that

$$\begin{aligned}&\bar{s}\bar{t}\int_{\mathbb{R}^3}[\nabla u_0\nabla w_\epsilon + V(x)u_0w_\epsilon - \lambda\int_{\mathbb{R}^3} f(x)u_0w_\epsilon]dx\\ =\ &\bar{s}^5\bar{t}\int_{\mathbb{R}^3} u_0^5 w_\epsilon dx - \bar{s}^3\bar{t}\int_{\mathbb{R}^3} k(x)\phi_{u_0}u_0w_\epsilon dx - \bar{s}^3\bar{t}b\|\nabla u_0\|_2^2\int_{\mathbb{R}^3}|\nabla u_0||\nabla w_\epsilon|dx.\end{aligned}$$

It follows from $k \in L^\infty(\mathbb{R}^3)$, $u_0 \in L^\infty(\mathbb{R}^3)$, $b > 0$, Hölder's inequality, (2.29) and the boundedness of s, t that

$$\bar{s}^5\bar{t}\int_{\mathbb{R}^3} u_0^5 w_\epsilon dx \le \bar{s}^5\bar{t}|u_0|_\infty^5\int_{|x|\le 2R} w_\epsilon dx \le C(\int_{|x|\le 2R} w_\epsilon^2 dx)^{\frac{1}{2}} \le C'\epsilon^{\frac{1}{4}}, \tag{3.19}$$

$$\bar{s}^3\bar{t}\int_{\mathbb{R}^3} k(x)\phi_{u_0}u_0w_\epsilon dx \le \bar{s}^3\bar{t}|k|_\infty|\phi_{u_0}|_6|u_0|_{\frac{12}{5}}|w_\epsilon|_{\frac{12}{5}} \le C|w_\epsilon|_{\frac{12}{5}} \le C'\epsilon^{\frac{1}{4}}, \tag{3.20}$$

$$\begin{aligned}\bar{s}^3\bar{t}b\|\nabla u_0\|_2^2\int_{\mathbb{R}^3}|\nabla u_0||\nabla w_\epsilon|dx &= \bar{s}^3\bar{t}b\|u_0\|^3_{D^{1,2}(\mathbb{R}^3)}(\int_{\mathbb{R}^3}|\nabla w_\epsilon|^2dx)^{\frac{1}{2}}\\ &\le C(\int_{\mathbb{R}^3}|\nabla w_\epsilon|^2dx)^{\frac{1}{2}} \le C'\epsilon^{\frac{1}{4}}.\end{aligned} \tag{3.21}$$

By (3.19), (3.20) and (3.21), we have that as $\epsilon \to 0$

$$-\bar{s}\bar{t}\int_{\mathbb{R}^3}[\nabla u_0\nabla w_\epsilon + V(x)u_0w_\epsilon - \lambda\int_{\mathbb{R}^3} f(x)u_0w_\epsilon]dx \le C'\epsilon^{\frac{1}{4}}. \tag{3.22}$$

For F, by a calculation we have that

$$\frac{1}{4}[T_{\phi_{(\bar{s}u_0-\bar{t}w_\epsilon)}}(\bar{s}u_0 - \bar{t}w_\epsilon) - T_{\phi_{\bar{s}u_0}}(\bar{s}u_0)] = H + Q + P + R,$$

where

$$H = \bar{s}\bar{t}\left(\int_{\mathbb{R}^3} k(x)\phi_{\bar{s}u_0}u_0w_\epsilon dx + \int_{\mathbb{R}^3} k(x)\phi_{\bar{t}w_\epsilon}u_0w_\epsilon dx\right),$$

$$Q = \frac{1}{2}\bar{t}^2T_{\phi_{\bar{s}u_0}}(w_\epsilon), \quad P = \frac{1}{4}\bar{t}^2T_{\phi_{\bar{t}w_\epsilon}}(w_\epsilon),$$

$$R = \bar{s}^2\bar{t}^2\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{k(x)k(y)u_0(x)u_0(y)w_\epsilon(x)w_\epsilon(y)}{|x-y|}dxdy.$$

Hölder's inequality, (2.29), Lemma 2.1(i) and (2.29), together with $k \in L^\infty(\mathbb{R}^3)$, $u_0 \in L^\infty(\mathbb{R}^3)$, imply that as $\epsilon \to 0$

$$\bar{s}\bar{t}\int_{\mathbb{R}^3} k(x)\phi_{\bar{t}w_\epsilon}u_0w_\epsilon dx) \le \bar{s}\bar{t}|k|_\infty|\phi_{\bar{t}w_\epsilon}|_6|u_0|_{\frac{12}{5}}|w_\epsilon|_{\frac{12}{5}} \le C|w_\epsilon|^3_{\frac{12}{5}} \le C'\epsilon^{\frac{3}{4}}.$$

(3.20) implies that

$$H \le C'(\epsilon^{\frac{1}{4}} + \epsilon^{\frac{3}{4}}). \tag{3.23}$$

Similarly,

$$Q \le \frac{1}{2}\bar{t}^2|k|_\infty|\phi_{\bar{s}u_0}|_6|w_\epsilon|^2_{\frac{12}{5}} \le C|w_\epsilon|^2_{\frac{12}{5}} \le C'\epsilon^{\frac{1}{2}}. \tag{3.24}$$

$$P \leq \frac{1}{2}\bar{t}^2|k|_\infty|\phi_{\bar{t}w_\epsilon}|_6|w_\epsilon|^2_{\frac{12}{5}} \leq C|w_\epsilon|^4_{\frac{12}{5}} \leq C'\epsilon. \tag{3.25}$$

For R, in view of the Hardy-Littlewood-Sobolev inequality, one has that

$$R \leq \bar{s}^2\bar{t}^2\frac{1}{4\pi}\left(\int_{\mathbb{R}^3}|k(x)u_0(x)w_\epsilon(x)|^{\frac{6}{5}}dx\right)^{\frac{5}{3}} \leq C|u_0|^2_{\frac{12}{5}}|w_\epsilon|^2_{\frac{12}{5}} \leq C'\epsilon^{\frac{1}{2}}. \tag{3.26}$$

Therefore, it follows from (3.23), (3.24), (3.25), (3.26) that as $\epsilon \to 0$

$$F \leq C'\epsilon^{\frac{1}{4}}. \tag{3.27}$$

For G, since $|x-y|^6 - x^6 - y^6 \geq -C(x^5y + xy^5)$ for all $x, y > 0$. Then it follows from $u_0 \in L^\infty(\mathbb{R}^3)$, Hölder's inequality and (2.29) that, as $\epsilon \to 0$,

$$\begin{aligned} -G &= -\frac{1}{6}\int_{\mathbb{R}^3}(|\bar{s}u_0 - \bar{t}w_\epsilon|^6 - |\bar{s}u_0|^6 - \bar{t}w_\epsilon^6)dx \\ &\leq C\int_{\mathbb{R}^3}(|\bar{s}u_0|^5|\bar{t}w_\epsilon| + |\bar{s}u_0||\bar{t}w_\epsilon|^5)dx \leq C\epsilon^{\frac{1}{4}} + C|w_\epsilon|_5^5 \leq C'\epsilon^{\frac{1}{4}}. \end{aligned} \tag{3.28}$$

By above discussions, it follows from (3.15), (3.16), (3.17), (3.18), (3.22), (3.27), (3.28), Lemma 2.7(ii) and $\frac{3}{2} < \alpha < 2$ that,

$$m_\lambda \leq I_\lambda(\bar{s}u_0 - \bar{t}w_\epsilon) \leq I_\lambda(u_0) + c^* + C'\epsilon^{\frac{1}{4}} - C'\epsilon^{1-\frac{\alpha}{2}} < c_\lambda + c^*.$$

Consequently, we complete the proof. □

3.2. Proof of Theorem 1.1

By Lemma 3.2, there is a sequence $\{u_n\} \subset \Theta_\lambda$, such that $I_\lambda(u_n) \to m_\lambda$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$, and in view of Lemma 2.6(iii), we have that $m_\lambda > 0$. Combining Lemma 3.3 and Lemma 3.4, one has that $\{u_n\}$ contains a convergent subsequence, still denoted by $\{u_n\}$, which means the energy functional satisfies condition of $(PS)_{m_\lambda}$. Therefore by the continuity of I'_λ and I_λ, there exists $u \in M_\lambda$ such that $I_\lambda(u) = m_\lambda$ and $I'_\lambda(u) = 0$. Moreover, by $\{u_n\} \subset \Theta_\lambda$, it is obvious to see that $\frac{1}{2} < \Gamma_\lambda(u_n^+, u_n^-) < \frac{3}{2}$ and $\frac{1}{2} < \Gamma_\lambda(u_n^-, u_n^+) < \frac{3}{2}$. So similar to the proof of (2.25), we have that

$$\|u^\pm\| = \lim_{n\to\infty}\|u_n^\pm\| > \left[\frac{1}{2}S^3(1 - \frac{\lambda_1}{\lambda})\right]^{\frac{1}{4}} > 0.$$

Thus u is a ground state sign-changing solution for the system (SKP).

Now, we prove that u has exactly two nodal domains. We suppose by contradiction that $u = u_1 + u_2 + u_3$, with $u_i \neq 0$, $u_1 \geq 0, u_2 \leq 0$,

$$supp(u_i) \cap supp(u_j) = \emptyset \quad \text{for } i \neq j \ (i, j = 1, 2, 3)$$

and

$$\langle I'_\lambda(u), u_i\rangle = 0 \quad \text{for } i = 1, 2, 3.$$

By a simple calculation, it is easy to see that

$$T_{\phi_u}(u) = T_{\phi_{u_1}}(u_1) + T_{\phi_{u_2}}(u_2) + T_{\phi_{u_3}}(u_3) + 2[T_{\phi_{u_1}}(u_2) + T_{\phi_{u_1}}(u_3) + T_{\phi_{u_2}}(u_3)],$$

$$\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2 = \|\nabla u_1\|_2^4+\|\nabla u_2\|_2^4+\|\nabla u_3\|_2^4+2\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2 + 2\|\nabla u_1\|_2^2\|\nabla u_3\|_2^2+2\|\nabla u_2\|_2^2\|\nabla u_3\|_2^2.$$

Setting $w = u_1 + u_2$, it is clear to see that $w^+ = u_1$ and $w^- = u_2$. Then by Lemma 2.4, there exists a unique pair (s_w, t_w) of positive numbers such that

$$s_w w^+ + t_w w^- \in M_\lambda, \quad i.e., \quad s_w u_1 + t_w u_2 \in M_\lambda$$

hence

$$I_\lambda(s_w u_1 + t_w u_2) \geq m_\lambda. \tag{3.29}$$

By the fact that $\langle I'_\lambda(u), u_i\rangle = 0$, there hold

$$\langle I'_\lambda(w), w^+\rangle = \langle I'_\lambda(u_1+u_2), u_1\rangle = \langle I'_\lambda(u), u_1\rangle - T_{\phi_{u_3}}(u_1) - b\|\nabla u_3\|_2^2\|\nabla u_1\|_2^2 < 0,$$

$$\langle I'_\lambda(w), w^-\rangle = \langle I'_\lambda(u_1+u_2), u_2\rangle = \langle I'_\lambda(u), u_2\rangle - T_{\phi_{u_3}}(u_2) - b\|\nabla u_3\|_2^2\|\nabla u_2\|_2^2 < 0.$$

Consequently, in view of Lemma 2.5, we have that

$$(s_w, t_w) \in (0,1]\times(0,1].$$

On the other hand,

$$\begin{aligned} 0 &= \frac{1}{6}\langle I'_\lambda(u), u_3\rangle \\ &= \frac{1}{6}\int_{\mathbb{R}^3}(|\nabla u_3|^2+v(x)u_3^2)dx + \frac{b}{6}(\|\nabla u_1\|_2^2+\|\nabla u_2\|_2^2+\|\nabla u_3\|_2^2)\int_{\mathbb{R}^3}|\nabla u_3|^2dx \\ &\quad + \frac{1}{6}[T_{\phi_{u_3}}(u_3)+T_{\phi_{u_1}}(u_3)+T_{\phi_{u_2}}(u_3)] - \frac{\lambda}{6}\int_{\mathbb{R}^3}f(x)u_3^2dx - \frac{1}{6}\int_{\mathbb{R}^3}|u_3|^6dx \\ &< I_\lambda(u_3) + \frac{1}{4}T_{\phi_{u_1}}(u_3) + \frac{1}{4}T_{\phi_{u_2}}(u_3) + \frac{b}{4}[\|\nabla u_1\|_2^2\|\nabla u_3\|_2^2+\|\nabla u_2\|_2^2\|\nabla u_3\|_2^2], \end{aligned} \tag{3.30}$$

$$\begin{aligned} &I_\lambda(s_w u_1 + t_w u_2) \\ &= I_\lambda(s_w u_1) + I_\lambda(t_w u_2) + \frac{1}{4}s_w^2t_w^2T_{\phi_{u_1}}(u_2) + \frac{1}{4}s_w^2t_w^2T_{\phi_{u_2}}(u_1) \\ &\quad + \frac{b}{2}s_w^2t_w^2\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2 \\ &= \frac{1}{2}s_w^2\|u_1\|^2 + \frac{1}{4}s_w^4T_{\phi_{u_1}}(u_1) + \frac{b}{4}s_w^4\|\nabla u_1\|_2^4 - \frac{\lambda}{2}s_w^2\int_{\mathbb{R}^3}f(x)u_1^2dx \\ &\quad - \frac{1}{6}s_w^6\int_{\mathbb{R}^3}|u_1|^6dx + \frac{1}{2}t_w^2\|u_2\|^2 + \frac{1}{4}t_w^4T_{\phi_{u_2}}(u_2) + \frac{b}{4}t_w^4\|\nabla u_2\|_2^4 \\ &\quad - \frac{\lambda}{2}t_w^2\int_{\mathbb{R}^3}f(x)u_2^2dx - \frac{1}{6}t_w^6\int_{\mathbb{R}^3}|u_2|^6dx \\ &\quad + \frac{1}{4}s_w^2t_w^2T_{\phi_{u_1}}(u_2) + \frac{1}{4}s_w^2t_w^2T_{\phi_{u_2}}(u_1) + \frac{b}{2}s_w^2t_w^2\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2 \end{aligned} \tag{3.31}$$

$$
\begin{aligned}
&\le \frac{1}{2}[\|u_1\|^2+\|u_2\|^2]+\frac{1}{4}[T_{\phi_{u_1}}(u_1)+T_{\phi_{u_2}}(u_2)]\\
&\quad+\frac{b}{4}[\|\nabla u_1\|_2^4+\|\nabla u_2\|_2^4]-\frac{\lambda}{2}\int_{\mathbb{R}^3} f(x)(u_1^2+u_2^2)dx\\
&\quad-\frac{1}{6}\int_{\mathbb{R}^3}(|u_1|^6+|u_2|^6)dx+\frac{1}{4}[T_{\phi_{u_1}}(u_2)+T_{\phi_{u_2}}(u_1)]+\frac{b}{2}[\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2]\\
&= I_\lambda(u_1)+I_\lambda(u_2)+\frac{1}{4}[T_{\phi_{u_1}}(u_2)+T_{\phi_{u_2}}(u_1)]+\frac{b}{2}[\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2]\\
&< I_\lambda(u_1)+I_\lambda(u_2)+\frac{1}{4}[T_{\phi_{u_1}}(u_2)+T_{\phi_{u_2}}(u_1)+T_{\phi_{u_3}}(u_2)+T_{\phi_{u_3}}(u_1)]\\
&\quad+\frac{b}{2}[\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2]+\frac{b}{4}[\|\nabla u_1\|_2^2\|\nabla u_3\|_2^2+\|\nabla u_2\|_2^2\|\nabla u_3\|_2^2].
\end{aligned}
$$

Then it follows from (3.29), (3.30), (3.31) that

$$
\begin{aligned}
m_\lambda \le I_\lambda(s_wu_1+t_wu_2) &< I_\lambda(u_1)+I_\lambda(u_2)+I_\lambda(u_3)\\
&\quad+\frac{1}{4}[T_{\phi_{u_1}}(u_2)+T_{\phi_{u_2}}(u_1)+T_{\phi_{u_3}}(u_2)+T_{\phi_{u_2}}(u_3)+T_{\phi_{u_1}}(u_3)+T_{\phi_{u_3}}(u_1)]\\
&\quad+\frac{b}{2}[\|\nabla u_1\|_2^2\|\nabla u_3\|_2^2+\|\nabla u_2\|_2^2\|\nabla u_3\|_2^2+\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2]=I_\lambda(u)=m_\lambda,
\end{aligned}
$$

which leads to a contradiction, therefore $u_3=0$ and u changes only once. That is, the ground state sign-changing solution u has precisely two nodal domains.

Remark 3.5. Under the same conditions, we give the second method of calculation.

Since $0<\lambda<\lambda_1$, $b>0$ $\langle I'_\lambda(u),u\rangle=0$ and $\langle I'_\lambda(s_wu_1+t_wu_2),s_wu_1+t_wu_2\rangle=0$, we have that

$$
\begin{aligned}
m_\lambda &= I_\lambda(u)-\frac{1}{6}\langle I'_\lambda(u),u\rangle=\frac{1}{3}(\|u\|^2-\lambda\int_{\mathbb{R}^3} f(x)u^2dx)+\frac{1}{12}[b\|\nabla u\|_2^4+T_{\phi_u}(u)]\\
&> \frac{1}{3}(\|u\|^2-\lambda\int_{\mathbb{R}^3} f(x)u_1^2dx)+\frac{1}{3}(\|u_2\|^2-\lambda\int_{\mathbb{R}^3} f(x)u_2^2dx)\\
&\quad+\frac{1}{12}[T_{\phi_{u_1}}(u_1)+T_{\phi_{u_2}}(u_2)]+\frac{1}{12}b[\|\nabla u_1\|_2^4+\|\nabla u_2\|_2^4]\\
&\quad+\frac{1}{6}[T_{\phi_{u_1}}(u_2)+b\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2]\\
&\ge \frac{1}{3}s_w^2(\|u_1\|^2-\lambda\int_{\mathbb{R}^3} f(x)u_1^2dx)+\frac{1}{12}s_w^4[T_{\phi_{u_1}}(u_1)+b\|\nabla u_1\|_2^4]\\
&\quad+\frac{1}{6}s_w^2t_w^2[T_{\phi_{u_1}}(u_2)+b\|\nabla u_1\|_2^2\|\nabla u_2\|_2^2]+\frac{1}{3}t_w^2(\|u_2\|^2-\lambda\int_{\mathbb{R}^3} f(x)u_2^2dx)\\
&\quad+\frac{1}{12}t_w^4[T_{\phi_{u_2}}(u_2)+b\|\nabla u_2\|_2^4]\\
&= \frac{1}{3}(\|s_wu_1+t_wu_2\|^2-\lambda\int_{\mathbb{R}^3} f(x)|s_wu_1+t_wu_2|^2dx)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{12}[T_{\phi_{s_w u_1+t_w u_2}}(s_w u_1+t_w u_2)]+\frac{1}{12}b\|\nabla(s_w u_1+t_w u_2)\|_2^4\\
&=I_\lambda(s_w u_1+t_w u_2)-\frac{1}{6}\langle I'_\lambda(s_w u_1+t_w u_2), s_w u_1+t_w u_2\rangle\\
&=I_\lambda(s_w u_1+t_w u_2)\geq m_\lambda
\end{aligned}
$$

which leads to a similar contradiction. Therefore the ground state sign-changing solution u has precisely two nodal domains.

3.3. Proof of Theorem 1.2

We have shown that (SKP) has a sign-changing solution u which changes sign only once. Next, we prove its energy is strictly large than two times of the least energy.

By Lemma 2.7(i), there exist $\tilde{s}, \tilde{t} > 0$, such that $\tilde{s}u^+, \tilde{t}u^- \in N_\lambda$. Combining with the Lemma 2.4, we have that

$$
\begin{aligned}
m_\lambda = I_\lambda(u) &\geq I_\lambda(\tilde{s}u^+ + \tilde{t}u^-)\\
&= I_\lambda(\tilde{s}u^+) + I_\lambda(\tilde{t}u^-) + \frac{1}{2}\tilde{s}^2\tilde{t}^2 T_{\phi_{u^+}}(u^-) + \frac{b}{2}\tilde{s}^2\tilde{t}^2\|\nabla u^+\|_2^2\|\nabla u^-\|_2^2\\
&> I_\lambda(\tilde{s}u^+) + I_\lambda(\tilde{t}u^-)\\
&\geq 2c_\lambda.
\end{aligned}
$$

Furthermore, it implies that $c_\lambda > 0$ cannot be achieved by a sign-changing function in H.

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Aixia Qian
School of Mathematical Sciences
Qufu Normal University
Shandong 273165, P. R. China
Email address: qaixia@qfnu.edu.cn

Mingming Zhang
School of Mathematical Sciences
Qufu Normal University
Shandong 273165, P. R. China
Email address: 3187636542@qq.com